Journal of Sound and Vibration (1986) **104**(2) 348–350

EXAMINATION OF THE VALIDITY OF PROPORTIONAL DAMPING APPROXIMATIONS WITH TWO FURTHER NUMERICAL INDICES

1. INTRODUCTION

Five numerical indices were reported recently by Prater and Singh [1] to quantify the extent of non-proportionality in discrete vibratory systems. Two of these were based on the knowledge of complex modes, and three were derived from the damping matrix transformed to real normal co-ordinates. In this note we extend the previous study by reporting results for two additional indices. The first new index is evaluated by using the constraint matrices in physical co-ordinates and the second one is based on the system frequency response characteristics in Nyquist plot form. The validity of these indices is tested by examining the frequency response errors of a six degree of freedom system, for two commonly used proportional damping approximations.

We consider an n-degree of freedom linear, viscously damped system where $[M]$, $[C]$ and $[K]$ are the inertia, viscous damping and stiffness matrices, respectively, and $\{X\}$ and $\{F\}$ are the generalized displacement and force vectors, respectively. Let the eigensolution for the undamped case be given by the normalized modal matrix $[\psi]$. For a non-proportionally damped system $[\psi]^T[C][\psi]$ is obviously not a diagonal matrix. The two damping approximations used widely in the literature [1] are as follows. Approximation A: $[C]=[[\psi]^T]^{-1}[C_{rd}][\psi]^{-1}$, where $[C_{rd}]$ is the matrix $[\psi]^T[C][\psi]$ with the off-diagonal elements neglected. Approximation B: $[C]=[[\psi]^T]^{-1}[2\xi_r\omega_r][\psi]^{-1}]$, where ξ_r, ω_r are the exact damping ratio and natural frequency respectively. These can be obtained experimentally or from the complex eigensolution.

2. DEVELOPMENT OF THE INDICES

2.1. *Index σ_1, based on the constraint matrices*

Caughey and O'Kelly [2] derived a condition for the most general case of classical (proportional) damping as $[C][M]^{-1}[K]=[K][M]^{-1}[C]$. Using this, one can define matrix $[E]$ and index σ_1 as follows: $[E]=[C][M]^{-1}[K]-[K][M]^{-1}[C]$ and $\sigma_1=\sum_{(1,\dots,n)}\sum_{(1,\dots,n)}\text{abs}(E_{ij})$. For the proportional damping case, $[E]=[0]$ and $\sigma_1=0$.

2.2. *Index σ_2, based on Nyquist plot*

The system frequency response characteristic in the Nyquist plot form accentuates the differences between the exact and approximate damping models near the resonant frequencies. Let A_d be the area between the exact and approximate solution curves. A normalized index can be defined as $\sigma_2=A_d/A$, where A is the area of the rectangle that envelops the exact solution curve with sides parallel to the co-ordinate axes. For the proportional damping case, $\sigma_2=0$.

3. EXAMPLE CASE

Our example case is shown in Figure 1 where masses M and springs K are identical and equal to 100 in consistent units. Select $[C]=\alpha[M]+\beta[K]$, where α and β are arbitrary constants. For the proportional damping case, all the damping coefficients must be equal. Non-proportionality is then introduced in 10 steps, by incrementing only C_1

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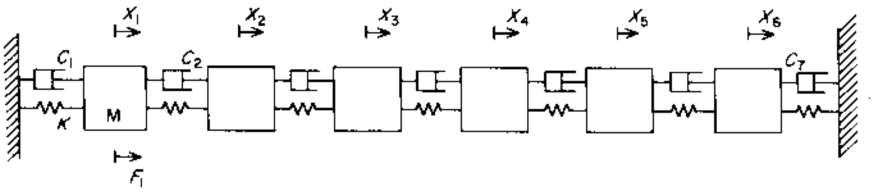

Figure 1. Example case.

and C_7 thus: $C_1 = C_7 = C_{(\text{prop. damped})} + k\Delta$, where $k = 1, \ldots, 10$ and Δ is the trial increment. Table 1 lists the undamped natural frequencies and modes. The following values of α, β and Δ were considered for the light, moderate and heavy non-proportional damping cases; the range of calculated modal damping ratios is also given below; in each case $\alpha = 0$: (1) light: $\beta = 0{\cdot}025$, $\Delta = 1{\cdot}25$, range of $\xi_r = 0{\cdot}005$–$0{\cdot}039$; (2) moderate: $\beta = 0{\cdot}25$, $\Delta = 6{\cdot}0$, range of $\xi_r = 0{\cdot}05$–$0{\cdot}3$; (3) heavy: $\beta = 0{\cdot}6$, $\Delta = 10{\cdot}0$, range of $\xi_r = 0{\cdot}133$–$0{\cdot}68$. The maximum errors in $X_1/F_1(\omega)$ were found to be 25% for the heavily damped case and only 0·8% for the lightly damped case. This is in agreement with the notion that the damping approximations work best for lightly damped models.

TABLE 1

Undamped modal data for the example case

ω_n	0·445	0·868	1·247	1·564	1·802	1·950
Mode shape ψ_n	0·2319	0·4180	0·5211	−0·5211	−0·4179	−0·2319
	0·4179	0·5211	0·2319	0·2319	0·5211	0·4179
	0·5211	0·2319	−0·4179	0·4179	−0·2319	−0·5211
	0·5211	−0·2319	−0·4179	−0·4179	−0·2319	0·5211
	0·4179	−0·5211	0·2319	−0·2319	0·5211	−0·4179
	0·2319	−0·4179	0·5211	0·5211	−0·4719	0·2319

The indices, plotted against the errors in computing the driving point frequency response $X_1/F_1(\omega)$, at $\omega = 0{\cdot}6$ rad/s, for the moderate damping case, are shown in Figure 2(a) and (b). This figure shows a linear variation of the indices with the frequency response errors, at one frequency, for both the approximations. Similar results were found over the entire frequency range; hence a high value of σ_1 or σ_2 indicates large errors associated with the approximations. The structure of the matrix $[E]$, corresponding to σ_1 was found to be as follows:

$$[E] = \begin{bmatrix} 0 & -p & 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & -p & 0 \end{bmatrix},$$

where p is proportional to $k\Delta$. This matrix highlights the location of non-proportionalities in the system, which could prove to be valuable information for the analyst.

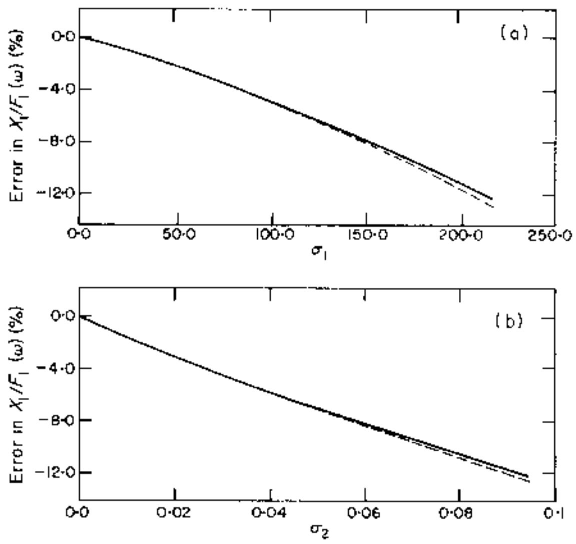

Figure 2. Errors induced by proportional damping approximations in the driving point frequency response, $X_1/F_1(\omega)$, at $\omega = 0{\cdot}6$ rad/s. (a) Index σ_1; (b) index σ_2. Damping approximations; A, ——; B, - - - -.

4. CONCLUDING REMARKS

Two indices σ_1 and σ_2 were developed to predict the errors induced by proportional damping approximations. The index σ_1 seems to be attractive as its computation requires knowledge only of the constraint matrices. Further, the example case illustrates that the corresponding matrix $[E]$ can potentially have information about the location of non-proportionality in a system provided that the mass or stiffness matrix are diagonal. A disadvantage of σ_1 is that it is not normalized. The index σ_2 also exhibits linear behaviour, though it is not as easy to compute as σ_1.

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(*Received* 23 *August* 1985)

REFERENCES

1. G. PRATER, JR. and R. SINGH *Journal of Sound and Vibration.* Quantification of the extent of non-proportional viscous damping in discrete vibratory systems.
2. T. K. CAUGHEY and M. E. J. O'KELLY 1965 *American Society of Mechanical Engineers, Journal of Applied Mechanics* **32**, 583–588. Classical normal modes in damped linear dynamic systems.
3. S. S. NAIR 1984 *M.Sc. Thesis, The Ohio State University.* Analysis of proportional damping approximations in discrete vibratory systems.