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COMPLEX EIGENSOLUTION FOR LONGITUDINALLY VIBRATING BARS WITH A VISCOUSLY DAMPED BOUNDARY

1. INTRODUCTION

Typical vibration texts and handbooks [1–6] tabulate only undamped natural frequencies and modes for standard structural elements. Exact solutions for distributed parameter systems with lumped dampers at the boundaries are generally not available, as the eigenvalue problem must be formulated and solved in the complex domain. In this note we investigate such configurations through the example case of a longitudinally vibrating bar which is fixed at one boundary, and constrained by a lumped damper of viscous damping coefficient B at the other (see Figure 1). A closed form eigensolution is generated, and results are verified by inspecting the predicted behavior at limiting damping values, and by comparing continuous system and discrete system modal parameters over a wide damping range.

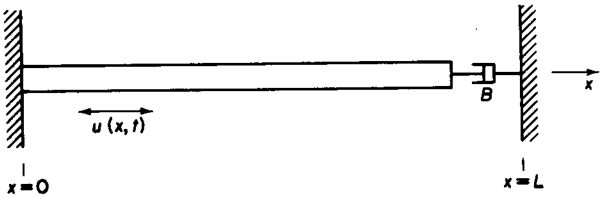

Figure 1. Longitudinally vibrating bar with fixed boundary at $x = 0$ and viscously damped boundary at $x = L$.

2. COMPLEX EIGENSOLUTION

The undamped wave equation describing the vibratory motion $u(x, t)$ of a bar is,

$$c^2\, \partial^2 u(x, t)/\partial x^2 = \partial^2 u(x, t)/\partial t^2, \tag{1}$$

where c is the speed of wave propagation and x is the axial co-ordinate. Separation of variables shows the solution to be harmonic, and of the form

$$u(x, t) = \Phi(x)\, e^{\gamma t}, \qquad \Phi(x) = X_1 \sin(i\gamma x/c) + X_2 \cos(i\gamma x/c). \tag{2a, b}$$

Here X_1 and X_2 are constants to be determined from the constraint conditions, i is the imaginary unit, $\Phi(x)$ is a complex eigenfunction, and γ is a complex eigenvalue given in terms of the system damping ratio (ζ), damped natural frequency (ω_d), and undamped natural frequency (ω) as

$$\gamma = -\zeta\omega \pm i\omega_d, \quad \omega_d = \omega\sqrt{1-\zeta^2}, \qquad 0 \leqslant \zeta \leqslant 1. \tag{3a, b}$$

Upon applying the no displacement boundary condition at $x = 0$ and recognizing that the amplitude of the eigenfunction is arbitrary, we obtain

$$u(0, t) = \Phi(0) = X_2 = 0, \qquad \Phi(x) = \sin(i\gamma x/c). \tag{4a, b}$$

The damping force at $x = L$ produces a compressive normal strain such that

$$B\, \partial u(L, t)/\partial t = -EA\, \partial u(L, t)/\partial x, \qquad -\gamma B \Phi(L) = EA\, \partial\Phi(L)/\partial x, \tag{5a, b}$$

where E is the modulus of elasticity and A is the cross-sectional area.

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Substituting equation (4b) into equation (5b) results in the following complex domain frequency equation:

$$\gamma_n = (\mathrm{i}c/L)\tan^{-1}(\mathrm{i}EAL/Bc). \tag{6}$$

This can be written, by using the definition of the complex arctangent function, as

$$\gamma_n = -\frac{c}{2L}\left[\ln\frac{\beta+1}{\beta-1}+\mathrm{i}\left(\tan^{-1}\left(\frac{0}{\beta+1}\right)+2n\pi\right)\right], \qquad n=1,2,\ldots. \tag{7}$$

In this expression the implicit complex conjugate terms have not been included, and β is a dimensionless damping coefficient given by

$$\beta = Bc/EA. \tag{8}$$

If we divide γ by the natural frequency corresponding to $\beta=0$, $(2n-1)\pi c/2L$, dimensionless damped and undamped natural frequencies can be defined as

$$\Omega_{nd} = \frac{2L\omega_{nd}}{(2n-1)\pi c} = \frac{1}{(2n-1)\pi}\left[\tan^{-1}\left(\frac{0}{\beta-1}\right)+2n\pi\right], \qquad n=1,2,\ldots, \tag{9a, b}$$

$$\Omega_n = \Omega_{nd}/\sqrt{(1-\zeta^2)} = 2L\omega_n/(2n-1)\pi c$$

$$= \frac{1}{(2n-1)\pi}\left[\left(\ln\left|\frac{\beta+1}{\beta-1}\right|\right)^2+\left(\tan^{-1}\left(\frac{0}{\beta+1}\right)+2n\pi\right)^2\right]^{1/2}, \qquad n=1,2,\ldots, \tag{10a–c}$$

$$\zeta_n = \alpha/(\alpha^2+\delta^2), \quad \alpha = \ln|(\beta+1)/(\beta-1)|, \quad \delta = \tan^{-1}\{0/(\beta+1)\}+2n\pi, \qquad n=1,2,\ldots. \tag{11a–c}$$

After substituting equations (7)–(11) into equation (46), the magnitude and phase of the complex eigenfunctions can be written as

$$|\Phi_n(x)| = [\sin^2(\omega_{nd}x/c)+\sinh^2(\omega_n x/c)]^{1/2}, \tag{12a}$$

$$\theta_n(x) = \tan^{-1}[\tanh(\zeta_n\omega_n x/c)/\tanh(\omega_{nd}x/c)], \tag{12b}$$

thereby completing the eigensolution.

3. LIMITING DAMPING CASES

It is now instructive to consider the behavior of the eigensolution for the following limiting values of the damping coefficient B: (i) $B \ll EA/c(\beta\to 0)$,

$$\zeta_n = 0, \quad \Omega_n = \Omega_{nd} = 1{\cdot}0, \quad \omega_n = (2n-1)\pi c/2L,$$

$$\Phi_n(x) = \sin[(2n-1)\pi x/2L], \qquad n=1,2,\ldots; \tag{13a–e}$$

(ii) $B \gg EA/c(\beta\to\infty)$,

$$\zeta_n = 0, \quad \Omega_n = \Omega_{nd} = 2n/(2n-1), \quad \omega_n = n\pi c/L,$$

$$\Phi_n(x) = \sin[n\pi x/L], \qquad n=1,2,\ldots; \tag{14a–e}$$

(iii) $B \to EA/c(\beta\to 1{\cdot}0)$,

$$\zeta_n \to 1{\cdot}0, \quad \Omega_n\to\infty, \quad \Omega_{nd}\to\infty, \quad n=1,2,\ldots. \tag{15a–c}$$

Clearly, the results of case (i) correspond to the eigensolution of a bar fixed at $x=0$ and free at $x=L$, while those of case (ii) represent a fixed-fixed bar. The eigenfunctions predict real domain modes and are consistent with classical theory. For finite damping values the system will be non-proportionally damped and have a complex domain eigensolution. The special case of $\beta = 1{\cdot}0$ in addition yields a critically damped system configuration, and exhibits an eigenvalue singularity which is characteristic of an infinite natural frequency and no oscillation. To obtain a physical feel for this behavior, we examine the damped boundary in terms of its impedance properties and note that, when $B = EA/c$, the constraint offers characteristic impedance. A damper with this coefficient can be thought of as an anechoic termination; the longitudinally propagating wave is not being reflected at $x = L$, and hence no standing waves or normal modes are possible [7]. The bar can be considered infinitely long, and the associated natural frequency will be unbounded.

4. DISCRETE SYSTEM MODEL AND RESULTS

A ten lump discretized model of the bar was developed, and analyzed in the complex domain [8-10]. Material and geometric properties were chosen such that $E = 2 \times 10^{11}$ Pa, density $\rho = 7700\ \mathrm{kg/m^3}$, $c = 5096$ m/s, $L = 0{\cdot}2$ m, and $A = 0{\cdot}0004\ \mathrm{m^2}$. The damping coefficient B was varied from 8 to 8×10^{12} N s/m, resulting in a β range from 5×10^{-4} to 5×10^{-8}.

The continuous and discrete system Ω_1 results are presented in Figure 2(a) as a function of β. Predictions for the limiting cases of very small and very large damping values compare well with the classical solutions of fixed-free and fixed-fixed bars, respectively. For mid-range β values, Ω_1 increases from the fixed-free value to the fixed-fixed value. Note that the discrete model yields a large though finite natural frequency value at $\beta = 1$, while the continuous model predicts an infinite value.

First mode damping ratio results (ζ_1) for both models are displayed in Figure 2(b). The maximum value occurs in both cases at $\beta = 1$, and while the values here are slightly

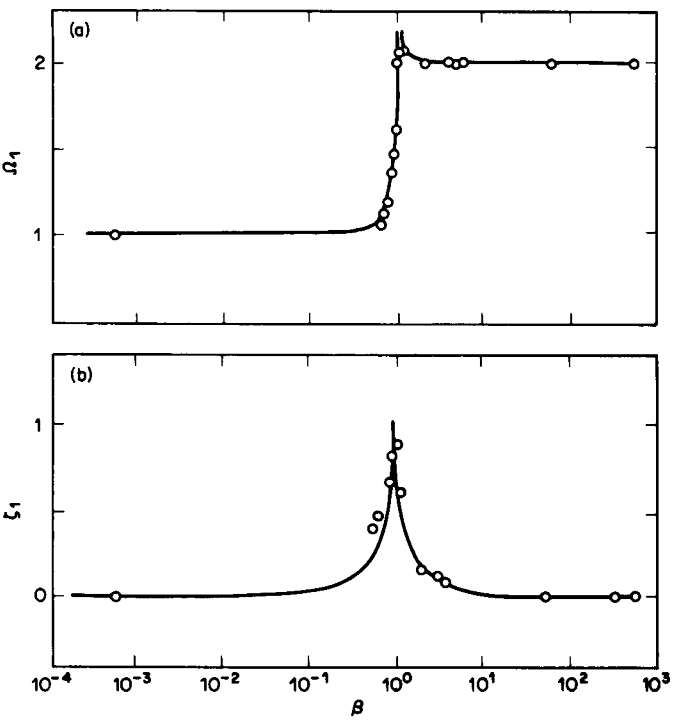

Figure 2. First mode modal parameters versus dimensionless damping coefficient. (a) Dimensionless undamped natural frequency; (b) damping ratio. ——, Continuous system results; ○, discrete system results.

different, the critical damping prediction of the continuous system is consistent with the infinite natural frequency seen at this point in Figure 2(a). Although not included here, plots of the eigenfunctions given by equation (4b) show that the normal modes lie within the complex domain.

In concluding, we should state that further work on this problem is in progress, and that other structural elements and systems are being investigated for complex eigensolutions.

Department of Mechanical Engineering,
The Ohio State University,
206 *West* 18*th Avenue,*
Columbus, Ohio 43210, *U.S.A.*

R. SINGH
W. M. LYONS

Department of Mechanical Engineering,
Speed Scientific School,
University of Louisville,
Louisville, Kentucky 40292, *U.S.A.*

G. PRATER, JR.

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