An improved brake squeal source model in the presence of kinematic and friction nonlinearities

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ABSTRACT

The goal of this paper is to investigate the source characteristics of brake squeal. An improved model is suggested where a point mass is in contact with a belt moving at constant velocity. First, the governing equations with kinematic and friction nonlinearities are formulated. The kinematic nonlinearities arise from an arrangement of the springs that support the point mass, as well as from a loss of contact between the belt and the mass (due to its vertical motion). Second, the nonlinear equations are numerically solved, and a wide range of dynamic responses are observed. Results show that some assumptions made in prior articles, where a linearized model was utilized, are not valid. Third, the nonlinear equations are simplified by ignoring the contact loss nonlinearity, and then linearized about an operating point for stability considerations. Instability regimes are then obtained for a set of parameters. Further, coupled modes are found even though some contradictions between the model assumptions and linearized system solutions are observed. It is concluded that the contact loss nonlinearity is crucial, and it must not be ignored for squeal source investigation.

Keywords: Brake noise and vibration, Friction-induced noise, Nonlinear mechanical system

1. INTRODUCTION

Brake squeal is a friction-induced noise problem over the higher frequency range (beyond 1 kHz) [1, 2]. This problem has been analytically and computationally studied by many researchers, though minimal models are often used for the investigation of the source characteristics [1, 2]. The following squeal source mechanisms are proposed in the literature: 1) Stick-slip vibrations due to negative damping [3]; 2) sprag-slip phenomenon [4]; 3) self-excited vibrations with constant friction coefficient [5]; and 4) mode coupling/splitting phenomenon [6]. Some of these mechanisms have also been experimentally examined [7, 8].

In particular, Hamabe et al. [9] and Hoffmann et al. [10] studied the mode coupling/splitting phenomenon (mode lock-in/mode lock-out) with a two degree of freedom model. The stability of the system was investigated with linearized equations, and the possibility of a loss in contact between the elements that represent brake pad and disc was ignored. The current study attempts to examine
the oversimplifications made in prior work [9, 10] and seeks better solutions and interpretations. Accordingly, the main objectives are listed as follows: 1. Develop a nonlinear model of the squeal source as defined in [9, 10] by considering kinematic and friction nonlinearities; 2. Calculate the steady state dynamic response of the nonlinear model with numerical integration for different parameter sets; 3. Analyze the stability of the linearized system and compare the results of the stability analysis to those from numerical method.

2. BRAKE SQUEAL SOURCE MODEL

A two degree of freedom model proposed for the brake squeal source investigation is shown in Fig. 1. As seen, the brake pad (m) is modeled as a particle doing plane motion in \( \mathbf{e}_x - \mathbf{e}_y \) plane where \( \mathbf{e}_x \) and \( \mathbf{e}_y \) are the unit vectors in the \( X \) and \( Y \) directions, respectively. In this model, \( m \) is attached to ground with two linear springs (\( k_1 \) and \( k_2 \)) with arbitrary angles (\( \alpha_1 \) and \( \alpha_2 \)). In addition, \( m \) is positioned over a sliding belt with a constant velocity \( V_b \), and the single point contact interface is defined with a linear spring (\( k_3 \)) in between. The actuation (brake) pressure on \( m \) is simulated with a pretension (\( l_{pre} \)) applied on \( k_3 \); hence, a constant normal force on \( m \) is provided in the \( \mathbf{e}_y \) direction, though a possible separation between \( m \) and the belt is also considered.

![Fig. 1 – Brake squeal source model: Two degree of freedom mechanical system schematics](image)

The governing equations of the system of Fig. 1 are derived by defining the elastic force vectors acting upon \( m \) and resolving them on the \( \mathbf{e}_x \) and \( \mathbf{e}_y \) axes. The force equilibria in both axes yield the following governing equations:

\[
\begin{align*}
    m\ddot{X} + \frac{k_1}{\sqrt{X^2 - 2XL\cos(\alpha_1) + Y^2 - 2YL\sin(\alpha_1) + L_1^2 - L_2^2}} (X - L_2 \cos(\alpha_1)) \\
    + \frac{k_2}{\sqrt{X^2 - 2XL\cos(\alpha_2) + Y^2 - 2YL\sin(\alpha_2) + L_2^2 - L_3^2}} (X - L_3 \cos(\alpha_2)) \\
    + \frac{0.5\mu(V_r)k_3(l_{pre} - Y)\left(1 + \text{sign}(l_{pre} - Y)\right)}{l_{pre} - Y} = 0
\end{align*}
\]

\[
\begin{align*}
    m\ddot{Y} + \frac{k_1}{\sqrt{X^2 - 2XL\cos(\alpha_1) + Y^2 - 2YL\sin(\alpha_1) + L_1^2 - L_2^2}} (Y - L_2 \sin(\alpha_1)) \\
    + \frac{k_2}{\sqrt{X^2 - 2XL\cos(\alpha_2) + Y^2 - 2YL\sin(\alpha_2) + L_2^2 - L_3^2}} (Y - L_3 \sin(\alpha_2)) \\
    + \frac{-0.5\mu(V_r)k_3(l_{pre} - Y)\left(1 + \text{sign}(l_{pre} - Y)\right)}{l_{pre} - Y} = 0
\end{align*}
\]

From above, first observe that the configuration of \( k_1 \) and \( k_2 \) (with angles \( \alpha_1 \) and \( \alpha_2 \)) creates a kinematic nonlinearity. Second, the contact loss nonlinearity is described with a ‘sign’ function with the \( k_3 \) term; this represents the contact stiffness, and thus, it cannot work as a two-way spring. This means that \( k_3 \) can only generate force when \( Y < l_{pre} \), and for \( Y \geq l_{pre} \), \( m \) loses contact with belt. Third, the friction coefficient \( \mu \) between \( m \) and the belt in Eq. (1) is defined as a function of relative velocity \( V_r = X - V_b \). This is due to the directional behavior and speed dependency of the friction force.
However, the current study focuses only on the steady sliding motion where $V_b > X$ at any time. Consequently the friction force acting on $m$ is always in the $+e_x$ direction, and the friction coefficient becomes only a function of the relative velocity magnitude. The first friction model considers the Coulomb formulation ($\mu = \mu_s$), where the friction coefficient is constant; hence, the friction model does not introduce an additional nonlinearity. However for the second friction model, Striebeck friction model ($\mu = \mu_k + (\mu_s - \mu_k) \exp(-\gamma|V_r|)$), the friction coefficient is a function of the relative motion between $m$ and the belt. In both friction models, $\mu_s$, $\mu_k$, and $\gamma$ represent the static friction coefficient, dynamic friction coefficient, and exponential decay coefficient, respectively. Note that the Coulomb model could be written as a special case of the Striebeck friction model where $\mu_s = \mu_k$.

### 3. NUMERICAL SOLUTION OF THE NONLINEAR SOURCE MODEL

In order to understand the dynamic behavior of the system of Fig. 1, Eqs. (1) and (2) are numerically solved for different cases using an event detection algorithm [11], i.e. two sets of equations for contact and contact loss cases are simultaneously solved by tracking the $Y - l_{pre} = 0$ condition. First, the difference between Coulomb and Striebeck friction models is investigated by solving the governing equations with the same set of parameters except $\mu_k$, and the results are displayed in Fig. 2 in terms of $X(t)$. As expected, the negative damping nature of the Striebeck friction model causes a growth in the $X(t)$ amplitude; hence, an unstable motion is observed (Fig. 2(b)). Even though the $Y(t)$ results are not given, the parameters used for the solutions of Fig. 2 are specifically chosen to ensure a full contact between $m$ and belt, i.e. $Y < l_{pre}$ at all times.

Since the Striebeck friction model leads to unstable motions, Coulomb formulation is considered for further investigation. First, the $X(t)$ and $Y(t)$ results of the previous case are examined in the frequency domain, as evident from the spectra of Fig. 3. Two main frequencies of $X(t)$ and $Y(t)$ motions are 987 Hz and 1208 Hz. Since there is no contact loss and $\alpha_1 = \pi/2$ and $\alpha_2 = 0$, these frequencies can be analytically determined by the natural frequencies, i.e. $f_1 = \sqrt{k_2/m}/2\pi = 987$ Hz and $f_2 = \sqrt{(k_1+k_3)/m}/2\pi = 1208$ Hz. Due to relatively close natural frequencies, $X(t)$ motion shows beating behavior as found from the zoomed view in Fig. 2(a). In addition, the second super-harmonic of $f_2$ appears in Fig. 3(a) due to the kinematic nonlinearity; however, it does not have a significant contribution being 70 dB below as indicated in Fig. 3(a). Next, the value of $k_1$ is decreased by an order of magnitude while keeping $\alpha_1$ and $\alpha_2$ intact in order to achieve a loss of contact. Time histories of $X(t)$ and $Y(t)$ along with their frequency spectra are shown in Figs. 4 and 5.
Fig. 3 – Source motion spectra for $k_1 = k_2 = 2k_3$, $\alpha_1 = 0.5\pi$, $\alpha_2 = 0$ and $\mu_s = 0.4$. a) $X(t)$; b) $Y(t)$.

Fig. 4 – Predicted motions for $10k_1 = k_2 = 2k_3$, $\alpha_1 = 0.5\pi$, $\alpha_2 = 0$ and $\mu_s = 0.4$. a) $X(t)$ vs. $t$; b) $Y(t)$ vs. $t$. The red dashed line represents the threshold of separation.

Yet another beating type oscillation is observed in Fig. 4(a) for $X(t)$, though the spectra of both $X$ and $Y$ motions (Fig. 5) suggest that these are amplitude modulated signals with a carrier frequency $f_c = 537$ Hz and modulation frequency of $f_m = 87$ Hz. Observing the $Y(t)$ motion, it is seen that there is separation between $m$ and the belt for 60% of the time in one period. Hence, the natural frequency of the system is $f_2 = \sqrt{k_1/m/2\pi} = 987$ Hz for 60% of the period and $f_2 = \sqrt{(k_1+k_3)/m/2\pi} = 1208$ Hz for 40% of the period. Therefore, the carrier frequency can be approximated as $f_c \approx 0.5 \times ((0.6 \times 987) + (0.4 \times 1208)) = 537$ Hz. Moreover, super-harmonics of $f_c$ and its side bands are also found in $X(t)$ and $Y(t)$ signals (Fig. 5) due to strong nonlinearities.

Fig. 5 – Motion spectra for $10k_1 = k_2 = 2k_3$, $\alpha_1 = 0.5\pi$, $\alpha_2 = 0$ and $\mu_s = 0.4$. a) $X(t)$; b) $Y(t)$.

Finally, yet another unstable $X(t)$ motion is observed when $\alpha_1$ and $\alpha_2$ are varied. Here, it should be noted that this unstable motion is obtained with the Coulomb formulation. As seen in Fig. 6(a), the amplitudes of $X(t)$ motion gradually increase, and a peak at 1387 Hz is observed in its spectrum (Fig. 7(a)). However, the $Y(t)$ motion reaches a steady state eventually, though it shows a beating phenomenon due to closely spaced frequencies of 1387 Hz and 1427 Hz. Again, due to strong nonlinearities, super-harmonics of 1387 Hz are also excited as seen in Fig. 7. As a final note, it is observed that $V_b > \dot{X}$ at any time during simulations. Therefore, the steady sliding assumption is
4. LINEAR SOURCE MODEL

In order to estimate the stability of the system, first it is assumed that \( m \) is always in contact with the belt as in prior work [9, 10]. This assumption allows the ‘sign’ functions to be dropped in Eqs. (1) and (2). However, this assumption may contradict numerical predictions, since separation is achieved for certain cases as already shown with Figs. 4 and 6. After dropping the ‘sign’ terms in Eqs. (1) and (2), the nonlinear equations are linearized about the origin (X, Y) = (0, 0). Using the Taylor series expansion, the following is obtained:

\[
mx'' + k_{11}x + (k_{12} - \mu k_3) y = -\mu k_3 l_{pre},
\]

\[
m\ddot{y} + k_{21}x + k_{22}y = k_3 l_{pre},
\]

where \( k_{11}, k_{12}, k_{21} \) and \( k_{22} \) are:

\[
k_{11} = k_1 \cos^2(\alpha_1) + k_2 \cos^2(\alpha_2),
\]

\[
k_{12} = k_{21} = k_1 \sin(\alpha_1) \cos(\alpha_1) + k_2 \sin(\alpha_2) \cos(\alpha_2),
\]

\[
k_{22} = k_1 \sin^2(\alpha_1) + k_2 \sin^2(\alpha_2).
\]

For the sake of simplicity, Eqs. (3) and (4) are normalized with the following dimensionless parameters:

\[
\omega_{11}^2 = k_{11}/m, \quad \omega_{12}^2 = k_{12}/m, \quad \omega_{21}^2 = k_{21}/m, \quad \omega_{22}^2 = k_{22}/m, \quad \tau = \omega_{11}^2, \quad x = X/l_{pre}, \quad y = Y/l_{pre},
\]

and the linearized governing equations in non-dimensional form are obtained as follows where \( \dot{\eta} = d\eta/d\tau \).

\[
x'' + x + \left( (\omega_{12}^2 - \mu \omega_{11}^2) / \omega_{11}^2 \right) y = -\mu \omega_{11}^2 / \omega_{11}^2,
\]

\[
y'' + \left( \omega_{12}^2 / \omega_{11}^2 \right) x + \left( \omega_{22}^2 / \omega_{11}^2 \right) y = \omega_{22}^2 / \omega_{11}^2.
\]

The Jacobian matrix should be calculated around the static equilibrium to check the system stability. The static equilibrium point and Jacobian matrix are calculated only for the Stribeck
friction model. In non-dimensional form, the Striebeck friction model is now 
\[ \mu = -\mu_s - (\mu_c - \mu_s) \exp(-\gamma l_{pre} \omega_1(v_b - x')) \], where \( v_b = V/\omega_{pre} \omega_1 \). Using this expression in Eqs. (7) and (8) and solving them for \( x \) and \( y \) for \( x' = 0 \) and \( y' = 0 \), the static equilibrium coordinates \( (x^*, y^*) \) are obtained as:

\[
x^* = \frac{\omega_1^2 \left( \omega_1^2 - \omega_1^3 \right) + \left( \omega_1^2 - \omega_1^3 \right) \exp \left[ \gamma l_{pre} \omega_1 \omega_1 \right] \right)}{\left( \omega_1^2 - \omega_1^3 \omega_1^2 + \mu \omega_1 \omega_1^3 \exp \left[ \gamma l_{pre} \omega_1 \omega_1 \right] \right) - \left( \mu - \mu_s \right) \omega_1^2 \omega_1^3} \], \hspace{1cm} (9a)

\[
y^* = \frac{\omega_1^3 \left( \omega_1 - \mu \right) + \left( \omega_1^2 \omega_1 - \omega_1^3 \right) \exp \left[ \gamma l_{pre} \omega_1 \omega_1 \right] \right)}{\left( \omega_1^2 - \omega_1^3 \omega_1^2 + \mu \omega_1 \omega_1^3 \exp \left[ \gamma l_{pre} \omega_1 \omega_1 \right] \right) - \left( \mu - \mu_s \right) \omega_1^2 \omega_1^3} \]. \hspace{1cm} (9b)

By transforming the motion variables from the origin to the static equilibrium; i.e. \((x, y) \Rightarrow (x+x^*, y+y^*)\), Eqs. (7) and (8) become:

\[
x'' + \beta \gamma l_{pre} \omega_1 (y' - 1)x' + x + (\rho + \beta) y = -\beta y \sum_{n=1}^{\infty} \left( \gamma l_{pre} \omega_1 x' \right)^n / n! - \beta (y' - 1) \sum_{n=1}^{\infty} \left( \gamma l_{pre} \omega_1 x' \right)^n / n!,
\]

\[
y'' + \left( \omega_1^2 / \omega_1^2 \right) x + \left( \omega_1^2 / \omega_1^2 \right) y = 0,
\]

where \( \rho = (\omega_1^2 + \mu_s \omega_1^2) / \omega_1^2 \) and \( \beta = (\mu - \mu_s) \omega_1^2 / \omega_1^2 \exp(\gamma l_{pre} \omega_1 v_b) \). Note that the exponential term of the Striebeck friction model is expanded in power series in order to obtain Eqs. (10) and (11); hence, the sum of series appear in Eq. (10). Equations (10) and (11) are now the equations of motion around the static equilibrium, and the right hand sides of these equations only include the nonlinear terms; there are no mean or DC terms. Hence, the Jacobian matrix is obtained as:

\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & -\beta \gamma l_{pre} \omega_1 (y' - 1) & -\rho - \beta & 0 \\
0 & 0 & 0 & 0 \\
-\omega_1^2 / \omega_1^2 & 0 & -\omega_1^2 / \omega_1^2 & 0
\end{bmatrix}.
\] \hspace{1cm} (12)

Note that \((x^*, y^*)\) and \(J\) for the Coulomb friction model is obtained by setting \(\mu_s = \mu_c\) in Eqs. (9) and (12).

5. MODE COUPLING BEHAVIOR AND STABILITY MAPS BASED ON LINEAR MODEL

The Jacobian matrix obtained before as Eq. (12) is used for stability analyses. First, the stability of the system with Striebeck friction model is used where \(\mu_s = 0.4\) and \(\mu_c = 0.2\). In the analyses, \(\alpha_1\) is varied from 0 to \(\pi\) while \(\alpha_2 = 0.125\pi\), and the eigenvalues \(\lambda\) are calculated. The real and imaginary parts of these eigenvalues are shown in Fig. 8 with respect to \(\alpha_1\). Since \(\text{Re}(\lambda_i) > 0, i = 1, 2\) for \(0 \leq \alpha_1 \leq \pi\), both vibration modes are unstable. This is expected as the negative damping characteristics of the Striebeck friction model lead to dynamic instability. For the Coulomb friction model, calculated eigenvalues are shown in Fig. 9. Now, \(\text{Re}(\lambda_i) = 0, i = 1, 2\) for \(\alpha_1 \leq 0.68\pi\), and the system has two vibration modes at distinct frequencies and \(\text{Im}(\lambda_1) \neq \text{Im}(\lambda_2)\). Observe in the \(0.68\pi < \alpha_1 < 0.75\pi\) region, \(\text{Re}(\lambda_1) > 0\) and \(\text{Re}(\lambda_2) < 0\). Therefore, one stable and one unstable mode emerge at the same frequency since \(\text{Im}(\lambda_1) = \text{Im}(\lambda_2)\). Beyond \(\alpha_1 = 0.75\pi\), frequencies of vibration modes diverge again, and the system is now stable. This is known as the “mode coupling” phenomenon, and it is one of the well-known mechanisms for brake squeal source [1].
Fig. 8 – Calculated eigenvalues for the Stribeck model with $k_1 = k_2 = 2k_3$, $\alpha_2 = 0.125\pi$, $\mu_s = 0.4$, $\mu_k = 0.2$. a) Re($\lambda$); b) Im($\lambda$).

Fig. 9 – Calculated eigenvalues for the Coulomb model with $k_1 = k_2 = 2k_3$, $\alpha_2 = 0.125\pi$, $\mu_s = \mu_k = 0.4$. a) Re($\lambda$); b) Im($\lambda$).

As a further investigation, the stability map of the system is obtained by simultaneously varying $\alpha_1$ and $\alpha_2$ over the $[0, \pi]$ interval, and the results are shown in Fig. 10. Observe distinct instability regions (shaded areas) for different $(\alpha_1, \alpha_2)$ configurations, and the one at the lower right of Fig. 10 represents the mode coupling behavior of Fig. 9. In addition, the unstable motion of Fig. 6 (with $\alpha_1 = \alpha_2 = 0.4\pi$) is now located in the unstable region of Fig. 10 (in the middle). Even though the stability map and the numerical solution both suggest unstable motions, these two analyses do not represent the same physics. This is due to the fact that the stability analysis is valid only when $m$ and the belt are in contact.

Fig. 10 – Stability map of the linear system for $k_1 = k_2 = 2k_3$, $\mu_s = \mu_k = 0.4$. Shaded regions denote unstable motions.

Figure 11 shows numerical predictions for another $(\alpha_1, \alpha_2)$ arrangement from the lower right unstable region of Fig. 10. In this case, the stability map predicts that the motions are unstable, unlike the time histories of Fig. 11 that display stable motions with a loss of contact during the steady state. As a consequence, the results of the stability map (based on a linear model) do not represent the true physics of a system, and the perpetual contact assumption between $m$ and belt is
Fig. 11 – Predicted motions for \( k_1 = k_2 = 2k_3, \alpha_1 = 0.7\pi, \alpha_2 = 0.125\pi \) and \( \mu_s = 0.4 \). a) \( X(t) \) vs. \( t \); b) \( Y(t) \) vs. \( t \).

The red dashed line represents the threshold of separation.

6. CONCLUSION

In this study, the source regime of the brake squeal problem is investigated with a simple discrete model. First, the nonlinear governing equations are derived with the elastic force vectors. In the formulation, both kinematic and friction nonlinearities are considered. Second, numerical solutions are obtained for several cases in order to show the wide range of dynamic responses of such a simple system. Numerical integrations are carried out with event detection algorithms [11]. Observations include the following: a. The Stribeck friction model leads to unstable motions due to its inherent negative damping behavior; b. a weak coupling between the two orthogonal motions of \( m \) can be achieved for specific \((\alpha_1, \alpha_2)\) arrangements; c. a loss of contact between the mass \( m \) and the belt can occur even under a preload; and d. brake squeal type dynamic behavior, such as beating oscillations, amplitude modulation, and unstable motions are observed for different \((\alpha_1, \alpha_2)\) arrangements. Third, the nonlinear governing equations are linearized about the origin \((X, Y) = (0, 0)\) for the continual contact between \( m \) and belt, and a mode coupling is found. However, assumptions made to facilitate stability investigation show some contradictions when compared with the numerical results. Therefore, the contact loss nonlinearity must not be ignored in squeal source models.

ACKNOWLEDGEMENT

We acknowledge the Smart Vehicle Concepts Center (www.SmartVehicleCenter.org) and the National Science Foundation Industry/University Cooperative Research Centers program (www.nsf.gov/eng/iip/iucrc) for supporting this fundamental study.

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