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Frequency Response of a Nonlinear Pneumatic System

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This paper studies the dynamic behavior of a pneumatic chamber connected to a reservoir through an orifice and to a linear mechanical system. In this paper, we focus our attention mainly on the nonlinearity associated with mass flow rate through the orifice and its effect on the overall system behavior. Nonlinearities induced by the friction and gas compressibility are also considered. Using the method of harmonic balance, we reduce a difficult mathematical formulation of this nonlinear pneumatic system to a simple nonlinear governing equation and a set of algebraic equations which yield frequency response easily at the first, second, and third harmonics. An excellent agreement between the method of harmonic balance and numerical integration has been found. Results obtained using a quasi-linear model are also given.

1 Introduction

In this paper, we consider a pneumatic chamber connected to a reservoir through an orifice/valve and to a linear mechanical system as shown schematically in Fig. 1. This simple model could represent several practical applications dealing with single-sided cylinder type actuators or isolators, air suspension vehicles, gas bearings, etc. An analytical solution of such a system is not available because of the nonlinearities associated with the orifice, gas compressibility, and friction. We will apply the method of harmonic balance and validate the approximate analytical solution for frequency response by comparing results with those obtained from a quasi-linear model and with those predicted by the numerical integration of the governing equations.

2 Literature Review

For a gas bearing, similar to the system considered here, Doebelin (1980) used a linear model to demonstrate the stability considerations using the Routh criterion. Andersen (1967), Cavanaugh (1976), and Bachrach and Rivin (1983) have also treated open systems to be linear, especially the orifice or valve elements. Linear models for other pneumatic components and systems are also given by Anderson (1967), Blackburn et al. (1960), Cavanaugh (1976), and McCloy and Martin (1980). However, such linear models, while easy to use, may not be able to predict the true dynamic behavior. Furthermore, in some cases, a linearized mathematical description cannot be given or an operating point may not even exist. Hundal (1980) has also pointed out that the linear models are not suitable for shock isolation problems.

Nonlinear models of the pneumatic systems have been solved using the numerical integration techniques. For instance, Hundal (1980, 1982a, and 1982b) has analyzed pneumatic isolators and absorbers numerically. Wang et al. (1984) simulated a cushioning-type, double-acting shock-absorbing cylinder, and predicted time histories and performance indices; good correlations between simulation and experiment are reported. The numerical simulation technique is obviously very attractive and could lead to useful design information. But it could be very time consuming and difficult to implement from the computer algorithm viewpoint for complex systems; numerical convergence problems could also arise when many small control volumes are involved. It should be recognized that a numerical solution may not necessarily lead to an improved understanding of the system behavior.

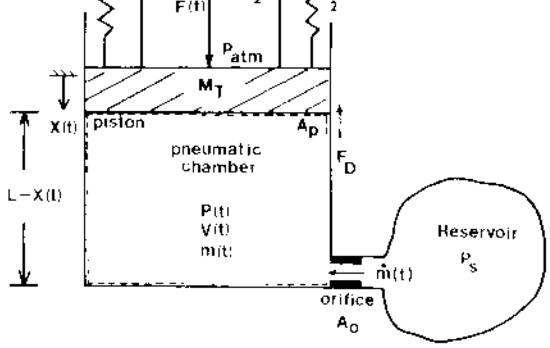

Fig. 1 Schematic of an example case: open pneumatic chamber

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The feasibility of finding approximate analytical solution of pneumatic systems using perturbation techniques has not been investigated properly with the exception of closed chamber analyses by Chen (1977) and Wang (1986). (However, it should be noted that the perturbation techniques have been applied to many other physical systems—see Mickens, 1981; Nayfeh and Mook, 1979; Siljak, 1969.) Chen (1977) analyzed a closed pneumatic chamber with only the nonlinearity induced by the gas compressibility. Recently, Wang (1986) examined the dynamic behavior of a closed pneumatic chamber coupled with a linear mechanical system; nonlinearities induced by the gas compressibility, sliding, and dry friction were included. He found that the nonlinear effects are large, as these are associated with the "hard" singularity (Bota and Mickens, 1984); he also showed the difficulties associated with the analytical solution of this system. The method of harmonic balance was then applied to this problem and an excellent agreement between the method of harmonic balance and numerical integration was evident at the zeroth, first, and second harmonics.

3 Mathematical Formulation

For the gas chamber control volume V as shown in Fig. 1, we assume an ideal equation of state with gas constant R and the polytropic thermodynamic process of exponent n

$$P(t)V(t)=m(t)RT(t) \tag{1}$$

$$P(t)\left[\frac{V(t)}{m(t)}\right]^n=P(0)\left[\frac{V(0)}{m(0)}\right]^n, P(t)T(t)^{\frac{n}{1-n}}=P(0)T(0)^{\frac{n}{1-n}} \tag{2}$$

where P, m, and T are pressure, mass, and temperature, respectively. The continuity equation is then used to relate $m(t)$ and mass flow rate $\dot{m}(t)$ through the valve, assumed to be an ideal orifice of discharging coefficient C and geometric area A_o.

$$m(t)=m(0)+\int_0^t \dot{m}(\zeta)d\zeta \tag{3}$$

$$\dot{m}=CA_oP_s\left[\left(\frac{P}{P_s}\right)^{\frac{2}{\gamma}}-\left(\frac{P}{P_s}\right)^{\frac{\gamma+1}{\gamma}}\right]^{\frac{1}{2}}\left[\frac{2\gamma g}{(\gamma-1)RT_s}\right]^{\frac{1}{2}},$$

$$\text{where } \frac{P}{P_s}>\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma}{\gamma+1}} \tag{4a}$$

$$\dot{m}^*=CA_oP_s\left[\frac{\gamma g}{RT_s}\right]^{\frac{1}{2}}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}},$$

$$\text{where } \frac{P}{P_s}\le\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma}{\gamma+1}} \tag{4b}$$

where γ is the specific heat ratio, g is the gravitational constant, the subscript s is the supply condition associated with the reservoir, and the superscript * indicates the critical value associated with the choking condition; hence, $\dot{m}^*$ is the maximum mass flow rate. Now, we can combine equations (1)–(3) to obtain a single equation for the thermofluid process.

$$\dot{P}(t)V(t)+nP(t)\dot{V}(t)-n\dot{m}(t)RT(0)\left[\frac{P(t)}{P(0)}\right]^{\frac{n-1}{n}}=0 \tag{5}$$

The equation of motion for the piston is

$$M_T\ddot{X}(t)+B\dot{X}(t)+KX(t)=F(t)+M_Tg$$
$$+P(t)A_p+P_{atm}A_p-F_D(t) \tag{6}$$

where M_T is the total mass, B is the linear mechanical damping coefficient, K is the linear mechanical spring stiffness, $F(t)$ is the external force, P_{atm} is the atmosphere pressure, and F_D is the total damping force which is assumed to be of the following form: $F_D=\mu_{\dot{x}}\dot{X}+\mu_pP$, where $\mu_{\dot{x}}$ is the viscous friction coefficient (in force/velocity unit) and μ_p is the scaled sliding friction coefficient (in area unit) to account for P_{atm}. Note that $\mu_{\dot{x}}$ and μ_p are unique to the physical system and must be known (Andersen, 1967; Wang, 1986).

From the initial point ($t=0$), we define the excitation $F(t)$ and responses $X(t)$ and $P(t)$ as

$$F(t)=f_o+f(t),\ P(t)=p_o+p(t),\ X(t)=x_o+x(t),$$

$$f_o=\lim_{T\to\infty}\frac{1}{T}\int_0^T F(t)dt \tag{7}$$

Now, we define response operating points x_o and p_o corresponding to f_o. From equation (5), we get $\dot{m}_o=0$; therefore, the operating point p_o is uniquely defined by the supply conditions P_s as $p_o=P_s$. And, from equation (6), we obtain

$$Kx_o=f_o+M_Tg-(A_p+\mu_p)p_o+P_{atm}A_p \tag{8}$$

For x_o, equation (8) dictates that, when $K=0$, the operating point exists only when $0=f_o+M_Tg-(A_p+\mu_p)p_o+P_{atm}A_p$. But for $K\neq0$, a unique x_o value corresponds to f_o and p_o, i.e., $x_o\equiv x_o(f_o,p_o)$. These unique operating points will now be considered as the reference conditions for dynamic analysis. We also define nondimensional parameters and variables as follows:

$$\begin{aligned}
\bar{D}_o &= D_o/D_p,\ \bar{A}_o=A_o/A_p=\bar{D}_o^2,\ \bar{x}=x/(L-x_o),\\
&\bar{p}=p/p_o,\ \bar{f}=f/[p_oA_p],\\
\bar{K} &= K(L-x_o)/[p_oA_p],\ \omega_n=(K/M_T)^{1/2},\ \bar{\omega}=\omega/\omega_n,\\
&\bar{\mu}_p=\mu_pp_o/[p_oA_p],\\
\xi &= (B+\mu_{\dot{x}})/[2(KM_T)^{1/2}],\ \alpha=C_c(M_TgRT/K)^{1/2}/(L-x_o),\\
\tau &= \gamma^{1/2}[2/(\gamma+1)]^{(\gamma+1)/2(\gamma-1)},\ \bar{\dot{m}}=\dot{m}RT_o/[p_oA_p(L-x_o)]
\end{aligned}$$

where ω is the excitation frequency and ω_n is the undamped natural frequency of the mechanical system.

4 Nature of Nonlinearity

To study the basic nature of nonlinearity, the governing equations, equations (5) and (6), are written in the dimensionless form as

$$\dot{\bar{p}}[1-\bar{x}]-n[1+\bar{p}]\dot{\bar{x}}=n\bar{\dot{m}}[1+\bar{p}]^{\frac{n-1}{n}} \tag{9}$$

$$\ddot{\bar{x}}+2\xi\omega_n\dot{\bar{x}}+\omega_n^2\bar{x}=\frac{\omega_n^2}{\bar{K}}[\bar{f}-(1+\bar{\mu}_p)\bar{p}] \tag{10}$$

These equations could be combined to yield a single equation, as given below, which is a third order ordinary nonlinear differential equation in the following form; here H is defined as the nonlinear function.

$$\dddot{\bar{x}}(t)+2\xi\omega_n\ddot{\bar{x}}(t)+\omega_n^2\dot{\bar{x}}(t)-\frac{\omega_n^2}{\bar{K}}\dot{\bar{f}}(t)=H(\bar{f},\dot{\bar{f}},\bar{x},\dot{\bar{x}},\ddot{\bar{x}},\dddot{\bar{x}}), \tag{11}$$

where

$$H(\bar{f},\dot{\bar{f}},\bar{x},\dot{\bar{x}},\ddot{\bar{x}},\dddot{\bar{x}})=\left[\frac{\omega_n^2}{\bar{K}}\dot{\bar{f}}-\dddot{\bar{x}}-2\xi\omega_n\ddot{\bar{x}}-\omega_n^2\dot{\bar{x}}\right]\bar{x}$$

Table 1 Order analysis of the zeroth and first five harmonics at $\bar{f}_1 = 0.4$ and $\bar{\omega} = 0.419$

Harmonic coefficient	Numerical integration		The method of harmonic balance		Quasi-linear model
	$\bar{x}_m/\bar{x}_1$	$\bar{p}_k/\bar{p}_1$	$\bar{x}_m/\bar{x}_1$	$\bar{p}_k/\bar{p}_1$	$\bar{x}_m/\bar{x}_1$, $\bar{p}_k/\bar{p}_1$
0	−0.026	−0.019	0	0	0
1	1	1	1	1	1
2	0.061	0.053	0.101	0.057	0
3	0.034	0.048	0.070	0.066	0
4	0.009	0.019	0	0	0
5	0.007	0.024	0	0	0
Reference values	$\bar{x}_1 = 0.445$	$\bar{p}_1 = 0.208$	$\bar{x}_1 = 0.507$	$\bar{p}_1 = 0.237$	$\bar{x}_1 = 0.349$ $\bar{p}_1 = 0.276$

$$+n\left[\frac{\omega_n^2}{\bar{K}}(1+\bar{\mu}_p+\bar{f})-\ddot{\bar{x}}-2\xi\omega_n\dot{\bar{x}}-\omega_n^2\bar{x}\right]\dot{\bar{x}}$$

$$+n\dot{\bar{m}}\left[\frac{\omega_n^2}{\bar{K}}(1+\bar{\mu}_p+\bar{f})-\ddot{\bar{x}}-2\xi\omega_n\dot{\bar{x}}-\omega_n^2\bar{x}\right]^{\frac{n-1}{n}}\left[\frac{\omega_n^2}{\bar{K}}(1+\bar{\mu}_p)\right]^{\frac{1}{n}}$$

First, consider the case when $\dot{\bar{m}}=0$. Equation (11) is reduced to the closed chamber case of Wang (1986) as

$$\ddot{\bar{x}}(t)+2\xi\omega_n\dot{\bar{x}}(t)+\omega_n^2\bar{x}(t)=\frac{\omega_n^2}{\bar{K}}\bar{f}(t)-H(x), \tag{12}$$

where

$$H(\bar{x})=\frac{\omega_n^2}{\bar{K}}(1+\bar{\mu}_p)\bar{p}=\frac{\omega_n^2}{\bar{K}}(1+\bar{\mu}_p)[(1-\bar{x})^{-n}-1]$$

This equation clearly demonstrates the "hard" singularity as $H\to\infty$ when $\bar{x}\to 1$. We also note the "anharmonic" type oscillations as H is not symmetric about $\bar{x}=0$ (Mahaffey, 1976).

Second, examine the case when $\dot{\bar{m}}\neq 0$. A clear identification of the nature of nonlinearity seems very difficult as all variables and their products appear in equation (11). This is the result of adding the nonlinear orifice element to the pneumatic chamber. Since for the closed chamber, Wang (1986) found large nonlinear effects and "hard" singularity, we must assume that these are still in evidence for the open chamber case as well.

5 The Method of Harmonic Balance

We will now find an approximate solution for equation (11) using the method of harmonic balance. Our selection of this method is based on the claim by Mickens (1984) who stated that the method of harmonic balance is the only suitable technique of analyzing a system with very strong nonlinear interactions. Wang (1986) has already used this method successfully for the closed pneumatic chamber case.

A general analytical solution for equations (4)–(6) or (11) is very difficult, if not impossible. Therefore, we must simplify them in order to perform the perturbation analysis. Accordingly, we assume that (*i*) the flow is unchoked, $\dot{m}\le\dot{m}^*$, (*ii*) the thermodynamic process is isothermal, $n=1$; and (*iii*) the discharge coefficient C is a constant, $C=C_c$. Such conditions are found in many practical situations and therefore our assumptions should be valid. Equation (4) is now simplified and generalized as

$$\dot{m}(t)=C_cA_o\left[|P_s^2-P^2(t)|\frac{g}{RT_s}\right]^{\frac{1}{2}}\operatorname{sign}(P_s-P(t)) \tag{13}$$

$$\text{where sign }(P_s-P)\begin{cases}=-1\\=0\\=1\end{cases}\text{ at }P_s-P\begin{cases}<0\\=0\\>0\end{cases}$$

and equation (5) is reduced to

$$\dot{P}(t)V(t)+P(t)\dot{V}(t)=\dot{m}(t)RT \tag{14}$$

Therefore, the resulting equations in the nondimensional form are

$$\dot{\bar{p}}(t)[1-\bar{x}(t)]-[1+\bar{p}(t)]\dot{\bar{x}}(t)$$
$$=\alpha\bar{D}_o^2\omega_n[|1-(1+\bar{p}(t))^2|]^{1/2}\operatorname{sign}(-\bar{p}(t)) \tag{15}$$

$$\ddot{\bar{x}}(t)+2\xi\omega_n\dot{\bar{x}}(t)+\omega_n^2\bar{x}(t)=\frac{\omega_n^2}{\bar{K}}[\bar{f}(t)-(1+\bar{\mu}_p)\bar{p}(t)] \tag{16}$$

Now, we apply the method of harmonic balance to predict the frequency response. According to this method, the excitation and responses are simultaneously assumed as follows:

$$\bar{f}(t)=\bar{f}_1\cos(\omega t+\theta_{f1}),$$

$$\bar{p}(t)=\sum_{k=0}^{\infty}\bar{p}_k\cos(k\omega t+\theta_{pk}),$$

$$\bar{x}(t)=\sum_{m=0}^{\infty}\bar{x}_m\cos(m\omega t+\theta_{xm}) \tag{17}$$

Since there is some ambiguity regarding the order approximation associated with the application of the method of harmonic balance to a pneumatic system as shown by Wang (1986), we first examine the harmonic orders using the results obtained by the numerical integration of equations (4)–(6) or (11). Table 1 indicates that the zeroth harmonic is even less than the third harmonic and that the second and third harmonics are of the same order of magnitude (this table will be discussed in detail later). Furthermore, since the pneumatic system applications do not deal with very high frequencies, the interest here could be limited up to the third harmonic. Hence, $\bar{p}(t)$ and $x(t)$ are approximated as

$$\begin{aligned}\bar{p}(t)&\simeq\bar{p}_1\cos(\omega t+\theta_{p1})+\bar{p}_2\cos(2\omega t+\theta_{p2})+\bar{p}_3\cos(3\omega t+\theta_{p3}),\\ \bar{x}(t)&\simeq\bar{x}_1\cos(\omega t+\theta_{x1})+\bar{x}_2\cos(2\omega t+\theta_{x2})+\bar{x}_3\cos(3\omega t+\theta_{x3})\end{aligned} \tag{18}$$

where

$$\bar{p}_2/\bar{p}_1=\bar{x}_2/x_1=0(\epsilon),\ \bar{p}_3/\bar{p}_1=\bar{x}_3/\bar{x}_1=0(\epsilon^2)$$

and $\bar{p}_1=\bar{x}_1=0(\epsilon)$.

The procedure of determining response for equations (15) and (16) is: (*i*) substitute equations (17) for $\bar{f}(t)$ and (18) for $\bar{p}(t)$ and $\bar{x}(t)$ into equations (15) and (16); (*ii*) expand $|1-(1+\bar{p})^2|^{1/2}$ sign $(-p)$ to a sine and cosine series; and (*iii*) to reduce the mathematical difficulty, set θ_{p1} equal to zero, not θ_{f1}. Mathematical manipulations yield twelve coupled algebraic equations, which allow us to solve for the first, second, and third harmonic coefficients and phases, step by step, without any interference. Closed form solutions are as follows:

$$\bar f_1^2 = \sigma_1 \bar p_1^2 + \sigma_2 \bar p_1^{3/2} + \sigma_3 \bar p_1 \tag{19}$$

where

$$\sigma_1 = [1+\bar\mu_p+(1-\bar\omega^2)\bar K]^2 + [2\xi\bar\omega\bar K]^2$$
$$\sigma_2 = 6.468\alpha\xi\bar D_o^2\bar K(1+\bar\mu_p)$$
$$\sigma_3 = [3.234\alpha\xi\bar D_o^2\bar K]^2 + [1.617\alpha\bar D_o^2(1-\bar\omega^2)\bar K/\bar\omega]^2$$

$$\bar x_1 = \left[\bar p_1^2 + \left(\frac{1.617\alpha\bar D_o^2}{\bar\omega}\right)^2 \bar p_1\right]^{1/2} \tag{20}$$

$$\tan\theta_{x1} = \frac{-1.617\alpha\bar D_o^2}{\bar\omega\bar p_1^{1/2}} \tag{21}$$

$$\tan\theta_{p1} = \frac{(1-\bar\omega^2)\bar K\bar x_1\sin\theta_{x1} + 2\xi\bar\omega\bar K\bar x_1\cos\theta_{x1}}{(1+\bar\mu_p)\bar p_1 + (1-\bar\omega^2)\bar K\bar x_1\cos\theta_{x1} - 2\xi\bar\omega\bar K\bar x_1\sin\theta_{x1}} \tag{22}$$

$$\bar x_2 = \frac{(1+\bar\mu_p)\bar x_1\bar p_1}{2\{[1+\bar\mu_p+(1-4\bar\omega^2)\bar K]^2 + [4\xi\bar\omega\bar K]^2\}^{1/2}} \tag{23}$$

$$\bar p_2 = \frac{\bar K\bar x_2[(1-4\bar\omega^2)^2+(4\xi\bar\omega)^2]^{1/2}}{1+\bar\mu_p} \tag{24}$$

$$\tan\theta_{x2} = \frac{-\{[1+\bar\mu_p+(1-4\bar\omega^2)\bar K]\sin\theta_{x1} - 4\xi\bar\omega\bar K\cos\theta_{x1}\}}{\{[1+\bar\mu_p+(1-4\bar\omega^2)\bar K]\cos\theta_{x1} + 4\xi\bar\omega\bar K\sin\theta_{x1}\}} \tag{25}$$

$$\tan\theta_{p2} = \frac{-\{(1-4\bar\omega^2)\sin\theta_{x2} + 4\xi\bar\omega\cos\theta_{x2}\}}{\{(1-4\bar\omega^2)\cos\theta_{x2} - 4\xi\bar\omega\sin\theta_{x2}\}} \tag{26}$$

$$\bar x_3 = \left\{\frac{(1+\bar\mu_p)^2(\Gamma_1^2+\Gamma_2^2)}{[1+\bar\mu_p+(1-9\bar\omega^2)\bar K]^2 + [6\xi\bar\omega\bar K]^2}\right\}^{1/2} \tag{27}$$

where

$$\Gamma_1 = 0.333[\bar x_1\bar p_2\cos(\theta_{p2}+\theta_{x1}) + \bar x_2\bar p_1\cos\theta_{x2}]$$

$$\Gamma_2 = 0.333[\bar x_1\bar p_2\sin(\theta_{p2}+\theta_{x1}) + \bar x_2\bar p_1\sin\theta_{x2}] - \frac{0.114\alpha\bar D_o^2\bar p_1^{1/2}}{\bar\omega}$$

$$\bar p_3 = \frac{\bar K\bar x_3[(1-9\bar\omega^2)^2+(6\xi\bar\omega)^2]^{1/2}}{1+\bar\mu_p} \tag{28}$$

$$\tan\theta_{x3} = \frac{-\{[1+\bar\mu_p+(1-9\bar\omega^2)\bar K]\Gamma_2 - [6\xi\bar\omega\bar K]\Gamma_1\}}{-\{[1+\bar\mu_p+(1-9\bar\omega^2)\bar K]\Gamma_1 + [6\xi\bar\omega\bar K]\Gamma_2\}} \tag{29}$$

$$\tan\theta_{p3} = \frac{-\{(1-9\bar\omega^2)\sin\theta_{x3} + 6\xi\bar\omega\cos\theta_{x3}\}}{-\{(1-9\bar\omega^2)\cos\theta_{x3} - 6\xi\bar\omega\sin\theta_{x3}\}} \tag{30}$$

To examine the uniqueness of the solution, we consider equation (19) for $\bar p_1$, which is a nonlinear algebraic equation. Since ξ, K, and $\bar\mu_p$ are positive, σ_1, σ_2, and σ_3 must be definitely positive. Incidently, no negative $\bar p_1$ can be considered; hence $\bar p_1$ is unique and numerically convergent. After solving $\bar p_1$, other harmonic results as given by equations (2)–(30) could be uniquely solved for and all results converge numerically. Note that the uniqueness does not imply local stability of the solution as other stable motions may occur. The solution obtained here yields only the information which is presupposed by equation (18).

6 Quasi-Linear Model

Since our operating point for the nonlinear perturbation analysis is associated with $\dot m_o = 0$ or $p_o = P_s$, the linear system theory cannot describe equations (4) or (13). Thus we must perform the quasi-linearization by assuming the following: (*i*) $\dot p x \simeq 0$; (*ii*) $p\dot x \simeq 0$; and (*iii*) the orifice is modeled as a resistance of constant value R_c which is assumed to be half of the critical flow resistance, i.e.,

$$R_c = \frac{P_s - P}{\dot m} \simeq \frac{1}{2\left\{C_c A_o\left[\frac{\gamma g}{RT_s}\right]^{\frac{1}{2}}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}}\right\}} \tag{31}$$

In the nondimensional form, we define

$$\bar R_c = \frac{1-(1+\bar p)}{\dot m/\omega_n} \simeq \frac{1}{2\bar D_o^2\alpha\tau} \tag{32}$$

Hence, equation (9) is reduced to a first order differential equation

$$\dot{\bar p} + 2\bar D_o^2\alpha\tau\omega_n\bar p = \dot{\bar x} \tag{33}$$

Thus the governing linear equation, a third order ordinary differential equation, is as follow

$$\dddot{\bar x} + [2\xi + 2\bar D_o^2\alpha\tau]\omega_n\ddot{\bar x} + \left[1 + 4\bar D_o^2\alpha\tau\xi + \frac{1+\bar\mu_p}{\bar K}\right]\omega_n^2\dot{\bar x} + 2\bar D_o^2\alpha\tau\omega_n^3\bar x = \frac{\omega_n^2}{\bar K}\dot{\bar f} + 2\bar D_o^2\alpha\tau\omega_n^3\bar f \tag{34}$$

Applying the Routh criterion to equation (34), we determine that the roots are stable.

7 Results

Now we can compare the results obtained using the method of harmonic balance as given by equations (19)–(30) with the results predicted by the numerical integration of equations (9) and (10), and with the results given by the quasi-linear model of equations (33) and (34). Numerical parameters used for the demonstration of the validity of the analytical solution are chosen as:

$$\bar K = 0.511,\ \xi = 0.5,\ \bar\mu_p = 0,\ \bar D_o = 0.05,\ \alpha = 95.37,$$
$$\tau = 0.685,\ \omega_n = 0.5$$

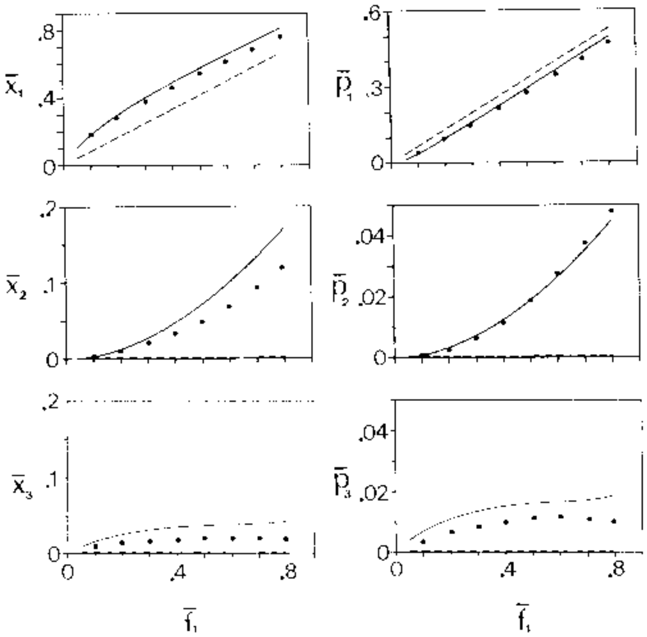

Fig. 2 Response versus excitation at $\bar\omega = 0.419$: (- - - -) quasi-linear model; (. .) numerical integration; (——) method of harmonic balance

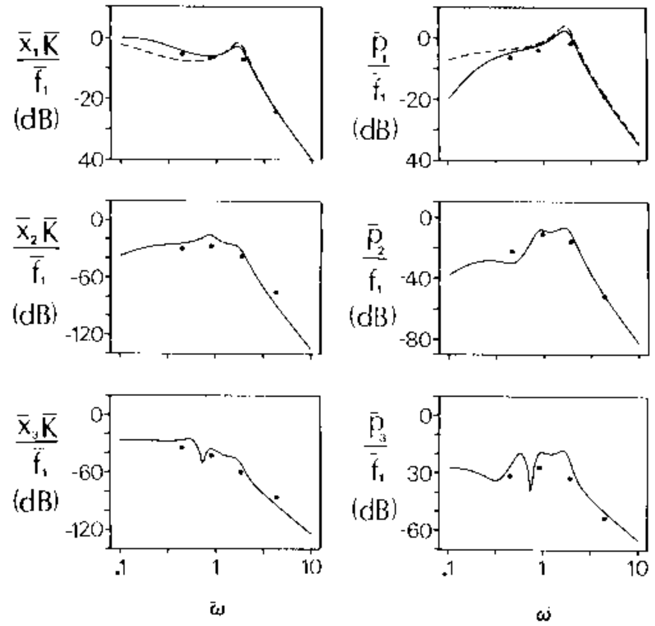

Fig. 3 Magnitude frequency response curves at $f_1 = 0.4$: (- - - -) quasi-linear model; (. .) numerical integration; (——) method of harmonic balance

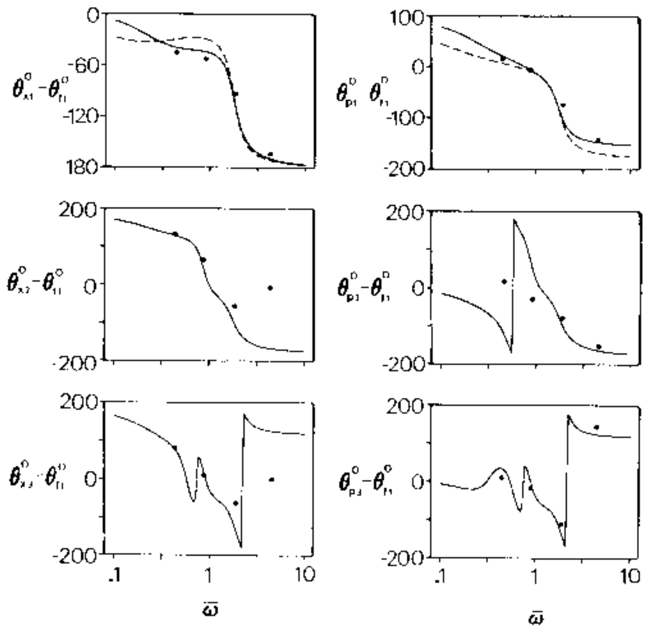

Fig. 4 Phase frequency response curves at $f_1 = 0.4$: (- - - -) quasi-linear model; (. .) numerical integration; (——) method of harmonic balance

In Table 1, an order analysis for both $\bar{x}$ and p at the zeroth and first five harmonics has been performed using the first harmonic as the reference. Although the harmonic coefficients predicted by the numerical integration do not follow our assumption for the method of harmonic balance perfectly, we find the results to be very reasonable considering the assumptions and simplifications involved.

Frequency response curves for a linear system are given by plotting the response versus frequency as the dimensionless response (normalized by dividing it by the excitation $\bar{f}$) is not a function of the excitation $\bar{f}$. For a nonlinear system, this is obviously not true, i.e., $\bar{x} = x(\bar{f}, \bar{\omega})$. Hence, a three-dimensional representation is required to sketch the sinusoidal response. But, since a three-dimensional graph is difficult to visualize,

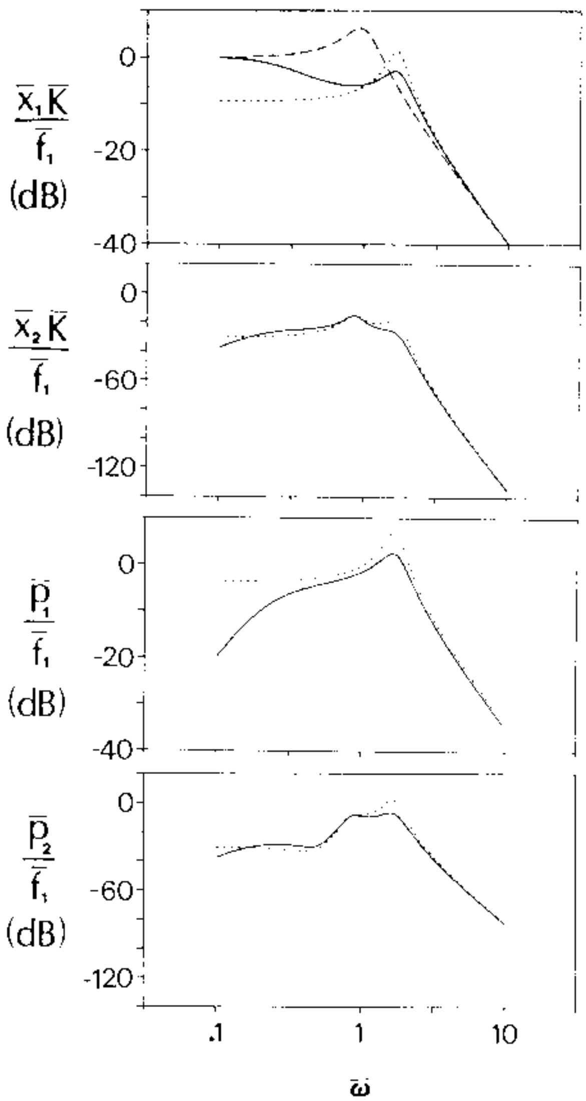

Fig. 5 Magnitude frequency response curves for various nonlinearities at $f_1 = 0.4$: (- - - -) pure mechanical system; (. . . .) closed chamber; (——) open chamber

we will illustrate the response through two-dimensional views by taking $\bar{x} = x(\bar{\omega})|_{\bar{f}=\text{constant}}$ and $\bar{x} = x(\bar{f})|_{\bar{\omega}=\text{constant}}$.

We compare response $\bar{x}$ and $\bar{p}$ with excitation $\bar{f}$ levels in Fig. 2 for the first, second, and third harmonics. Even though we had noted some ambiguity earlier regarding the harmonic order, the method of harmonic balance still predicts second and third harmonic results successfully. In addition, an excellent agreement is found between the method of harmonic balance and the numerical integration at the first harmonic. We also note that the quasi-linear model, while not capable of predicting response at second and third harmonics, yields reasonable results at the first harmonic.

Figures 3 and 4 show the frequency response curves with ω_n as the reference frequency. By adding an orifice, the overall system becomes a third order system, and its "resonant" frequency is shifted to $\bar{\omega} = 1.62$. We again find excellent agreement, with the exception of results for the phase at the higher frequencies and harmonics, between the predictions by the method of harmonic balance and by the numerical integration. And, at the first harmonic, the quasi-linear model matches well with other models.

All of the frequency response results predicted by the numerical integration technique and the method of harmonic balance did not show any instability or jump phenomenon.

8 Orifice Nonlinearity

In order to determine the effect of orifice nonlinearity, we compare frequency response magnitudes at first and second harmonics for the following cases: (*i*) pure mechanical system; (*ii*) closed pneumatic system ($\bar{D}_o = 0$); and (*iii*) open pneumatic system ($\bar{D}_o = 0.05$). The results obtained using the method of harmonic balance are shown in Fig. 5. Results for the first harmonic ($\bar{x}_1$ and $\bar{p}_1$) indicate that the nonlinear orifice has strong influence at very low frequencies ($\bar{\omega} \ll 1$) and in the vicinity of the resonance ($\bar{\omega} = 1.62$). At very low frequencies, the orifice completely eliminates the pneumatic spring or cushioning effect; therefore, the $\bar{x}_1$ response of open chamber case is the same as that of the pure mechanical system. In the vicinity of resonance, magnitudes associated with the open chamber are lower compared to the closed-chamber case. This demonstrates that the orifice acts mainly as the damper since the resonant frequency for both cases is about the same. Results for the second harmonic ($\bar{x}_2$ and $\bar{p}_2$) indicate that the open and closed chambers behave approximately in the same manner except that the orifice adds slightly more damping.

The orifice also changes the order of harmonics. For example, the response at the zeroth harmonic for the open system is considerably lower than the response at the first harmonic, i.e., $\bar{x}_0/\bar{x}_1 \leq 0(\epsilon^2)$. Conversely for the closed system, $\bar{x}_0/\bar{x}_1 - 0(\epsilon^1)$ was found. This illustrates that the orifice essentially diminishes the unsymmetrical nature of the compression process which will reduce the "anharmonic" type oscillations.

9 Conclusion

The major accomplishment of this study has been in gaining a better understanding of the effect of the nonlinear orifice and in eliminating the mathematical difficulty associated with zero mean flow rate case while using the method of harmonic balance. In spite of several simplifications involved and some doubt regarding the harmonic order, our analytical solution still provides accurate predictions when compared to the numerical integration technique. Of more importance is the fact that a difficult mathematical formulation of a nonlinear pneumatic system has been reduced to a set of simple algebraic equations which are much easier to deal with; also a substantial saving in computer time is evident. Further work in this area is in progress with emphasis in the modeling of "real" world components and systems.

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