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EXPERIMENTAL MODAL ANALYSIS OF NON-LINEAR SYSTEMS: A FEASIBILITY STUDY

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Experimental modal analysis techniques are based on the theory of linear systems. In this paper the feasibility of their application to non-linear systems is examined. A three degree of freedom non-linear mechanical system was modeled on an analog computer and system responses due to band limited random excitation were examined by using a two channel FFT analyzer. Frequency response functions, natural frequencies and model responses were studied experimentally for a number of example cases. This study shows that the measured transfer function plots can alert one to the fact that the structure is non-linear; measured modal response data base can still give a rough idea about the system behavior. In addition, measured coherence function estimates can be used as the "warning signals" for inherent non-linearities.

1. INTRODUCTION

1.1. GENERAL

Experimental modal analysis systems are now widely available and used extensively for system identification, diagnostics and mathematical model development. A typical user treats the system as a "black box" and generally relies on commercial software. Moreover, he may not be aware of the underlying assumptions and limitations associated with modal extraction and synthesis procedures which are based on the theory of linear systems [1–3]. This is especially true when the structure is inherently non-linear; hence incorrect results could be obtained. The focus of this study is to examine the feasibility of using conventional modal analysis techniques for a multi-degree of freedom non-linear system.

1.2. NON-LINEAR SYSTEMS

Non-linearities in a vibrating system generally result from such non-linear elastic mechanisms as the physical gap (dead zone) between a mass and spring [4], and non-linear damping elements [5, 6]. These non-linearities are enhanced when the response amplitudes are no longer small. For a non-linear system the following essential properties of linear systems are not valid: (i) amplitude scaling; (ii) the principle of superposition [7–9]. Consequently, one starts to question many of the steps associated with conventional modal testing.

1.3. SCOPE

An experimenter would obviously like to know if the system being tested is non-linear in nature. If the system is detected to be non-linear, then the experimenter is interested in determining the extent of non-linearities, and the identification of the system and the extraction of inherent properties. In addition, it is desirable to know whether or not such a non-linear system could be approximated as a linear system.

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In order to answer some of these questions, we first review the recent literature on experimental modal analysis techniques applied to both linear and non-linear systems. Then we describe a study of a multidegree of freedom mechanical system with several non-linear elastic elements. Conventional modal testing techniques are used to examine the following: (i) frequency response functions; (ii) coherence function; (iii) natural frequencies; (iv) modes. Since for a non-linear system modal data are dependent on response, we will be comparing the normalized response at a given resonant frequency. It should be noted that our study is limited in scope as only the frequency domain approach [3] will be used.

2. LITERATURE REVIEW ON EXPERIMENTAL MODAL ANALYSIS

2.1. LINEAR SYSTEMS

The general theory of modal analysis is strictly valid for linear systems [1-4]. Typically, dynamic compliance transfer functions $H(\omega)$ or impulse responses $h(t)$ are measured at a number of locations, and then, using the modal expansion theorem, as shown below, one can estimate natural frequencies ω_r, mode shapes ψ_r, damping ratios ζ_r, modal masses or modal stiffnesses over the frequency range of interest [2, 3] (see the List of Symbols for identification):

$$H_{jk}(\omega) = [A_{jkr}/(s-s_r)] + [A^*_{jkr}/(s-s^*_r)], \quad \text{or} \quad h_{jk}(t) = \sum_{r=1}^{\infty} A_{jkr}\, e^{s_r t} + A^*_{jkr}\, e^{s^*_r t}, \tag{1, 2}$$

$$\text{where} \quad s_r = -\zeta_r\omega_r + i\omega_r\sqrt{1-\zeta_r^2}. \tag{3}$$

Experimental modal analysis techniques have been applied to both symmetric and non-symmetric systems, structures with repeated roots, etc., [1-3]. However, in each case the vibrating system is considered linear with small perturbations about the operating point. Further, non-linear Coulomb and hysteretic damping cases have also been analyzed, by treating them as linear equivalent-viscous damping cases and using conventional techniques [5, 6].

2.2. NON-LINEAR SYSTEMS

An extensive amount of literature is available on analytical or computational techniques applied to non-linear systems; references [7-10] are typical. But the same cannot be said for the application of modern experimental techniques to non-linear vibratory systems. In fact, only a few studies have been published recently [11-28] in which the feasibility of using experimental techniques for non-linear vibratory systems has been examined.

Ulm and Morse [11] have examined Duffing's equation on an analog computer. Both swept sine and random inputs yielded sharp jumps in the amplitude and phase plots, shape distortion of the Nyquist plots, and poor coherence at the third harmonic of the system resonance. Moreover, it was noted that higher excitation levels generally led to more deviations in the transfer functions from the linear system response. Okubo [12] has also analyzed such a system numerically with an impulse excitation somewhat similar to that applied in experimental testing. He has also reported distortions in transfer functions such as split and sharp peaks in the magnitude plots, real and imaginary parts in reverse, and additional small circles in the Nyquist plane. Okubo and Honda [13] performed a similar analysis on a two degree of freedom system and found distortion in frequency response functions. Goyder [14] tested a single-degree-of-freedom non-linear system with harmonic excitation; measured frequency response functions had considerable distortion from the responses expected from a linear structure. Based on this Goyder stated that the use of sinusoidal pulse or sweep excitation does not seem appropriate for non-linear systems. He also used random excitation and noted that the frequency response function looks like that produced by a linear structure and hence concluded that random excitation is suitable for non-linear structures. Moreover, he suggested that a non-linear model under random excitation can be obtained from the linear model plus an added response term determined by treating non-linearities as noise. Consequently, low coherence between excitation and response should be obtained.

Ewins [15] ran a series of modal tests on a specifically built test piece designed to simulate characteristics of helicopter structures; the test identified the presence of weak non-linearities. He stated that a systematic analysis can eliminate the non-linear effects from the results. But he pointed out that further refinements in modal testing are required to deal with non-linear problems.

Tamhane [17] looked at the feasibility of using the impact technique on a non-linear system by presenting certain ways to control the forcing function level. Lagnese and Koester [18] experimentally investigated dry friction damping and identified narrow frequency bandwidth quasi-linearities of a single turbine blade. Leete [19] used a finite element model of the TORS antenna module to match the measured modal frequencies at the hinges. Ito *et al.* [20] analyzed a complex structure using the building block approach combined with the non-linear modal transient response method.

Tomlinson and Kirk [21] have stated that the principal limitation to date, regardless of the testing procedure employed, is that in order to analyze non-linear structures important assumptions have to be made; these refer to an assumed model, or models of the non-linearity and to the degree of non-linearity present. They presented a Hilbert transform method which overcomes some of these shortcomings as this method provides a relationship between the real and imaginary parts of a complex frequency response function. This technique has been shown to be useful in identifying and characterizing non-linearities through application to a single-degree-of-freedom system. Vinh *et al.* [22] also applied the Hilbert transform method to a single degree of freedom mechanical structure by linearization. Further, Simon and Tomlinson [23] have shown that typical structural non-linearities can be detected and identified directly without generating an explicit model. A discrete Hilbert transform algorithm was also developed and truncation effects arising from a finite frequency range were accounted for by employing correction terms in the frequency domain. Tomlinson [24] has extended the Hilbert transform technique to investigate multi-mode non-linear systems. He also showed that if power weighted frequency response functions are employed then it is possible to detect and identify non-linearities without the use of correction terms.

Yang and Ibrahim [25] have described a non-parametric identification technique based on the use of a power series expansion. Fillod *et al.* [26] took the non-linearities into account in the experimental modal analysis by curve fitting the transfer functions (1) before the resonance only, (2) after the resonance only, and (3) before and after the resonance. The proposed method appears to be efficient, even for large non-linearities, when a mode is well isolated. Hunter [27] also used a curve fit technique based on the assumption that the system is linear. Ratcliffe [28] indicated that it may not be possible to identify non-linearities easily from a single frequency response function if random or step excitation is employed.

Based on the literature survey, it is evident that many of the proposed methods have certain inherent limitations and work only for weakly non-linear systems, and that research on experimental identification of non-linear systems is just being initiated.

3. THREE DEGREE OF FREEDOM SYSTEM

Figure 1 shows the example case which allows a combination of translational (q_1, q_2) and rotational (q_3) motions, with three degrees of freedom. The dampers are assumed here to be linear, characterized by an equivalent viscous damping coefficient c. Translational springs are considered to be of the hardening type such that the restoring force f_s

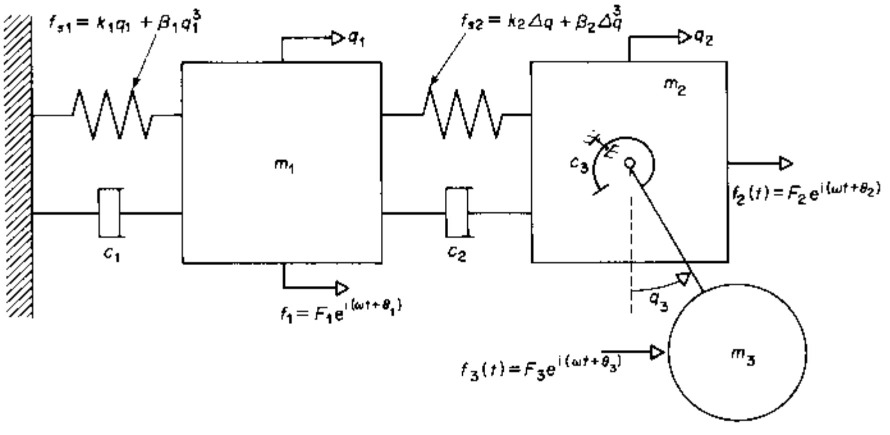

Figure 1. Example case. C_3 is a rotary damper.

is equal to $k\Delta q + \beta \Delta q^3$ (Duffing type) where k and β are spring constants and Δq is the relative displacement. The forcing functions are applied at the masses and the initial conditions are assumed to be zero. The equations of motion, in matrix form, show clearly the extent of the non-linearities:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2+m_3 & m_3 l \cos q_3 \\ 0 & m_3 l \cos q_3 & m_3 l^2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2 & -m_3 l \dot{q}_3 \sin q_3 \\ 0 & 0 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}$$

$$+ \begin{bmatrix} k_1+k_2+\beta_1 q_1^2 + \beta_2 (q_1-q_2)^3/q_1 & k_2 & 0 \\ -k_2 & k_2+\beta_2(q_2-q_1)^3/q_2 & 0 \\ 0 & 0 & m_3 q l (\sin q_3)/q_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_1 e^{i\theta_1} \\ F_2 e^{i\theta_2} + F_3 e^{i\theta_3} \\ F_3 e^{i\theta_3} l \cos q_3 \end{Bmatrix} e^{i\omega t}. \quad (4)$$

Many cases of this non-linear system can be examined by varying spring constants (k and β) and force levels (F). This example case can also be reduced to single and two degree of freedom systems. Of course, in each case one can also examine the linear system by considering q_1, q_2, and q_3 to be small and by setting $\beta_1 = \beta_2 = 0$.

The analytical solution for sinusoidal excitation is given in Appendix A.

4. EXPERIMENT

Since it is difficult, if not impossible, to build a physical system with known and controlled non-linearities for testing purposes, we have conducted a simulated experiment. The physical system was simulated on an analog computer circuit, as shown in Figure 2,

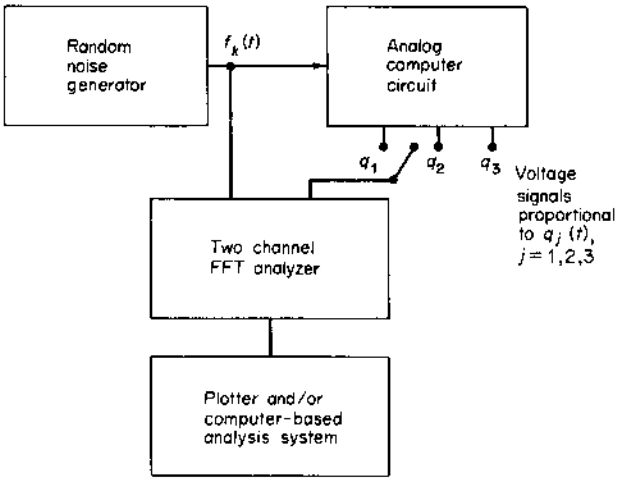

Figure 2. Experimental setup.

and the computer outputs considered as physical system outputs. The forcing function $f_k(t)$ was band-limited white noise which would excite the system over all frequencies of interest. This input along with an output q_j were acquired and processed by a two channel FFT analyzer, in a manner identical to that which one would employ for a real structure. Only transfer functions over the frequency range of interest, in various forms, were analyzed and plotted; standard modal extraction techniques [2, 3] were not used.

5. EXPERIMENTS, RESULTS AND DISCUSSION

5.1. PRESENTATION OF RESULTS

In order to perform some parametric studies the general system was simplified and numerical values assigned to the system parameters and variables, as follows (several dimensionless quantities (given by an overbar) are also defined, as the results are presented in dimensionless form): $m_1 = m_2 = m_3 = m$; $k_1 = k_2 = k$; $c_1 = c_2 = c$; $c_3 = 0$; $c = 0{\cdot}10$; $\beta_1 = \bar{\beta}$, $\beta_2 = \bar{\beta}$, $\bar{\beta} = 0$, 0·3, 0·5, and 5·0; $F_2 = F_3 = 0$, $F_1 = F$; $\theta_1 = 0$; $\bar{q} = q/l$, $\bar{F} = F/k$, $\bar{\beta} = \beta l^2/k$, $\bar{c} = c/2\sqrt{mk}$, $\bar{\omega} = m^2/k^2$, $\bar{\omega}_r = \omega_r m^2/k^2$, $\bar{H}_{jk}(\bar{\omega}) = \bar{q}_j/\bar{F}_k = M(\bar{\omega})\,\mathrm{e}^{\mathrm{i}\phi(\bar{\omega})} = \mathrm{Re}\,(\bar{\omega}) + \mathrm{i}\,\mathrm{Im}\,(\bar{\omega})$, where $j, k = 1, 2, 3$.

The following modal parameters were defined, which could be used to indicate the extent of non-linearity: (1) modal response ratio, $\bar{R}_{jr} = \bar{H}_{1kr}/\bar{H}_{jkr}$, $j, k = 1, 2, 3$; for the forcing function case employed here $\{\bar{R}\}_r^{\mathrm{T}} = [1, \bar{R}_{2r}, \bar{R}_{3r}]$; for the linear case where modes are well separated, $\psi_r^{\mathrm{T}} = [1, \bar{R}_{2r}, \bar{R}_{3r}]$, where ψ_r^{T} is the transpose of the rth mode, normalized by taking the modal response of m_1 equal to unity; (2) "backbone" natural frequency $(\bar{\omega}_r)_0$; this serves as a reference frequency for the rth mode as this would be equal to the natural frequency $\bar{\omega}_r$ if the system were linear; all $\bar{R}_{jr}$ values are computed at frequency $(\bar{\omega}_r)_0$; (3) modal frequency of highest magnitude $(\bar{\omega})_{\max}$; at this frequency the normalized transfer function magnitude M has the maximum value for the rth mode. Note that for a non-linear system $(\bar{\omega}_r)_0$ may not be equal to $(\bar{\omega}_r)_{\max}$ which one would extract from the measured or computed $M(\bar{\omega})$ data. For a linear system $(\bar{\omega}_r)_0 = (\bar{\omega}_r)_{\max}$. For typical results of the linear case, see Table 1.

5.2. CASE I: NON-LINEAR SPRING ($\bar{\beta}_2$)

In the first case the second spring is treated as the only non-linear spring with values of $\bar{\beta}_2 = 0{\cdot}3$, 0·5 and 5·0. For this case the response amplitudes are considered small,

TABLE 1

Experimental natural frequency and modal response results for linear case and non-linear cases I, II and III

Type	$\bar{\beta}_1$	$\bar{\beta}_2$	r	$(\bar{\omega}_r)_0$	$(\bar{\omega}_r)_{max}$	$\{\bar{R}_j\}^T$ at $(\bar{\omega}_r)_0$
Linear case	0·0	0·0	1	0·377	0·377	1, 2·090, 2·400
			2	0·817	0·817	1, 1·131, −2·300
			3	1·634	1·634	1, −0·757, 0·826
Non-linear case I	0·0	0·3	1	0·377	0·377	1, 1·941, 2·350
			2	0·817	0·817	1, 1·355, −1·822
			3	1·634	1·760	1, −1·061, 1·150
	0·0	0·5	1	0·377	0·377	1, 1·960, 2·514
			2	0·817	0·817	1, 1·603, −2·450
			3	1·634	1·885	1, −1·410, 1·480
	0·0	0·5	1	0·377	0·377	1, 2·000, 1·810
			2	0·817	0·817	1, 1·080, −2·320
			3	1·634	2·796	1, −3·402, 4·320
Non-linear case II	0·3	0·0	1	0·377	0·377	1, 1·480, 2·100
			2	0·817	0·817	1, 1·870, −3·188
			3	1·634	1·665	1, −0·747, 0·840
	0·5	0·0	1	0·377	0·377	1, 1·400, 1·303
			2	0·817	0·880	1, 1·208, −3·154
			3	1·634	1·634	1, −0·462, 0·738
	5·0	0·0	1	0·377	0·408	1, 1·495, 4·380
			2	0·817	0·974	1, 1·521, −1·977
			3	1·634	1·916	1, −0·576, 0·780
Non-linear case III	0·3	0·3	1	0·377	0·408	1, 1·161, 1·830
			2	0·817	0·848	1, 1·635, −2·138
			3	1·634	1·760	1, −1·131, 1·416
	0·5	0·5	1	0·377	0·408	1, 2·447, 4·136
			2	0·817	0·942	1, 1·552, −2·713
			3	1·634	1·980	1, −1·615, 1·809
	5·0	5·0	1	0·377	0·440	1, 1·866, 3·630
			2	0·817	1·225	1, 1·338, −1·000
			3	1·634	2·765	1, −1·217, 1·000

especially for q_3: i.e., $\sin \bar{q}_3 \cong \bar{q}_3$ and $\cos \bar{q}_3 \cong 1$. The dimensionless damping $\bar{c}$ is taken to be equal to 0·10. For $\bar{\beta}_2 = 0$ the system of course reduces to the linear case.

Figures 3 and 4 present measured driving point compliance $\bar{H}_{11}(\bar{\omega})$ magnitude and phase spectra obtained with random excitation for the non-linear ($\bar{\beta}_2 = 5{\cdot}0$) case. The corresponding figures for the analytical solution for sinusoidal excitation are given in Appendix A, Figures A1 and A2. Experimentally, with random excitation one does not see any of the jump phenomena associated with sinusoidal excitation solutions. The measured coherence function $\gamma_{11}^2(\bar{\omega})$ is also shown here in Figure 5; the γ_{11}^2 shape is considerably distorted for the non-linear case.

Table 1 lists the experimental results for the natural frequencies $\bar{\omega}_r$ and the modal response ratios $\bar{R}_{jr}$. The corresponding values for the analytical solution are given in Table A1. Note that $(\bar{\omega}_r)_0$ differs from $(\bar{\omega}_r)_{max}$ only for the third mode, the deviation being directly related to the $\bar{\beta}_2$ value. In the experiments the relative amplitudes are distorted more and the relative phases less.

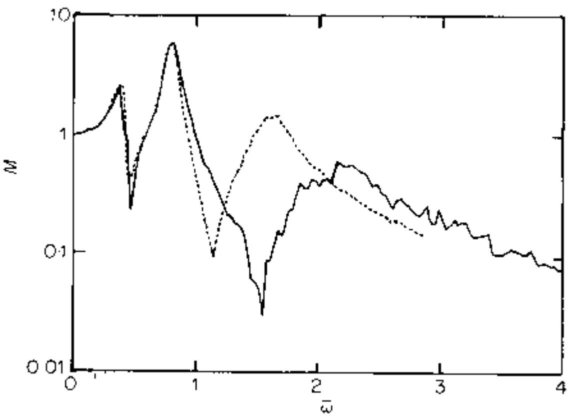

Figure 3. Magnitude of $\bar{H}_{11}(\bar{\omega})$, for example Case I. $\bar{H}_{11}(\bar{\omega}) = \bar{q}_1/\bar{F}_1(\bar{\omega})$. ——, $\bar{\beta}_2 = 5{\cdot}0$ (non-linear case); ---, $\bar{\beta}_2 = 0$ (linear case).

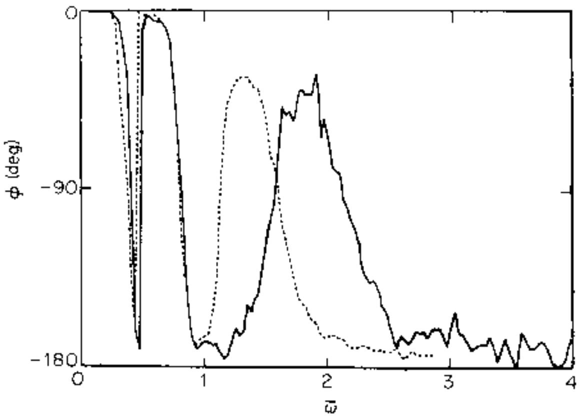

Figure 4. Phase of $\bar{H}_{11}(\bar{\omega})$, Key as Figure 3.

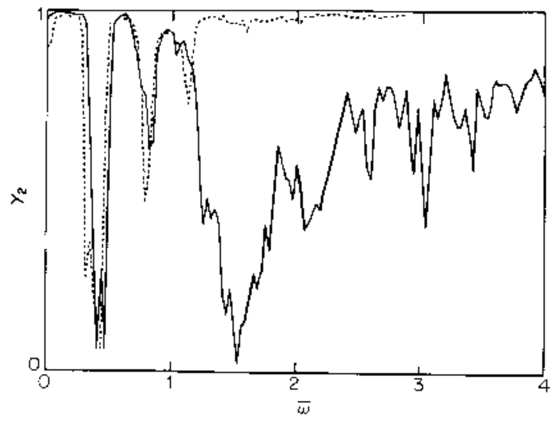

Figure 5. Experimentally measured coherence function $\gamma_{11}^2(\bar{\omega})$ for the transfer function $\bar{H}_{11}(\bar{\omega})$.

5.3. CASE II: NON-LINEAR SPRING ($\bar{\beta}_1$)

In this case the first spring is assumed to be the only non-linear spring with values of $\bar{\beta}_1 = 0{\cdot}3$, 0·5 and 5·0. Again the response amplitudes are considered small as in Case I, the dimensionless damping $\bar{c}$ is taken to be equal to 0·10, and for $\bar{\beta}_1 = 0$, the system reduces to the linear case. Table 1 shows the experimental results and Table A1 the analytical results for the natural frequencies $\bar{\omega}_r$ and modal response ratios $\bar{R}_{jr}$. Note that this case is similar to Case I with the exception that the first mode instead of the third mode shows the maximum deviation. Typical vibration patterns can still be recognized.

5.4. CASE III: NON-LINEAR SPRINGS ($\bar{\beta}_1, \bar{\beta}_2$)

For this case both the first and second springs are assumed to be non-linear with values of $\bar{\beta}_1 = \bar{\beta}_2 = 0{\cdot}3$, 0·5 and 5·0. Again the response amplitudes are considered small and the dimensionless damping $\bar{c}$ is taken to be 0·10. Table 1 lists the frequency and modal response results and Table A1 the corresponding analytical results. For this case, discrepancies are noted for all three frequencies and especially the mode shape discrepancies are more severe than those seen in Cases I and II.

5.5. CASE IV: FINITE AMPLITUDES OF THE PENDULUM ($\bar{q}_3$)

As the final case, both springs are considered to be linear, by setting $\bar{\beta}_1 = \bar{\beta}_2 = 0$, but the amplitude of the pendulum motion ($\bar{q}_3$) is finite. Although for the analytical part sin q and cos q are retained in the equations of motion, they were approximated in the analog computer experiment by taking the only first two terms of the series: i.e., $\sin \bar{q}_3 \cong \bar{q}_3 - \bar{q}_3^3/6$ and $\cos \bar{q}_3 \cong 1 - \bar{q}_3^2/2$. This was necessitated by the fact that more function generators for the analog computer circuits were not available; this approximation is however valid as long as q_3 is less than or equal to approximately $\pi/4$. Figures 6 and 7 show the driving point compliance spectrum $\bar{H}_{11}(\bar{\omega})$ in real and imaginary forms. Note that the finite amplitudes introduce a spring softening effect and thus the curves are shifted to the left and somewhat distorted. Thus, in a real structural experiment, one can expect softening or hardening effects if the excitation force levels are high enough to produce large motions. The coherence function γ_{11}^2, as shown in Figure 8, is again distorted for the finite amplitude case.

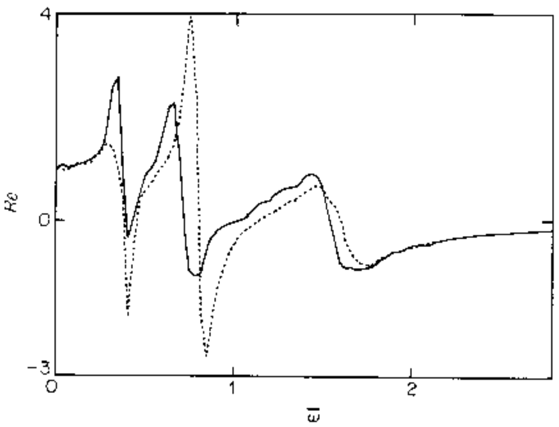

Figure 6. Real part of the experimentally measured driving point compliance $\bar{H}_{11}(\bar{\omega}) = \bar{q}_1/\bar{F}_1(\bar{\omega})$ for example case IV.

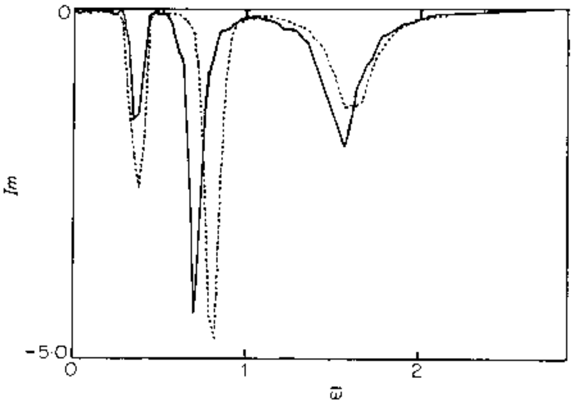

Figure 7. Imaginary part of the measured $\bar{H}_{11}(\bar{\omega})$.

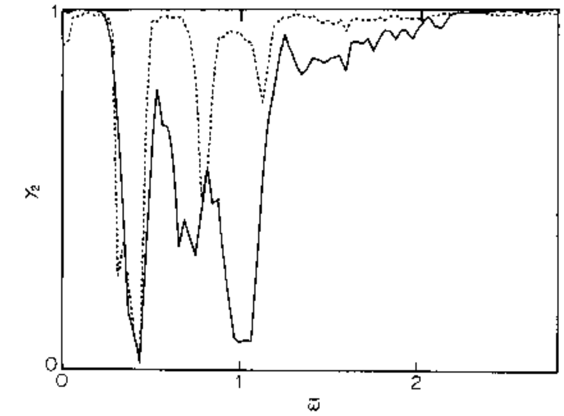

Figure 8. Experimentally measured coherence function $\gamma_{11}^2(\bar{\omega})$ for $\bar{H}_{11}(\bar{\omega})$.

Table 2 lists the experimental results and Table A2 the analytical results for the natural frequencies $\bar{\omega}_r$ and modal response ratios $\bar{R}_{jr}$. Note that $(\bar{\omega}_r)_0$ differs from $(\bar{\omega}_r)_{max}$ for all three modes but is mainly dominant for the second mode. The modal response results again show significant discrepancies as experimentally witnessed in Cases I–III; however, the relative phase information is such that each mode can still be identified.

TABLE 2

Experimental natural frequency and modal response results for linear case and non-linear case IV

Type	r	$(\bar{\omega}_r)_0$	$(\bar{\omega}_r)_{max}$	$\{\bar{R}_j\}^T$ at $(\bar{\omega}_r)_0$
Linear case	1	0·377	0·377	1, 2·090, 2·400
	2	0·817	0·817	1, 1·131, −2·300
	3	1·634	1·634	1, −0·757, 0·826
Case IV	1	0·377	0·346	1, 2·120, 3·630
	2	0·817	0·691	1, 1·910, −1·820
	3	1·634	1·571	1, −0·540, 0·270

6. CONCLUDING REMARKS

Based on the examples presented here and others worked out by us, including the single and two degree of freedom non-linear systems problems, we can draw the following conclusions. (1) Measured transfer function plots, for band-limited random excitation, can alert one to the fact that the structure is non-linear. (2) Modal response data can give a rough idea about the system behavior, including natural frequencies and modes; this information could be sufficient for solving vibration and noise problems but certainly not suitable for mathematical model building. (3) Measured coherence function estimates can be used as the "warning signals" for inherent non-linearities, including finite amplitude motion; this conclusion is in agreement with those expressed previously [3, 11, 12, 14].

Overall, it seems that an experimenter has to be very careful and should use conventional experimental modal analysis techniques with some discretion. Also, excitation signals, especially those which contain spectral energy over a broad frequency range, along with levels, should be chosen judiciously; single frequency excitation or slow sinusoidal sweeps could be more suitable. Further research work in this area is definitely required.

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APPENDIX A: ANALYTICAL RESULTS FOR THE EXAMPLE CASES OBTAINED FOR SINUSOIDAL EXCITATION

Only sinusoidal excitation and the associated steady state solution were considered. Response only at the forcing frequency ω was assumed and higher harmonic terms were ignored. The frequency equations were obtained and solved numerically by using the Newton–Raphson method. Frequency response functions of normalized response, with dimensionless frequency, were then generated over the frequency range of interest, which covers essentially the first three modes of the linear system. Results are given in Tables A1 and A2 and Figures A1 and A2.

TABLE A1

Analytical natural frequency and modal response results with sinusoidal excitation for linear case and non-linear cases I, II and III; refer to Table 1 for analogous experimental results obtained with random excitation

Type	$\bar{\beta}_1$	$\bar{\beta}_2$	r	$(\bar{\omega}_r)_0$	$(\bar{\omega}_r)_{max}$	$\{\bar{R}_j\}^T$ at $(\bar{\omega}_r)_0$
Linear case	0·0	0·0	1	0·377	0·377	1, 1·854, 2·314
			2	0·817	0·817	1, 1·334, −2·164
			3	1·634	1·634	1, −0·726, 0·803
Non-linear case I	0·0	0·3	1	0·377	0·408	1, 1·391, 1·724
			2	0·817	0·817	1, 1·121, −1·819
			3	1·634	2·105	1, −1·460, 1·612
	0·0	0·5	1	0·377	0·408	1, 1·345, 1·679
			2	0·817	0·817	1, 1·109, −1·798
			3	1·634	2·262	1, −1·675, 1·853
	0·0	0·5	1	0·377	0·408	1, 1·204, 1·500
			2	0·817	0·817	1, 1·061, 1·722
			3	1·634	3·518	1, −6·764, 7·481

TABLE A1 (*cont.*)

Type	$\bar{\beta}_1$	$\bar{\beta}_2$	r	$(\bar{\omega}_r)_0$	$(\bar{\omega}_r)_{max}$	$\{\bar{R}_j\}^T$ at $(\bar{\omega}_r)_0$
Non-linear case II	0·3	0·0	1	0·377	0·408	1, 1·858, 2·318
			2	0·817	0·974	1, 1·337, −2·170
			3	1·634	1·882	1, −0·721, 0·748
	0·5	0·0	1	0·377	0·408	1, 1·868, 2·319
			2	0·817	0·934	1, 1·337, −2·170
			3	1·634	1·884	1, −0·701, 0·798
	5·0	0·0	1	0·377	0·408	1, 1·858, 2·319
			2	0·817	1·037	1, 1·337, −2·170
			3	1·634	3·047	1, −0·721, 0·798
Non-linear case III	0·3	0·3	1	0·377	0·471	1, 1·620, 2·022
			2	0·817	1·382	1, 1·313, −2·130
			3	1·634	2·136	1, −1·430, 1·582
	0·5	0·5	1	0·377	0·471	1, 1·600, 2·000
			2	0·817	1·539	1, 1·110, −2·713
			3	1·634	2·324	1, −1·650, 1·820
	5·0	5.0	1	0·377	0·471	1, 1·607, 1·881
			2	0·817	2·607	1, 1·288, −2·090
			3	1·634	3·613	1, −6·685, 7·453

TABLE A2

Analytical natural frequency and modal response results for linear case and non-linear case IV; refer to Table 2 for analogous experimental results

Type	r	$(\bar{\omega}_r)_0$	$(\bar{\omega}_r)_{max}$	$\{\bar{R}_j\}^T$ at $(\bar{\omega}_r)_0$
Linear case	1	0·377	0·377	1, 1·854, 2·314
	2	0·817	0·817	1, 1·334, −2·164
	3	1·634	1·634	1, −0·726, 0·803
Case IV	1	0·377	0·283	1, 1·016, 2·080
	2	0·817	0·503	1, 2·460, −2·730
	3	1·634	1·634	1, −0·683, 0·758

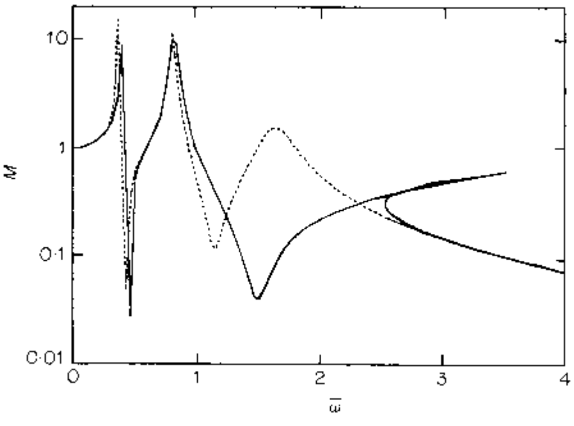

Figure A1. Magnitude of $\bar{H}_{11}(\bar{\omega})$, for example Case I. $\bar{H}_{11}(\bar{\omega}) = \bar{q}_1/\bar{F}_1(\bar{\omega})$. ——, $\bar{\beta}_2 = 5{\cdot}0$ (non-linear case); ---, $\bar{\beta}_2 = 0$ (linear case).

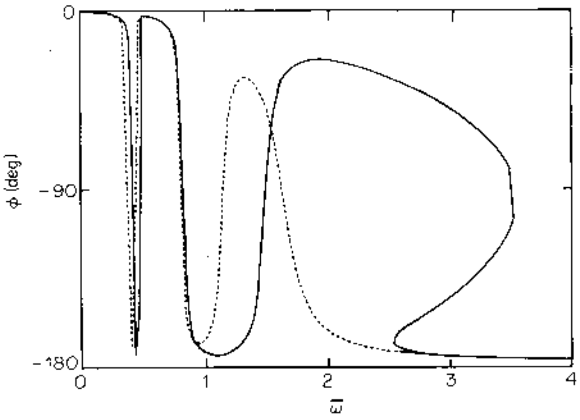

Figure A2. Phase of $\bar{H}_{11}(\bar{\omega})$. Key as Figure A1.

APPENDIX B: LIST OF SYMBOLS

A modal residue (complex valued)
c damping coefficient
f force
F force amplitude
g acceleration due to gravity
h impulse response
H dynamic compliance (complex valued)
i imaginary unit
Im imaginary part of $\bar{H}(\omega)$
k linear spring constant
l pendulum length
m mass
M magnitude of $\bar{H}(\omega)$
q generalized displacement
R modal response ratio
Re real part of $\bar{H}(\omega)$
s Laplace variable
t time
β non-linear spring constant
γ coherence function
ϕ phase of $\bar{H}(\omega)$
θ phase of force f
ω radian frequency
ζ damping ratio
ψ mode shape

Subscripts
j response location
k excitation point
r modal index
0 backbone frequency
max modal frequency of highest magnitude

Superscripts
$*$ complex conjugate
$\bar{}$ dimensionless or normalized
$\dot{}$ $\mathrm{d}/\mathrm{d}t$
$\ddot{}$ $\mathrm{d}^2/\mathrm{d}t^2$
T transpose