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QUANTIFICATION OF THE EXTENT OF NON-PROPORTIONAL VISCOUS DAMPING IN DISCRETE VIBRATORY SYSTEMS

G. PRATER, JR. AND R. SINGH

Department of Mechanical Engineering, The Ohio State University, Columbus, Ohio 43210, *U.S.A.*

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The primary objective of this paper is to present several numerical indices developed to determine quantitatively the extent of non-proportional damping present within a discrete vibratory system. A total of five distinct indices are considered. Two of these are based on the complex modes of a generally damped system, and three are based on the configuration of the system damping matrix after transformation into real normal co-ordinates. Each index has been normalized so that it assumes values between zero (proportional damping case) and one. Application of the indices is illustrated through a four degree of freedom system example problem. As part of this exercise, an effort is made to relate the magnitudes of the indices to the frequency response errors induced by two proportional damping approximations often found in the literature.

1. INTRODUCTION

The equations of motion for an n-degree of freedom linear vibratory system can be written in matrix form as

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{Q\}, \tag{1}$$

where $[M]$, $[C]$ and $[K]$ are inertia, damping, and stiffness matrices, and $\{x\}$ and $\{Q\}$ are generalized displacement and force vectors, respectively. There are three widely used mathematical descriptions of the damping matrix: (i) the undamped case, $[C]=[0]$; (ii) the proportionally damped case such as given by Rayleigh, $[C]=\alpha[M]+\beta[K]$, where α and β are constants; (iii) the non-proportionally damped case.

Case (i) models can give good results for lightly damped systems, or systems where the excitation frequency is well removed from a natural frequency; but for real systems subjected to transient inputs, or harmonic inputs close to a natural frequency, large errors can occur. The Case (ii) approximation gives better results even though the additional computations involved are not significant. However, proportional damping is not physically realizable, and therefore this model tends to work best for systems whose natural frequencies are widely separated [1]. The most difficult method to implement and the most expensive computationally is the exact solution of the generally damped system given by Case (iii). Here $2n$ ordinary differential equations must be solved simultaneously, and the solution itself lies within the complex domain. For these reasons, this model is rarely used, but it does find some applications in heavily damped structures whose natural frequencies are closely spaced [2].

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2. LITERATURE REVIEW

It can be shown that a set of n identity equations can be combined with a discrete system's n equations of motion and thereby allow one to find co-ordinates that will uncouple the equations of motion [3, 4]. Several investigators [1, 5–8] have introduced differing forms of the identity equations and used the method to simplify the modification and synthesis of discrete systems. These applications gave rise to the so called eigenvalue modification [1, 8, 9] and component mode synthesis techniques [10–16]. Considerable work has also been done with the numerical solution techniques associated with these methods [17–20].

There is extensive literature on techniques for approximating a non-proportional damping configuration with a proportional configuration. The most straightforward of these is to apply the same co-ordinate transformation that uncouples the undamped equations of motion to the damping matrix and then ignore the resulting off-diagonal terms. Cronin [21] has suggested using the diagonal elements from this approximation as the first term in a series solution representing the true system response. Thomson *et al.* [22] investigated approximations where a diagonal damping matrix is established by an optimization algorithm that minimizes the mean square of the frequency response errors, and by adjusting the modal damping ratios so that the peaks of the coupled and uncoupled systems are matched. Finally, Ozguven [23] developed a method that approximates the total system response with the modal response at the excitation mode.

Clough and Mojtahedi [24] considered several methods for treating generally damped systems, and concluded that proportional damping approximations may give unreliable results for many systems, thus underscoring the need for the predictive capabilities that are a primary objective of this study. Similarly, Duncan and Taylor [25] showed that significant errors can be incurred when the dynamic analysis of a non-proportionally damped system is based on a truncated set of modes, as is commonly done when modeling continuous systems.

Warburton and Soni [26], and Hasselman [1], independently developed similar criteria for deciding if significant errors are caused by neglecting the coupling between two given modes. Hasselman's criterion suffers because it can give the same results for both a proportionally damped and non-proportionally damped system, and also because he made a restrictive assumption during its development (namely, that the ratio of the individual off-diagonal terms and the diagonal terms of the modal damping matrix are less than or equal to unity for the modes being considered). Warburton and Soni avoided these difficulties, but being only criteria, neither theirs nor Hasselman's has the capability of explicitly estimating the magnitude of the errors.

3. OBJECTIVES

There is a well defined need for analytical tools that can be used to determine the validity of a proportional damping assumption. Therefore, the specific objectives of this study are to (i) develop several numerical indices that can be used to estimate the extent of non-proportional damping present within a system, (ii) illustrate the application of these indices through an example case, and (iii) investigate the nature and magnitude of errors caused by proportional damping assumptions. The indices of item (i), referred to as non-proportionality indices, are of two general classes: one best suited for experimental applications, and the other for analytical applications. Use of these indices and the associated response errors are illustrated by computing both the exact and approximate frequency domain responses of a linear, viscously damped, four degree of freedom system.

4. EIGENVALUE PROBLEM FORMULATION AND MODAL TRANSFORMATIONS

4.1. CASE (i): $[C]=[0]$

The natural frequencies ω_i and modes $\{X\}_i$ of an undamped system are given by the expressions

$$|[M]^{-1}[K]-\omega^2[I]|=0, \qquad \omega_i^2[M]\{X\}_i=[K]\{X\}_i, \qquad i=1,2,\ldots,n. \tag{2a, b}$$

If one defines a normal co-ordinate matrix $\{n\}$ such that

$$\{x\}=[X][M_r]^{-1}\{n\}, \tag{3}$$

where $[X]=[\{X\}_1\{X\}_2\cdots\{X\}_n], \qquad [M_r]=[X]^t[M][X], \qquad [K_r]=[X]^t[K][X],$ (4a–c)

one can see that the equations of motion become uncoupled in the normal co-ordinates, and can be written as

$$[I]\{\ddot{n}\}-[\omega_r^2]\{n\}=\{0\}. \tag{5}$$

4.2. CASE (ii): $[C]=\alpha[M]+\beta[K]$

For the proportionally damped case, $[C]$ is diagonalized by the same transformation as was used for Case (i). The equations of motion in normal co-ordinates are now

$$[I]\{\ddot{n}\}+[2\xi_r\omega_r]\{\dot{n}\}+[\omega_r^2]\{n\}=\{0\}, \tag{6}$$

$$\text{where} \quad \xi_r=(\alpha/\omega_r+\beta\omega_r)/2. \tag{7}$$

4.3. CASE (iii): $[C]\neq\alpha[M]+\beta[K]$

For this general damping case, the equations of motion must be manipulated in such a way that only two symmetric matrices exist. This may be accomplished by transforming the equations from n-dimensional Euclidean space to $2n$-dimensional Euclidean space through one of the following sets of identity equations:

$$[M]\{\dot{x}\}-[M]\{\dot{x}\}=\{0\}, \quad \text{or} \quad [K]\{\dot{x}\}-[K]\{\dot{x}\}=\{0\}. \tag{8a, b}$$

A similar identity involving $[C]$ is not considered because its use results in non-symmetric matrices. The choice between expressions (8a) and (8b) depends on several factors. Since the inertia matrix is usually less complicated than the stiffness matrix, equation (8a) is often computationally more efficient. However, structural modification algorithms may use either, depending on the type of modification being implemented [9]. In the following developments, equation (8a) is used, but the procedure for using (8b) is quite similar. Equations (1) and (8a) can be written together in composite matrix form as

$$[A]\{\dot{y}\}+[B]\{y\}=\{\tilde{0}\}, \tag{9}$$

where $[A]=\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix}, \qquad [B]=\begin{bmatrix}-[M] & [0]\\ [0] & [K]\end{bmatrix}, \qquad \{y\}=\begin{Bmatrix}\{\dot{x}\}\\ \{x\}\end{Bmatrix}, \qquad \{\tilde{0}\}=\begin{Bmatrix}\{0\}\\ \{0\}\end{Bmatrix}.$ (10a–d)

The matrices $[A]$ and $[B]$ are sometimes referred to as the augmented mass and stiffness matrices, respectively [1]. The eigenproblem can now be expressed as

$$(-\lambda[I]-[A]^{-1}[B])\{Y\}=\{\tilde{0}\}. \tag{11}$$

The solution to this equation is $2n$ eigenvalues, λ_i, and $2n$ eigenvectors, $\{Y\}_i$. Here

$$\lambda_i = \xi_i\omega_i \pm \mathrm{j}\omega_i\sqrt{1-\xi_i} \quad \text{and} \quad \{Y\} = \begin{Bmatrix} \lambda_i\{X\}_i \\ \{X\}_i \end{Bmatrix}, \tag{12a, b}$$

where ω_i is the undamped natural frequency of the ith mode, $\{X\}_i$ is the modal displacements of the ith mode, ξ_i is the damping ratio of the ith mode, and $\mathrm{j} = \sqrt{-1}$. Now one can define a normal co-ordinate vector $\{z\}$ such that

$$\{y\} = [Y][A_r]^{-1}\{z\}, \tag{13}$$

where $[Y] = [\{Y\}_1\{Y\}_2 \cdots \{Y\}_{2n}]$, $\quad [Y]^t[A][Y] = [A_r]$, $\quad [Y]^t[B][Y] = [B_r]$. (14a–c)

By using these relations, the equations of motion can be written in uncoupled form as

$$[I]\{\dot{z}\} + [\lambda_r]\{z\} = \{\tilde{0}\}. \tag{15}$$

The preceding developments underscore the computational advantages of a proportionally damped system model over a generally damped model. Another advantage of the proportional damping model is that the eigenproblem solution lies within the real domain, thus simplifying the task of physically interpreting the results.

5. DEVELOPMENT OF NON-PROPORTIONALITY INDICES

5.1. INDICES BASED ON THE SYSTEM NATURAL MODES

The complex eigenvectors of non-proportionally damped systems are a result of coupling between the equations of motion after their transformation into real normal co-ordinates. At resonance this coupling leads to phase differences between the displacements at the various co-ordinate locations. Because of these lags (or leads) the system modes must be described with a phase angle as well as a magnitude, and hence the modal matrix consists of complex numbers. On the other hand, motions of a proportionally damped system are all in phase or exactly out of phase at resonance, resulting in modal displacements that simultaneously reach extrema. It is therefore possible to describe completely the natural modes of such a system with real numbers alone, these representing the relative magnitudes of the individual displacements.

If the modal displacements of a generally damped system are plotted in the complex plane, the effects of the phase differences become readily apparent. Each displacement of a proportionally damped system lies on a straight line, while those of non-proportionally damped systems do not. Instead, they exhibit an angular dispersion equivalent to the phase differences. If the individual displacements are connected by straight lines, an n-sided polygon is formed. As the degree of non-proportionality in the damping matrix (and thus the modal phase differences) increase, the area of this modal polygon also increases.

Figure 1 shows the first mode modal polygons for a four degree of freedom system (see section 6) with increasingly non-proportional damping. As noted, the displacements for trial 1, the proportionally damped case, lie on a straight line, and of course, the modal polygon has zero area. As the degree of non-proportionality and the phase differences increase, the modal area likewise increases. The gross rotations of the polygons are a result of normalizing the eigenvectors with different complex constants, and were done solely for clarity.

The results of Figure 1 suggest that modal displacement plots can be used as a qualitative indication of the non-proportionality of the damping matrix. A system whose phase

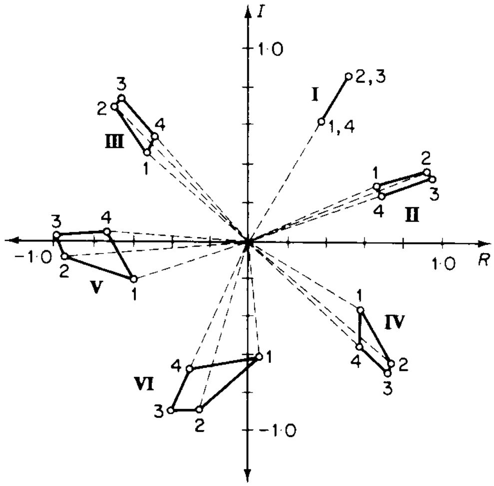


Figure 1. First mode modal polygons of a four degree of freedom system subjected to an increasingly non-proportional damping matrix. The arabic and roman numerals denote the trial and displacement numbers, respectively.

differences and modal areas are very small can be assumed either to be lightly damped, in which case energy dissipation can be neglected entirely, or proportionally damped. The correct interpretation may be readily made, since an analyst with access to the complex modes can also be expected to know the system damping ratios.

5.1.1. *Modal polygon areas*

By defining a consistent normalization procedure, a quantitative non-proportionality index based upon the modal area can be developed. First, one normalizes the modal displacements themselves so that, for a given mode,

$$\hat{a}_i + \mathrm{j}\hat{b}_i = (a_i + \mathrm{j}b_i)/(a_m + \mathrm{j}b_m), \qquad i = 1, 2, \ldots, n. \tag{16}$$

Here a and b are the real and imaginary parts of the modal displacements, and the subscript m denotes the displacement with the largest magnitude. This procedure yields normalized displacements that are clustered around the real axis and have magnitudes of less than or equal to unity. Although it in no way affects the modal phase differences, it does modify the modal area in such a way that it becomes possible to compare dissimilar systems. The modal area can be further normalized by dividing it by the maximum possible area. For the ith mode this gives

$$\gamma_{1i} = A_{1i}/A_{m1}. \tag{17}$$

The maximum modal area of an n-degree of freedom system can be shown to be given by

$$A_{m1} = n \cos(\pi/n) \sin(\pi/n). \tag{18}$$

This expression represents the area of an n-sided equilateral polygon formed by displacements of unit magnitude. Division by A_{m1} assures that the index of equation (17) is always less than or equal to unity.

For higher modes, when one or more displacements are nominally out of phase, the modal area as currently defined will be much larger than that of a mode where all displacements are approximately in phase. Such a situation is incompatible with the goal of a consistent index. It can be avoided by rotating each of the out-of-phase displacements by 180 degrees, thereby assuring that the entire polygon lies within the first and fourth quadrants of the complex plane. Upon denoting the revised modal area by A_2, the new non-proportionality index for the ith mode becomes

$$\gamma_{2i} = A_{2i}/A_{m2}, \tag{19}$$

and, since the modal polygon is confined to only half of the plane, the maximum modal area is

$$A_{m2} = (n-1)\cos(\pi/2(n-1))\sin(\pi/2(n-1)). \tag{20}$$

The non-proportionality index of equation (19) now provides a reasonable basis for making quantitative comparisons between modes.

5.1.2. *Modal phase differences*

Other non-proportionality indices can be developed from the displacement phase differences. The simplest, both conceptually and computationally, may be defined as the sum of the absolute values of the phase angles of the normalized modal displacements:

$$\rho_{1i} = \sum_{j=1}^{n} 2\phi_j/n\pi. \tag{21}$$

Here ϕ_j denotes the magnitude of the normalized phase angles of the ith mode, and a factor of $2/n\pi$ is included to insure that ρ_{1i} is less than unity. Although the index of equation (21) is simple, it is not independent of gross rotations of the displacements. If, for example, a given mode has two or more displacements with the same magnitude, and this magnitude also happens to be the maximum for that mode, ρ_{1i} will vary, depending upon which is used as the normalization constant. A phase based non-proportionality index which avoids this difficulty can be defined as

$$\rho_{2i} = (|\phi_{i,\max}| - |\phi_{i,\min}|)/\pi, \tag{22}$$

where ρ_{2i} is physically interpreted as the angle spanned by the normalized displacements. The factor $1/\pi$ again assures a value of less than unity.

The indices developed thus far are all mode dependent. If the frequency range of interest encompasses more than one natural frequency, it may be desirable to have an overall index for the entire system. Such an index can be developed by averaging the modal indices. Thus,

$$\gamma_1 = \sum_{i=1}^{n} \gamma_{1i}/n, \qquad \gamma_2 = \sum_{i=1}^{n} \gamma_{2i}/n, \qquad \rho_1 = \sum_{i=1}^{n} \rho_{1i}/n, \qquad \rho_2 = \sum_{i=1}^{n} \rho_{2i}/n. \tag{23a–d}$$

From these definitions, the following criteria for determining the suitability of a proportional damping approximation can be formulated:

$$\gamma_1 \ll 1\cdot 0, \qquad \gamma_2 \ll 1\cdot 0, \qquad \rho_1 \ll 1\cdot 0, \qquad \rho_2 \ll 1\cdot 0. \tag{24a–d}$$

Of course, when the damping matrix is proportional, each index will be identically zero. A similar set of criteria can be developed from definitions (17), (19), (21) and (22) if one is interested in only a particular mode.

5.2. INDICES BASED ON THE NORMAL CO-ORDINATE DAMPING MATRIX

In section 4 it was shown that the transformation which uncouples the undamped equations of motion will diagonalize the damping matrix of a proportionally damped system with the same mass and stiffness matrices. Conversely, if the damping matrix is non-proportional, its transformation will generally not be diagonal. The presence of off-diagonal terms means that the system response in principal or normal co-ordinates (as defined for the real domain) will no longer be dependent on a single co-ordinate.

It is possible to use the configuration of the normal co-ordinate damping matrix to develop a new class of non-proportionality indices. Assume the system response to be given by $\{x\}=\{X\}\,e^{j\Omega t}$, apply the transformation $\{X\}=[X][M_r]^{-1}\{N\}$, and premultiply by $[X]^t$ to express equation (1) as

$$(-\Omega^2[I]+j\Omega[\tilde{C}]+[\omega_r^2])\{N\}=\{Q\}_n. \tag{25}$$

Here $[\tilde{C}]=[X]^t[C][X][M_r]^{-1}$ and $\{Q\}_n=[X]^t\{Q\}$. In general, the matrix $[\tilde{C}]$ can be written as $[\tilde{C}]=[\tilde{C}_r]+[\Gamma]$, where $[\tilde{C}_r]$ is diagonal, and $[\Gamma]$ is non-diagonal. From this fact and from the form of equation (25), it is possible to develop a number of empirical indices. Several of these will now be considered.

5.2.1. *Summation based indices*

An extremely simple index can be defined as the quotient of the sum of the non-diagonal terms and the sum of all the terms of the transformed damping matrix:

$$\delta_1=\sum_{i=1}^{n}\sum_{j=1}^{n}|\Gamma_{ij}|\Big/\sum_{i=1}^{n}\sum_{j=1}^{n}|\tilde{c}_{ij}|. \tag{26}$$

For $\delta_1 \ll 1$, a proportional damping approximation will give good results. Again, the limiting case, $\delta_1=0$, corresponds to a proportional damping configuration. Equation (26) gives an indication of the non-proportionality of the entire damping matrix; it is a system index "averaged" over all the modes. However, it is quite possible for a system to be heavily coupled at some modes and lightly coupled at others. Since a designer is often interested in a specific frequency or frequency range, a method for determining the degree of coupling present at a particular mode is also needed. Such an index can be developed by modifying expression (26) so that

$$\delta_{1i}=\sum_{j=1}^{n}|\Gamma_{ij}|\Big/\sum_{j=1}^{n}|\tilde{c}_{ij}|, \qquad i=1,2,\ldots,n. \tag{27}$$

δ_{1i} is now the ratio of the sum of the off-diagonal terms of the ith row to the sum of all the terms of the ith row. Note that equation (27) yields n different indices, each corresponding to a different mode.

5.2.2. *Determinant based indices*

Another overall index can be defined as

$$\delta_2=|[\Gamma]|/|[\tilde{C}]|. \tag{28}$$

Once again, as the original damping matrix becomes increasingly proportional, δ_2 approaches zero. Since the determinant of a matrix is a scalar, it is not possible to develop separate modal indices from this definition.

5.2.3. *Response based indices*

The indices of equations (26) and (27) involve two inherent assumptions that prevent them from giving better estimates of the degree of coupling present in real normal co-ordinates. First, each term in the transformed damping matrix is weighted equally. In reality the contribution of each element to the total system response depends upon the magnitude of its corresponding normal displacement. Relations (26) and (27) also neglect the contribution of the real part of equation (25) to the total response.

These two shortcomings can be overcome, but only at the cost of a considerable increase in computational effort. To do so, one defines a new index as the quotient of the change in the magnitude of the normal co-ordinate frequency response when the off-diagonal terms are neglected, and the exact normal frequency response. The excitation function is chosen as a unit sinusoidal force at each normal mode.

To develop the necessary relations, first consider equation (25), the frequency response equation for a sinusoidal forcing function. This relation represents n coupled, linear equations of the form

$$\lfloor(\omega_i^2-\Omega_i^2)+\mathrm{j}\Omega(\tilde{c}_{i1}),\mathrm{j}\Omega(\tilde{c}_{i2}),\ldots,\mathrm{j}\Omega(\tilde{c}_{in})\rfloor\{N\}=1\cdot 0, \tag{29}$$

where $\{Q\}_n$ has been defined as

$$\{Q\}_n=\lfloor 1,1,\ldots,1\rfloor^{\mathrm{t}}. \tag{30}$$

For a given excitation frequency Ω, equation (29) can easily be solved for the normal displacements. The question as to which excitation frequency to use now arises. The errors in the response magnitude tend to be highest in the vicinity of the system natural frequencies. This, and the fact that the response at the natural frequencies is often of special concern to the analyst, suggests that Ω be chosen as the ith damped natural frequency (denoted by μ_i) when computing the ith modal index.

After solving for the exact responses ($\{N\}$), one needs to solve for the response corresponding to the proportional damping configuration (arrived at by neglecting the off-diagonal terms in equation (29)). For the ith mode, this is

$$\tilde{N}_i=1\cdot 0/|(\mu_i^2-\omega_i^2)+\mathrm{j}(\mu_i\tilde{c}_{ii})|. \tag{31}$$

The new modal non-proportionality indices can now be defined as

$$\delta_{3i}=|(|N_i|-|\tilde{N}_i|)|/|N_i|,\qquad i=1,2,\ldots,n. \tag{32}$$

It is also possible to define an overall system index as

$$\delta_3=\sum_{i=1}^{n}\delta_{3i}/n. \tag{33}$$

A set of criteria analogous to those of the preceding section can be developed for determining the suitability of a proportional damping approximation. From relations (28), (29) and (33), these are

$$\delta_1\ll 1\cdot 0,\qquad \delta_2\ll 1\cdot 0,\qquad \delta_3\ll 1\cdot 0. \tag{34a–c}$$

If desired, similar criteria can be formulated for the modal indices of equations (27) and (32).

Because of the empirical nature of the non-proportionality indices just discussed, they are by no means the only ones that could be developed. For instance, it would be possible to devise indices based on the system time domain response. Even so, those presented here are quite comprehensive in that they encompass a wide range of complexities and applications.

6. EXAMPLE CASE

The example case is based on a vibratory system of the configuration depicted in Figure 2. To yield natural frequencies that are closely spaced, and thereby introduce a large degree of coupling between modes, numerical values of the system elements were chosen

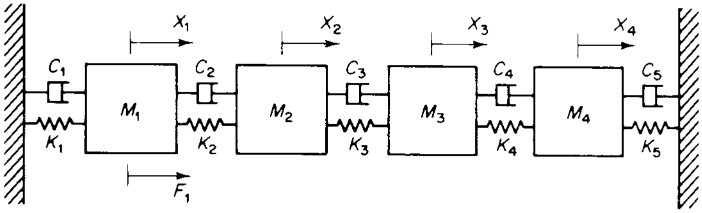


Figure 2. Example case: linear, viscously damped, four degree of freedom system.

to give mass and stiffness matrices of

$$[M]=\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix},\qquad [K]=\begin{bmatrix}5&-3&0&0\\-3&7&-4&0\\0&-4&7&-3\\0&0&-3&5\end{bmatrix}.$$

Table 1 lists the natural frequencies and modes of the system as computed from equation (11). Note that the natural frequencies are indeed closely spaced, particularly those corresponding to modes three and four.

TABLE 1

Undamped natural frequencies and modes of the example problem

Mode	$\omega_i(r/s)$	X_{1i}	X_{2i}	X_{3i}	X_{4i}
1	·7071	·3922	·5883	·5883	·3922
2	1·7647	·6345	·3989	−·3989	−·6345
3	2·4495	·6953	−·2318	−·2318	·6953
4	2·7177	·5534	−·4402	·4402	−·5534

The damping matrix was chosen as

$$[C]=\sigma\begin{bmatrix}c_{11}&-1\cdot5&0\cdot0&0\cdot0\\-1\cdot5&3\cdot5&-2\cdot0&0\cdot0\\0\cdot0&-2\cdot0&3\cdot5&-1\cdot5\\0\cdot0&0\cdot0&-1\cdot5&2\cdot5\end{bmatrix},$$

where σ is a scalar constant. For $c_{11}=2\cdot5$, $[C]$ is proportional with $\alpha=0\cdot0$ and $\beta=0\cdot5$. By incrementing c_{11}, the non-proportionality of the entire matrix can be varied, and by changing σ the overall system damping ratios can be modified. Beginning with a value of 2·5, c_{11} was increased by increments of 0·2 for a total of 20 trials. The sinusoidal forcing function for the frequency response computation was chosen as $\{Q\}=\lfloor F_1 \sin t\ 0\ 0\ 0\rfloor^t$, with $F_1=1\cdot0$.

A total of three σ values were considered. The first, $\sigma=0\cdot05$, yields a lightly damped system with modal damping ratios that range from 0·009 to 0·039 over the course of the non-proportionality increments. The second, $\sigma=0\cdot25$, results in a moderately damped system with damping ratios from 0·044 to 0·196, while for the final, heavily damped case, $\sigma=1\cdot0$, and damping ratios range from 0·177 to 0·983.

In addition, two different proportional damping approximations were considered. The first, referred to as damping approximation 1, is arrived at by neglecting the off-diagonal terms of the normal co-ordinate damping matrix. It would be the logical choice if an analytical study were being performed. The second, damping approximation 2, is defined as the matrix given by

$$[C]=[[X]^t]^{-1}[[2\xi_r\omega_r][M_r]][X]^{-1}. \tag{35}$$

Here ξ_r corresponds to the exact system damping ratios computed from the complex domain eigenproblem solution, and is equivalent to that which might be measured experimentally.

7. RESULTS AND DISCUSSION

Since all three σ cases exhibit similar trends, only results from the moderately damped configuration will be presented here. Figures 3–7 show that the non-proportionality indices behave as predicted in section 5. As the system damping matrix becomes increasingly non-proportional, the indices invariably increase. They also increase most rapidly for the more heavily damped configurations. The nature and rate of change of these increases vary considerably. For example, the modal areas and phase differences ($\{\gamma_{2i}\}$ and $\{\rho_{2i}\}$), exhibit a parabolic shape when plotted against δ_1. Figure 3 shows that the first and second modes are more lightly coupled than the third and fourth. There is, however, some ambiguity in the relative position of the traces for the third and fourth modes. The corresponding plots of $\{\gamma_{2i}\}$ and $\{\rho_{2i}\}$ against δ_2 and δ_3 show similar trends. The curves for δ_2 have an exponential-like shape (Figure 4), while those for δ_3 are essentially linear (Figure 5).

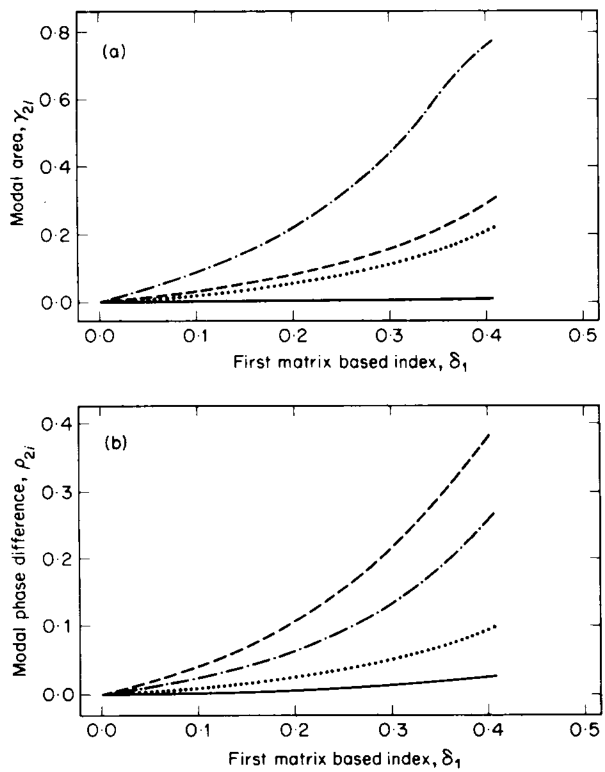


Figure 3 (a) Modal areas and (b) modal phase differences versus the overall first matrix based non-proportionality index. ——, Mode 1; · · · ·, mode 2; - - -, mode 3; - · - · -, mode 4.

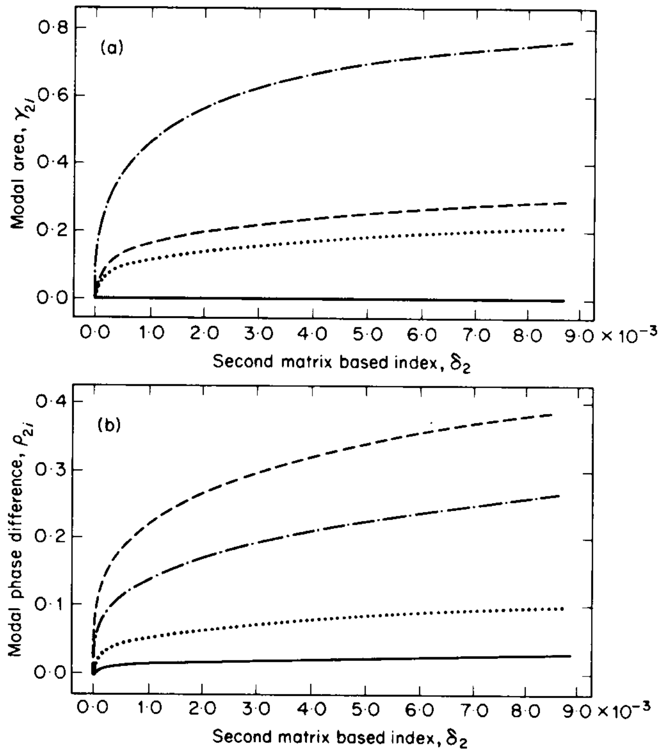


Figure 4 (a) Modal areas and (b) modal phase differences versus the second matrix based non-proportionality index. ——, Mode 1; · · · ·, mode 2; - - -, mode 3; - · - · -, mode 4.

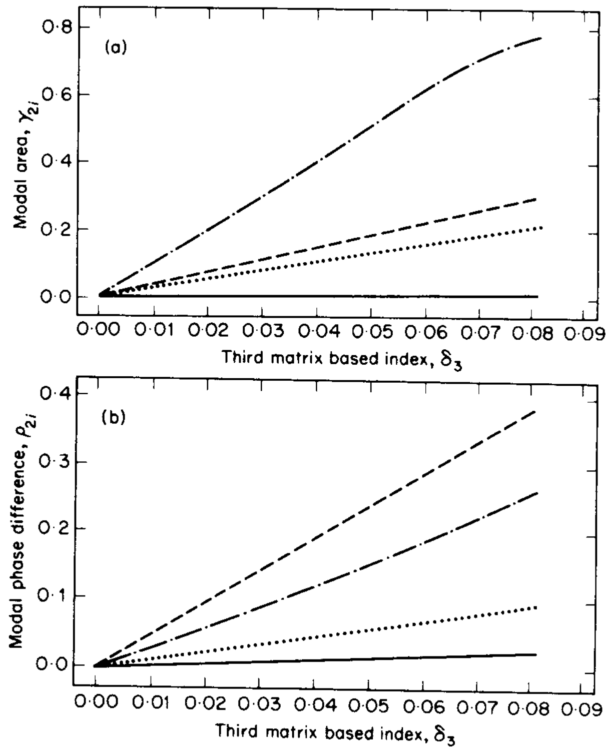


Figure 5 (a) Modal areas and (b) modal phase differences versus the overall third matrix based non-proportionality index. ——, Mode 1; · · · ·, mode 2; - - -, mode 3; · · ·, mode 4.

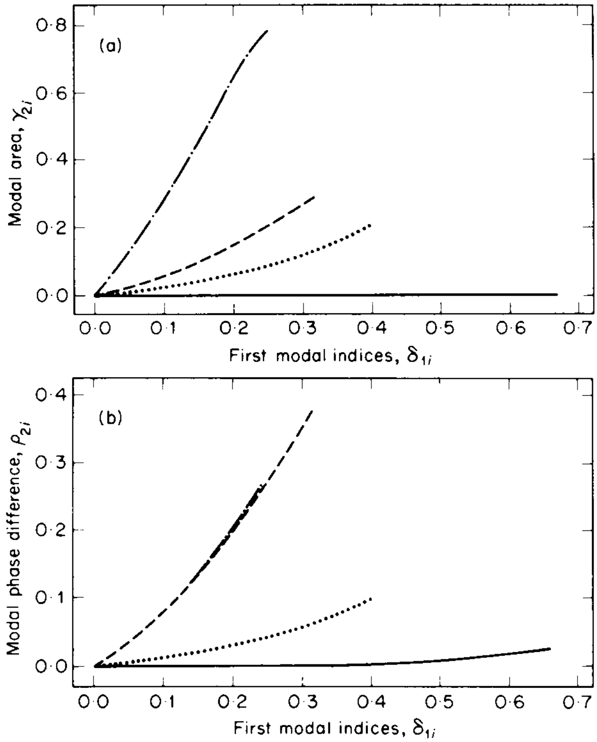


Figure 6. (a) Modal areas and (b) modal phase differences versus the first matrix based non-proportionality indices. ——, Mode 1; · · · ·, mode 2; - - -, mode 3; ·   ·   ·, mode 4.

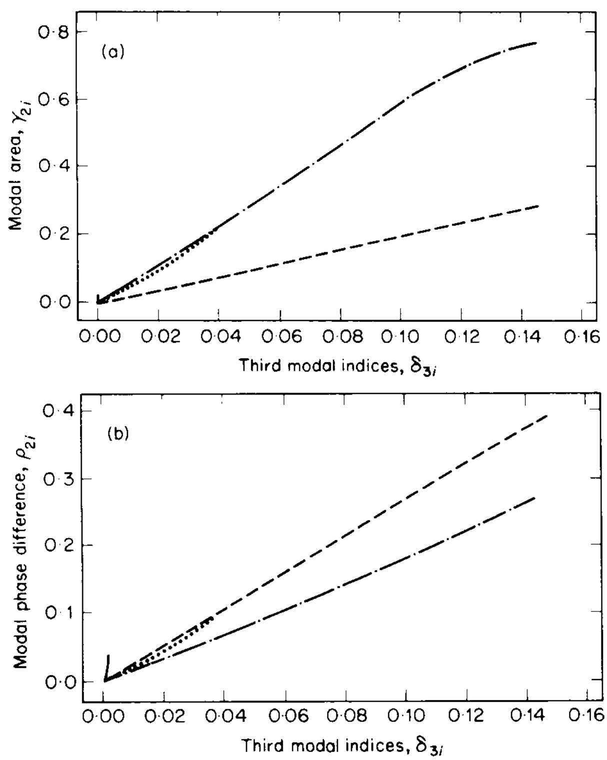


Figure 7. (a) Modal areas, and (b) modal phase differences versus the third matrix based non-proportionality indices. ——, Mode 1; · · · ·, mode 2; - - -, mode 3; - · - · -, mode 4.

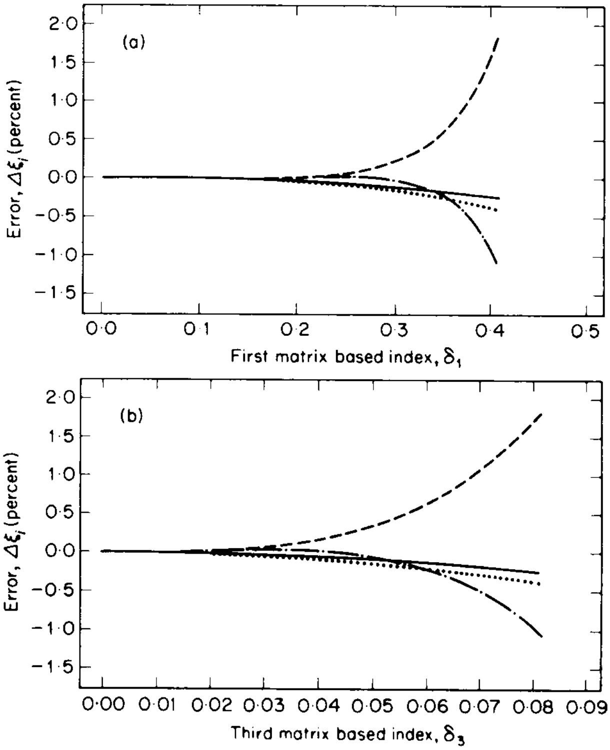


Figure 8. Errors in damping ratio caused by the first damping approximation versus (a) the overall first matrix based non-proportionality index, and (b) the overall third matrix based non-proportionality index. ——, Mode 1; · · · ·, mode 2; - - -, mode 3; - · - · -, mode 4.

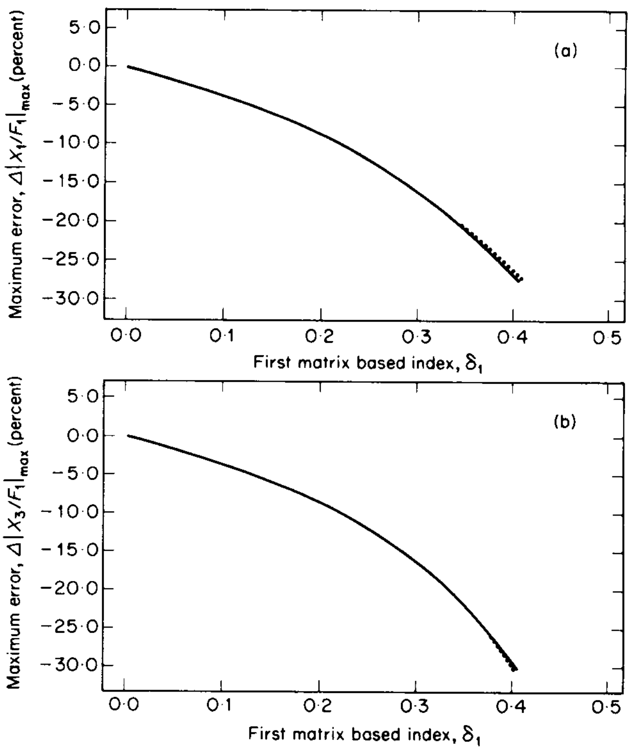
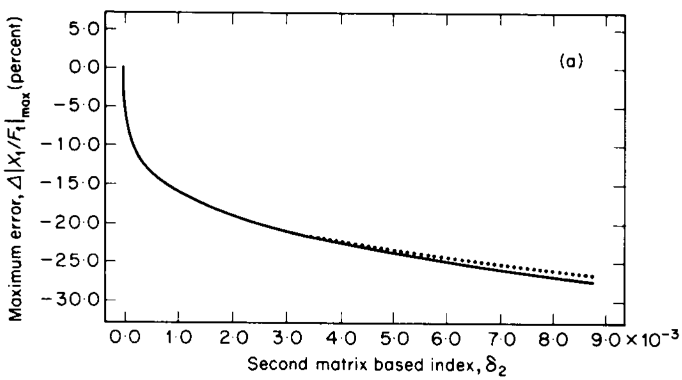


Figure 9. Maximum frequency response magnitude errors versus the overall first matrix based non-proportionality index for (a) mass 1, and (b) mass 3. ——, Damping approximation 1; · · · ·, damping approximation 2.

Figures 6 and 7 show the functional relationships between the matrix based non-proportionality indices and the modal areas and phases. The results for $\{\delta_{1i}\}$ and $\{\delta_{3i}\}$ are quite dissimilar, with the first mode appearing to be lightly coupled in the $\{\delta_{3i}\}$ plots and heavily coupled in the $\{\delta_{1i}\}$ plots. In reality, the system is lightly coupled in this mode, but still exhibits very large, though localized, frequency response errors. This suggests that $\{\delta_{3i}\}$ be used as an indication of the degree of coupling, and $\{\delta_{1i}\}$ as an indication of the response errors.

The effect of the first damping approximation on the system's damping ratios is shown in Figure 8. Here the errors depicted are defined as the difference between the damping ratios computed from the diagonal terms of the normal co-ordinate damping matrix and those computed from the complex eigenproblem solutions. The plots show that the errors are very small, although they are largest for the higher modes and the more non-proportionally damped configurations.

Also included in the results is a set of plots showing the maximum magnitude errors for $X_1/F_1(\omega)$ and $X_3/F_1(\omega)$ as a function of the overall matrix based non-proportionality indices (Figures 9–11). These errors were computed over the entire frequency range of interest (0–60 rad/s), and tend to be maximized near the natural frequencies. The results are similar to the plots of the modal areas and phases against the same indices. Once again, δ_1 yields a parabolic trace, δ_2 an exponential-like trace, and δ_3 a linear trace. This set of plots also shows that the frequency response errors are largest for the more non-proportionally damped configurations. Although plots of the mode based indices against the maximum response errors are not included, the linearity of $\{\gamma_2\}$, $\{\rho_{2i}\}$, and



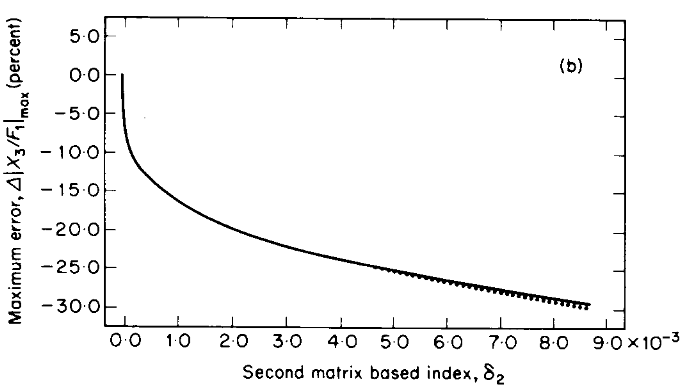


Figure 10. Maximum frequency response magnitude errors versus the second matrix based non-proportionality index for (a) mass 1, and (b) mass 3. ——, Damping approximation 1; · · · ·, damping approximation 2.

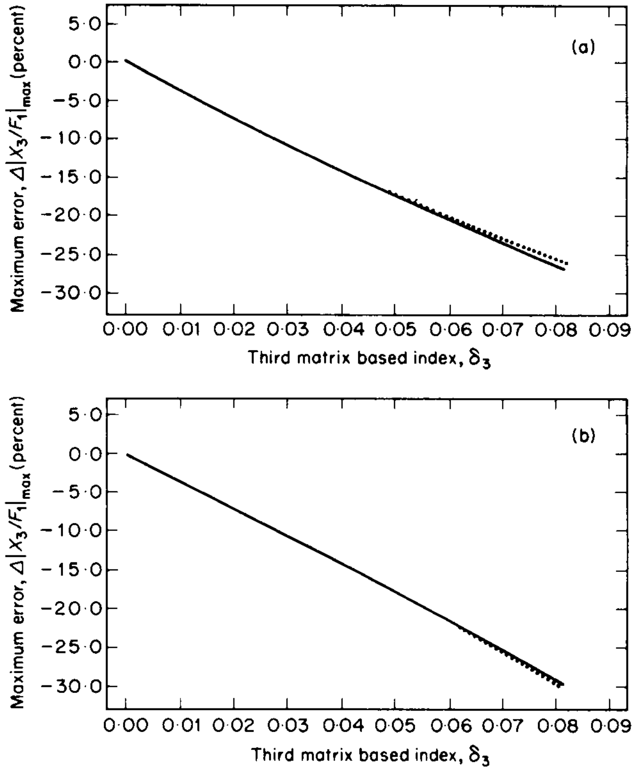


Figure 11. Maximum frequency response magnitude errors versus the overall third matrix based non-proportionality index for (a) mass 1, and (b) mass 3. ——, Damping approximation 1; · · · ·, damping approximation 2.

the response errors when plotted against δ_3 suggest that such relationships would also be linear.

In section 5, the criteria developed for predicting the suitability of a proportional damping approximation specified that each non-proportionality index be much less than unity. Although the availability of any numerical measure of non-proportionality permits comparisons to be made between different systems, the results of this section allow more specific guidelines to be developed.

The first matrix based non-proportionality index gives what is perhaps the best estimate of the error induced by a proportional damping assumption. Since it neglects the real part of the total response, its results are always conservative. Therefore, if an analyst can accept errors of, say, Δe, he is justified in using a proportional damping approximation when $\max(\{\delta_{1i}\}) \leqslant \Delta e$.

The second and third matrix based indices are not as easily interpreted as the first. Both exhibit numerical values that are much smaller than the corresponding response errors. Given the linear relationships that characterize δ_{3i}, it would be possible to specify an absolute criterion for the examples of this study as $\max(\{\delta_{3i}\}) \leqslant k\Delta e$, where k is nominally the reciprocal of the slope of the error plots. Unfortunately, the value of k is not necessarily constant for different systems.

This difficulty with $\{\delta_{3i}\}$ can be attributed to one particular assumption made during its development. The forcing function frequency was chosen as the ith natural frequency, and the response errors here, while large, may not be at a maximum. The third matrix based indices do have an advantage over the others in that they allow computation of the exact error (in normal co-ordinates) at a specific frequency.

The second matrix based index is not as useful as the first or the third. It has very little resolution at low degrees of non-proportionality, and is probably too sensitive at high degrees. It does not discriminate between different modes, and its computation is by no means simple, especially for complicated systems. Because of these limitations, it is best employed as an adjunct to δ_1 and δ_3.

The empirical nature of the normalization procedures used in their development means that the mode based indices, $\{\gamma_{2i}\}$ and $\{\rho_{2i}\}$, suffer from the same limitations as the third matrix based index. Like $\{\delta_{3i}\}$ they initially have linear functional relations to the response errors, but the data base provided by the example cases is not sufficient to determine the associated constants. Therefore, these indices are presently most useful for comparisons between different modes and systems.

8. CONCLUDING REMARKS

The non-proportionality indices presented here can provide both qualitative and quantitative indications of the extent of non-proportional damping present in a discrete vibratory system. Of the two classes developed, the mode based indices are best suited for processing experimental data, and the matrix based indices are best suited for analytical work.

The first matrix based non-proportionality index, δ_1, gives the most reliable estimate of the response errors. In addition, its predictions are always conservative, and it is very simple to compute. On the other hand, δ_3, the third matrix based index, is more useful for comparisons between physical systems, at least until an expanded data base is available for both discrete and continuous systems. It does, however, have the advantage of allowing computation of the exact normal co-ordinate response error for a specific frequency, but it is more difficult to compute than δ_1. The second matrix based index, δ_2, seems to be the least useful of the three in its current form. Computationally, its degree of difficulty falls between that of δ_1 and δ_3.

The mode based indices (modal areas and phases) also require further applications before their exact relationships to the response errors can be determined. At present they can provide a quantitative basis for comparisons between different modes and systems.

Finally, the concept of a complex domain plot of the modal displacements of a system can be useful for purposes other than the development of non-proportionality indices. For example, such plots can indicate the locations of any non-proportionalities in a system. If one is using subsystem analysis as part of a proportional damping approximation [24], they can suggest how the system should be divided. They can also illustrate the effects of inertia and stiffness modifications on a system.

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