Estimation of nonlinear frequency response for a system with backlash nonlinearity using multi-term harmonic balance method

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Most real-life structures include inherent nonlinearities, and their responses (such as unstable motions, limit cycles, bifurcations and chaotic motions) are of significant interest to researchers. The harmonic balance method is a powerful tool to approximate the dynamic response in a more tractable form since it converts the nonlinear ordinary differential equations into a set of manageable nonlinear algebraic equations.

The goal of this study is to apply the multi-term harmonic balance method to a mechanical oscillator with clearance nonlinearity as depicted in Fig. 1. As shown in Fig. 1, a mass \( m \) moves with \( x(t) \) due to the external excitation force \( F(t) \), and two sets of elastic \( (k) \) and dissipative \( (c) \) elements are located on both sides of \( m \) with a backlash (clearance) of \( b \). Elements of gear rattle in vehicles can be simulated using this model. The chief objective of the current study is to estimate the nonlinear frequency responses using the multi-term harmonic balance method.

**Figure 1.** Single degree of freedom model with clearance nonlinearity.

The governing equation of the system of Fig. 1 is given with \( g(x, \dot{x}) \) and \( h(x) \) as dissipative and elastic functions, respectively,
\[
mx + cg(x, \dot{x}) + kh(x) = F(t).
\]
(1)

The piecewise linear functions are described as:
\[
g(x, \dot{x}) = \begin{cases} \dot{x} & ; \dot{x} < 0 \\ 0 & ; \dot{x} = 0 \\ \dot{x} & ; \dot{x} > 0 \end{cases} \quad h(x) = \begin{cases} x - b & ; x > b \\ 0 & ; -b \leq x \leq b \\ x + b & ; x < -b \end{cases}.
\]
(2a, b)

The excitation force \( F(t) \) is assumed to be periodic of order \( M \) with a constant mean value, i.e. \( F(t) = F_0 + \sum_{m=1}^{M} F_m \sin(m\omega t) \). For the multi-term harmonic balance solution, Eq. (1) is first transformed from \( t \) domain to spatial \( \theta \) domain by assuming \( \theta = \omega t \) with \( \theta \in [0, 2\pi) \). Second, the Fourier series expansion is assumed for the solution of Eq. (1), as: \( x(\theta) = a_0 + \sum_{n=1}^{\infty} a_{2n-1} \sin(n\theta) + a_{2n} \cos(n\theta) \). Third,
the discrete Fourier transform operator ($\Gamma$) and differential operator ($D$) are defined, and the nonlinear elastic and dissipative functions along with $x(\theta)$ are discretized as $g(x,x') = \Gamma \gamma$, $h(x) = \Gamma \eta$ and $x(\theta) = \Gamma a$, where $' = d/d(\theta)$. Finally Eq. (1) is rewritten as follows.

$$m\omega^2 \Gamma D^2 a + c\omega\Gamma\gamma + k\Gamma \eta = \Gamma Q.$$  

(3)

Equation (3) can now be transformed to the frequency domain by pre-multiplying it with the pseudo-inverse of $\Gamma$ ($\Gamma^+ = (\Gamma^T \Gamma)^{-1} \Gamma^T$), and the residue equation is defined as:

$$R = m\omega^2 D^2 a + c\omega\gamma + k\eta - Q.$$  

(4)

Equation (4) represents a set of nonlinear algebraic equations; therefore it is iteratively solved using the Newton-Raphson method. In the solution, the pseudo arc-length continuation method is also applied in order to successfully track the nonlinear frequency response curve, especially in the vicinity of turning points [1, 2]. The calculated nonlinear frequency response curve is shown in Fig. 2 in terms of the maximum normalized $\bar{x}$ amplitudes, where $\bar{x} = x/b$. First, observe multiple peaks due to the multiple orders in $F(t)$. Peaks are well separated at the higher frequency range; however, they are closer in the lower frequency range. Second, higher orders of $F(t)$ do not excite the nonlinearity due to their lower amplitudes, hence the response is linear in the lower frequency region. The bending of the resonant peaks is clearly seen at first and second orders. This demonstrates the amplitude dependent response of a nonlinear system.

![Figure 2](image)

**Figure 2.** Nonlinear frequency response of the model of Fig. 1.

In order to better observe the interactions between adjacent peaks, the DC value of the $F(t)$ ($F_0$) is halved and the calculated nonlinear frequency response, in the lower frequency regime only, is displayed in Figure 3. An isolated branch emerges over the lower frequency range, which is away from the peaks of Fig. 2. It is believed that this branching is due to interactions between adjacent orders. Furthermore, the frequency response curve follows an interesting path as shown in the zoomed view of Fig. 3.
Figure 3. Isolated branch over lower frequency region.

In summary, this study examines the application of multi-term harmonic balance method for a system with backlash nonlinearity. It is shown that even a single degree of freedom system with a discontinuous nonlinearity can exhibit a wide class of dynamic responses. Furthermore, the stability of the solutions is also checked using the Hill’s method [1-3]. In addition, the method is utilized to track and identify the bifurcations, though such results are not included in the text due to page limitations.

References