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FREQUENCY RESPONSE CHARACTERISTICS OF A MULTI-DEGREE-OF-FREEDOM SYSTEM WITH CLEARANCES

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In an earlier study, the frequency response characteristics of a single-degree-of-freedom system with a clearance non-linearity were studied. The current study, an extension of the earlier work, is concerned with the frequency response characteristics of a multi-degree-of-freedom system with clearances. The method of harmonic balance is used to develop approximate analytical solutions of the undamped equations of motion of a multi-degree-of-freedom system composed of three coupled non-linear oscillators. For primary resonances and a harmonic excitation, general formulations are presented which can be used to study both the existence and the stability of the solutions. These general formulations are used to discuss a number of modal spacing and modal coupling issues. An analysis methodology for multi-degree-of-freedom systems is introduced and illustrated by two special but practical cases: a strongly non-linear system and a weakly non-linear system. The results of the analysis of the special cases are validated by using analog simulation. It is shown that an analysis of the special cases of the general multi-degree-of-freedom non-linear formulation can provide not only an improved understanding of the dynamic behavior of non-linear systems, but also can be used as the basis for the development of simplified approximate solutions. The limitations of the current study and areas for further research are discussed.

1. INTRODUCTION

Clearances exist in many complex mechanical systems either by design, due to manufacturing errors and wear, or as the result of mechanical failures. Vibration of a translational or rotational system with clearances can result in relative motion across the clearance space and impacting between the components. Repeated impacts, referred to as vibro-impacts, may lead to excessive noise, large dynamic loads, and large changes in the dynamic stiffness.

Vibro-impact problems have been studied by a number of investigators [1–28], most of whom use a simple single-degree-of-freedom (SDOF) model in which the clearance is modeled as a piecewise linear function. A number of analytical methods are available for studying the dynamic behavior of mechanical systems with clearances; these include piecewise linear techniques which couple a series of linear solutions, associated with the piecewise linear function, by using appropriate conditions at impact [1, 2, 4, 9–13, 15, 16], digital simultaion [1–3, 15–18, 20, 22], analog simulation [17, 18], or an approximate analytical technique based on the method of harmonic balance [19, 21, 22, 24, 26–28]. The emphasis of most of the vibro-impact analyses [1–6, 8, 9, 11, 12, 17, 18, 20] has been

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on the determination of the actual time domain response for a harmonic excitation and these have been limited to a SDOF system.

In an effort to gain a more fundamental understanding of the frequency response characteristics of a SDOF system with a clearance non-linearity, Comparin and Singh [28] relaxed the requirement for predicting the actual time domain response and instead considered the response in terms of r.m.s. values. By using the method of harmonic balance approximate analytical solutions for primary resonance and a harmonic excitation were obtained in terms of the system parameters. These solutions resulted in an improved understanding of the frequency response characteristics of a SDOF system with a clearance non-linearity subjected to a harmonic excitation.

Many mechanical systems are, however, better represented as multi-degree-of-freedom (MDOF) systems and can in fact be viewed as a set of coupled non-linear oscillators [27]. For the MDOF system, it is necessary to consider not only the characteristics of a single oscillator but also the coupling between the oscillators. To gain a more fundamental understanding of the frequency response characteristics of MDOF systems with clearances, in this study the SDOF analysis of Comparin and Singh [28] is extended to the case of coupled non-linear oscillators.

2. PROBLEM FORMULATION FOR ANALYTICAL STUDY

A generic four-degree-of-freedom system is shown in Figure 1. The system is semi-definite with three non-zero natural frequencies. There are three non-linear stiffness elements in this model, $K_1 f_1(\delta_1)$, $K_2 f_2(\delta_2)$ and $K_3 f_3(\delta_3)$. K_1, K_2 and K_3 are constants and $f(\delta)$ is a generic non-linear element, referred to as a clearance type non-linearity, which is shown in Figure 2. Here α is a measure of the strength of the non-linearity. When α is close to one the system is weakly non-linear and when α is much greater or

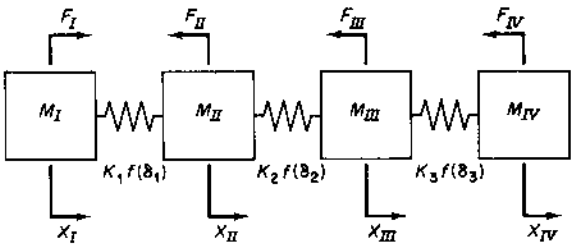

Figure 1. Generic multi-degree-of-freedom system.

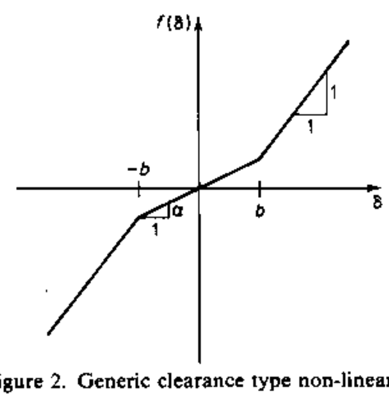

Figure 2. Generic clearance type non-linearity.

much less than one the system is strongly non-linear (a list of symbols is given in the Appendix). In the absence of damping, the equations of motion are

$$M_I\ddot{X}_I + K_1f_1(\delta_1) = F_I, \qquad M_{II}\ddot{X}_{II} - K_1f_1(\delta_1) + K_2f_2(\delta_2) = -F_{II}, \tag{1a, b}$$

$$M_{III}\ddot{X}_{III} - K_2f_2(\delta_2) + K_3f_3(\delta_3) = -F_{III}, \qquad M_{IV}\ddot{X}_{IV} - K_3f_3(\delta_3) = -F_{IV}, \tag{1c, d}$$

where

$$\delta_1 = X_I - X_{II}, \qquad \delta_2 = X_{II} - X_{III}, \qquad \delta_3 = X_{III} - X_{IV}, \tag{1e, f, g}$$

$$f_j(\delta_j) = \left\{\begin{matrix} \delta_j - (1-\alpha_j)b_j, & b_j < \delta_j \\ \alpha_j\delta_j, & -b_j \leq \delta_j \leq b_j \\ \delta_j + (1-\alpha_j)b_j, & \delta_j < -b_j \end{matrix}\right\}. \tag{1h}$$

The equations of motion for the semi-definite system can be simplified by rewriting equations (1) in terms of the relative displacement, δ_j, between the inertia elements. The simplified equations for the relative displacements are

$$M_1\ddot{\delta}_1 + K_1f_1(\delta_1) - K_2(M_1/M_{II})f_2(\delta_2) = F_{m1} + F_p(t), \tag{2a}$$

$$M_2\ddot{\delta}_2 + K_2f_2(\delta_2) - K_1(M_2/M_{II})f_1(\delta_1) - K_3(M_2/M_{III})f_3(\delta_3) = F_{m2}, \tag{2b}$$

$$M_3\ddot{\delta}_3 + K_3f_3(\delta_3) - K_2(M_3/M_{III})f_2(\delta_2) = F_{m3}, \tag{2c}$$

where:

$$M_1 = \frac{M_IM_{II}}{M_I + M_{II}}, \qquad M_2 = \frac{M_{II}M_{III}}{M_{II} + M_{III}}, \qquad M_3 = \frac{M_{III}M_{IV}}{M_{III} + M_{IV}}, \tag{2d–f}$$

$$F_{m1} = \frac{F_{mI}M_1}{M_I} + \frac{F_{mII}M_1}{M_{II}}, \qquad F_{m2} = -\frac{F_{mII}M_2}{M_{II}} + \frac{F_{mIII}M_2}{M_{III}}, \tag{2g, h}$$

$$F_{m3} = -\frac{F_{mIII}M_3}{M_{III_3}} + \frac{F_{mIV}M_2}{M_{IV}}, \qquad F_p = \frac{F_{pI}M_1}{M_I}. \tag{2i, j}$$

The non-linear functions $f_j(\delta_j)$, $j = 1, 2, 3$, are defined in terms of a stiffness break point b_j and a relative stiffness between the stages α_j. The equations of motion are non-dimensionalized as follows: length $\bar{\delta}_j = \delta_j/b$, $\bar{b}_j = b_j/b$, time $\bar{t} = t\omega_{11}$, where $\omega_{11}^2 = K_1/M_1$, force $\bar{F}_{mj} = F_{mj}/M_jb\omega_{11}^2$, $\bar{F}_p = F_p/M_1b\omega_{11}^2$ and frequency $\bar{\Omega} = \Omega/\omega_{11}$, $\bar{\Omega}_{ij} = K_{ij}/M_i\omega_{11}$, where K_{ij}, $i = 1, 2, 3$ and $j = 1, 2, 3$, are the coefficients of $f_j(\delta_j)$ in equation (2). The non-dimensional equations of motion become

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\bar{\delta}}_1 \\ \ddot{\bar{\delta}}_2 \\ \ddot{\bar{\delta}}_3 \end{Bmatrix} + \begin{bmatrix} \bar{\Omega}_{11}^2 & -\bar{\Omega}_{12}^2 & 0 \\ -\bar{\Omega}_{21}^2 & \bar{\Omega}_{22}^2 & -\bar{\Omega}_{23}^2 \\ 0 & -\bar{\Omega}_{32}^2 & \bar{\Omega}_{33}^2 \end{bmatrix}\begin{Bmatrix} f_1(\bar{\delta}_1) \\ f_2(\bar{\delta}_2) \\ f_3(\bar{\delta}_3) \end{Bmatrix} = \begin{Bmatrix} \bar{F}_{m1} + \bar{F}_p \\ \bar{F}_{m2} \\ \bar{F}_{m3} \end{Bmatrix}, \tag{3a}$$

$$f_j(\bar{\delta}_j) = \left\{\begin{matrix} \bar{\delta}_j - (1-\alpha_j)\bar{b}_j, & \bar{b}_j < \bar{\delta}_j \\ \alpha_j\bar{\delta}_j, & -\bar{b}_j \leq \bar{\delta}_j \leq \bar{b}_j \\ \bar{\delta}_j + (1-\alpha_j)\bar{b}_j, & \bar{\delta}_j < -\bar{b}_j \end{matrix}\right\}. \tag{3b}$$

The primary interest of this study is the vibro-impact behavior which is described by the relative motion $\bar{\delta}_j$ between the inertia elements. A study of the dynamic behavior of the system, therefore, reduces to a study of the solution of equations (3). Once the relative motion is known, the absolute motion can be easily obtained.

The method of harmonic balance will be used to obtain the solutions and their stability will be studied by using a perturbation technique. The analysis will be limited in scope to consider the following: (1) a non-linear stiffness defined by a two-stage symmetric clearance non-linearity with constant K_j and a stiffness ratio $0 \leq \alpha_j \leq 1$; (2) no damping; (3) an external excitation composed of a mean component and a single frequency alternating component; (4) steady state frequency response for primary resonances only; and (5) a stability analysis associated with only the first harmonic of the response (primary resonance).

3. STEADY STATE SOLUTION

3.1. GOVERNING EQUATIONS

The method of harmonic balance is now used to obtain an approximate analytical solution of the MDOF system (3). The solution is developed in the same manner as for the SDOF system described by Comparin and Singh [28]. The basic steps are repeated here for clarity. The excitation is assumed to have the form

$$\bar{F}_1 = \bar{F}_{m1} + \bar{F}_p \cos(\bar{\Omega}_p \bar{t} + \phi_{F_p}), \qquad \bar{F}_2 = \bar{F}_{m2} \qquad \text{and} \qquad \bar{F}_3 = \bar{F}_{m3}. \tag{4}$$

Here $\bar{F}_{mj}$ represents the mean transmitted force associated with each relative displacement and $\bar{F}_p$ is the amplitude of the vibratory component acting on the first inertia at frequency $\bar{\Omega}_p$ and relative phase angle ϕ_{F_p}. The approximate solution is assumed to have the form

$$\bar{\delta}_j = \bar{\delta}_{mj} + \bar{\delta}_{pj} \cos(\bar{\Omega}_p \bar{t} + \phi_{pj}), \qquad j = 1, 2, 3. \tag{5}$$

The constant term $\bar{\delta}_{mj}$ is used to account for the steady state offset, or bias, introduced by the mean load component $\bar{F}_{mj}$. The term $\bar{\delta}_{pj}$ is a positive constant and represents the amplitude of the forced response due to the alternating force $\bar{F}_p$. For the solution given by equation (5) it is assumed that the forced response of equations (3) is given by the first harmonic only and, therefore, not only the highest harmonics and the possibility of superharmonics or subharmonics are neglected but also the possibility of combination resonances and internal resonances.

The non-linearities $f_j(\bar{\delta}_j)$ are expanded in a Fourier series, retaining only the mean and first harmonic terms for each non-linearity. The non-linear functions given in terms of the describing functions N_{fmj} and N_{fpj} are

$$f_j(\bar{\delta}_j) = N_{fmj}\bar{\delta}_{mj} + N_{fpj}\bar{\delta}_{pj} \cos \varphi_{pj}, \qquad j = 1, 2, 3, \tag{6a}$$

where

$$N_{fmj}(\bar{\delta}_{mj}, \bar{\delta}_{pj}) = \frac{1}{\pi \bar{\delta}_{mj}} \int_0^{\pi} f_j(\bar{\delta}_j)\, \mathrm{d}\varphi_{pj}, \tag{6b}$$

$$N_{fpj}(\bar{\delta}_{mj}, \bar{\delta}_{pj}) = \frac{2}{\pi \bar{\delta}_{pj}} \int_0^{\pi} f_j(\bar{\delta}_j) \cos \varphi_{pj}\, \mathrm{d}\varphi_{pj}, \qquad \varphi_{pj} = \bar{\Omega}_p \bar{t} + \phi_{pj}. \tag{6c, d}$$

Substituting equations (4), (5) and (6) into equations (3) and equating coefficients of like harmonics yields a set of coupled non-linear algebraic equations. Solving these equations for $\bar{\delta}_{mj}$ and $\bar{\delta}_{pj}$ and redefining the mean components relative to the center of their respective non-linearities yields the following set of non-linear algebraic equations:

$$\bar{\delta}_{mj} = \frac{\bar{F}_{mj}}{N_{fmj}}, \quad j = 1, 2, 3, \qquad \bar{F}_{m1} = \frac{F_{mI}}{bK_1}, \qquad \bar{F}_{m2} = \frac{F_{mI} - F_{mII}}{bK_2}, \qquad \bar{F}_{m3} = \frac{F_{mIV}}{bK_3}, \tag{7a–d}$$

$$\bar{\delta}_{pl}=\frac{\pm(\bar{F}_p/\bar{\Omega}_{11}^2)[(N_{fp2}-\bar{\Omega}_p^2/\bar{\Omega}_{22}^2)(N_{fp3}-\bar{\Omega}_p^2/\bar{\Omega}_{33}^2)-(\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2/\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2)N_{fp2}N_{fp3}]}{\Lambda}, \quad (7e)$$

$$\bar{\delta}_{p2}=\frac{\pm(\bar{F}_p\bar{\Omega}_{21}^2/\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2)N_{fp1}(N_{fp3}-\bar{\Omega}_p^2/\bar{\Omega}_{33}^2)}{\Lambda}, \quad (7f)$$

$$\bar{\delta}_{p3}=\frac{\pm(\bar{F}_p\bar{\Omega}_{32}^2\bar{\Omega}_{21}^2/\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2)N_{fp1}N_{fp2}}{\Lambda}, \qquad \tan(\phi_{F_p}-\phi_{p1})=\frac{0}{\bar{\delta}_{p1}\Lambda}, \quad (7g, h)$$

$$\Lambda=\left\{\left[\left(N_{fp1}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left(N_{fp2}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)-\frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}N_{fp1}N_{fp2}\right]\left(N_{fp3}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)\right.$$
$$\left.-\left(N_{fp1}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}N_{fp2}N_{fp3}\right\}. \quad (7i)$$

Here a positive sign is used when $\bar{\delta}_{p1}$ and $\bar{F}_p$ are in phase and a negative sign when they are out of phase where the relative phase, $(\phi_{F_p}-\phi_{p1})$, is defined by equation (7h). The describing functions N_{fmj} and N_{fpj} are given by

$$N_{fmj}=1+\frac{\bar{\delta}_{pj}}{2\bar{\delta}_{mj}}\left\{(1-\alpha_j)\left[G\left(\frac{\bar{b}_j-\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)-G\left(\frac{-\bar{b}_j-\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)\right]\right\}, \quad (8a)$$

$$N_{fpj}=1-\left(\frac{1-\alpha_j}{2}\right)\left[H\left(\frac{\bar{b}_j-\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)-H\left(\frac{-\bar{b}_j-\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)\right], \quad (8b)$$

$$G(\mu)=\begin{cases}(2/\pi)(\mu\sin^{-1}\mu+\sqrt{1-\mu^2}), & |\mu|\leqslant 1\\ |\mu|, & |\mu|>1\end{cases}, \quad (8c)$$

$$H(\mu)=\begin{cases}-1, & \mu<-1\\ (2/\pi)(\sin^{-1}\mu+\mu\sqrt{1-\mu^2}), & |\mu|\leqslant 1\\ +1, & \mu>1\end{cases}, \quad (8d)$$

$$\mu=(\pm\bar{b}_j-\bar{\delta}_{mj})/\bar{\delta}_{pj}. \quad (8e)$$

To facilitate the development of the solutions in terms of the design variables, the functions G and H are replaced with truncated series expansions. The truncated series expressions are obtained by retaining only the first two terms in each series and adjusting the coefficient of the second term to yield the actual value of the series when the argument $\mu=1$. This modification is necessary because when the argument is close to 1, the contribution from the higher terms is significant. For small values of μ, the contribution from the higher terms is small and the modification is not required as these terms have little or no effect on the functions G and H. The truncated series expressions which are within 5% of the original G and within 6% for H are found to be as follows for $|\mu|\leqslant 1$:

$$G(\mu)\cong\frac{2}{\pi}\left\{1+\left(\frac{\pi-2}{2}\right)\mu^2\right\}, \qquad H(\mu)\cong\frac{4}{\pi}\left\{\mu-\left(\frac{4-\pi}{4}\right)\mu^3\right\}. \quad (9a, b)$$

In all cases, when $\alpha_j=1$ the describing functions N_{fmj} and N_{fpj}, given by equations (8a) and (8b), reduce to one, which is the linear case.

The MDOF system can be viewed as essentially consisting of three coupled nonlinear oscillators; the I–II oscillator, the II–III oscillator and the III–IV oscillator. To understand better the type of behavior which may be expected, consider the physical effect of the non-linear stiffnesses on one of the oscillators. The overall response can be separated into three different regimes: a no-impact regime, a single-sided impact regime, and a two-sided impact regime. The different impact regimes are illustrated for $\alpha=0$ in Figure 3.

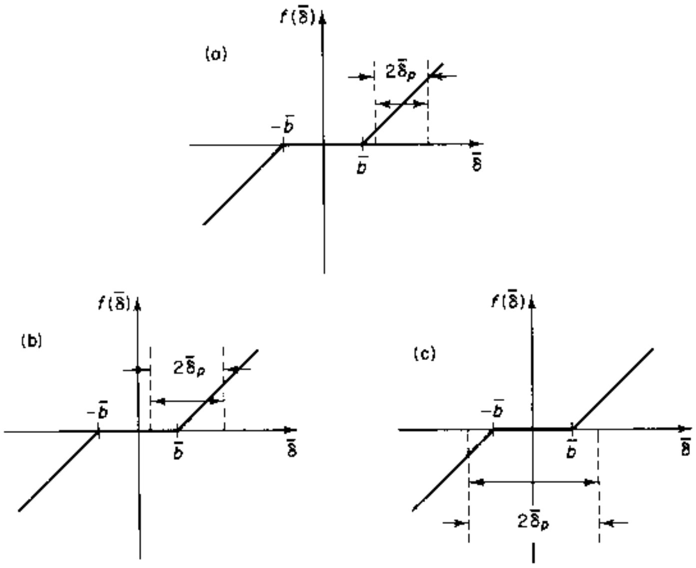

Figure 3. Illustration of different impact regimes: (a) no impacts; (b) single-sided impacts; (c) two-sided impacts.

The non-linearity can be viewed as an amplitude dependent stiffness with an average value given by the relative amount of time (over one period of vibration) that the oscillator is in one stage versus the other. As the amplitude of the alternating component $\bar{\delta}_p$ changes so does the stiffness. The oscillator is said to be hardening if the stiffness increases for increasing $\bar{\delta}_p$ and softening if the stiffness decreases for increasing $\bar{\delta}_p$. For the clearance type non-linearity, the hardening or softening character will depend on the location of the mean component and on whether the oscillator is undergoing single- or two-sided impacts (the system is linear for the no-impact case). For single-sided impacts if the mean component (static deflection) is in the second stage, as $\bar{\delta}_p$ increases the average stiffness decreases because the amount of time the system spends in the first stage increases and the system is softening. For the two-sided impact case and the single-sided case where the mean component is in the first stage the situation is reversed and the average stiffness will increase as $\bar{\delta}_p$ increases, resulting in a hardening system. Each oscillator may undergo any or all of the different impact regimes, so the MDOF system may have a total of $3\times3\times3=27$ cases. Further complicating the situation is the fact that the oscillators are dynamically coupled; therefore, the behavior of one will, in general, depend on the type of vibration experienced by the other two. With these factors in mind, the general behavior for the different impact regimes of the MDOF system are discussed in the following sections.

3.2. NO-IMPACT CASE

For a given oscillator, impacts will not occur if one of the following conditions is met:

$$\text{type 1:}\quad \bar{\delta}_{mj}+\bar{\delta}_{pj}<\bar{b}_j \quad\text{and}\quad \bar{\delta}_{mj}-\bar{\delta}_{pj}>-\bar{b}_j; \tag{10a}$$

$$\text{type 2:}\quad \bar{\delta}_{mj}+\bar{\delta}_{pj}>\bar{b}_j \quad\text{and}\quad \bar{\delta}_{mj}-\bar{\delta}_{pj}>\bar{b}_j. \tag{10b}$$

The describing functions reduce to

$$\text{type 1:}\quad N_{fmj}=\alpha_j \quad\text{and}\quad N_{fpj}=\alpha_j; \tag{11a}$$

$$\text{type 2:}\quad N_{fmj}=1-(1-\alpha_j)/\bar{\delta}_{mj} \quad\text{and}\quad N_{fpj}=1; \tag{11b}$$

these are of the same form as for a SDOF oscillator [28]. For illustration purposes, consider the I-II oscillator ($j=1$). Substituting equations (11) into equations (7) yield, for type 1,

$$\bar{\delta}_{m1}=\frac{\bar{F}_{m1}}{\alpha_1},\qquad \bar{\delta}_{p1}=\pm\frac{\bar{F}_p}{\bar{\Omega}_{11}^2}\left[\left(N_{fp2}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)\left(N_{fp3}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}N_{fp2}N_{fp3}\right]\Big/\Lambda, \quad \text{(12a, b)}$$

$$\Lambda=\left\{\left[\left(\alpha_1-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left(N_{fp2}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)-\frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\alpha_1 N_{fp2}\right]\left(N_{fp3}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)\right.$$
$$\left.-\left(\alpha_1-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}N_{fp2}N_{fp3}\right\}, \quad \text{(12c)}$$

and for type 2,

$$\bar{\delta}_{m1}=\bar{F}_{m1}+(1-\alpha_1)\bar{b}_1, \quad \text{(12d)}$$

$$\bar{\delta}_{p1}=\pm\frac{\bar{F}_p}{\bar{\Omega}_{11}^2}\left[\left(N_{fp2}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)\left(N_{fp3}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}N_{fp2}N_{fp3}\right]\Big/\Lambda, \quad \text{(12e)}$$

$$\Lambda=\left\{\left[\left(1-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left(N_{fp2}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)-\frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}N_{fp2}\right]\left(N_{fp3}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)\right.$$
$$\left.-\left(1-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}N_{fp2}N_{fp3}\right\}. \quad \text{(12f)}$$

The mean component of the displacement $\bar{\delta}_{m1}$ is constant and is the same as would be expected for a linear system. The alternating component $\bar{\delta}_{p1}$, however, is not necessarily linear for the no-impact case and will be linear only if the other two oscillators are also in a no-impact regime. Equation (12) is single valued in $\bar{\delta}_{m1}$ and $\bar{\delta}_{p1}$ and solutions will always exist.

The transition frequency from no impacts is calculated by using equations (12) when $\bar{\delta}_{p1}=|\bar{b}_1-\bar{\delta}_{m1}|$. The frequency equation has the general form

$$a_3\bar{\Omega}_p^6+a_2\bar{\Omega}_p^4+a_1\bar{\Omega}_p^2+a_0=0. \quad \text{(13)}$$

The coefficients will depend on the motion of the other two oscillators (the values of N_{fp2} and N_{fp3}). Because $\bar{F}_p$ can be positive or negative (in phase or out of phase), equation (13) will have a total of 12 roots. There can be as many as six positive real roots with one above and one below each resonant peak in equation (12). The transition frequencies are shown in Figure 4 for an idealized frequency response function.

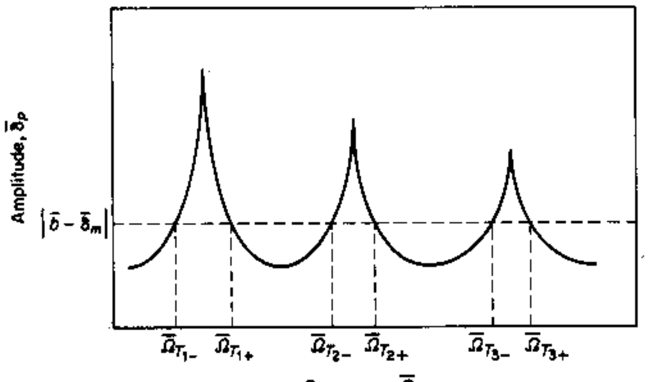

Figure 4. Transition frequencies between impact regimes.

Although the preceding discussion was given in terms of the I–II oscillator ($j=1$) the behavior described also applies to the other two oscillators but the equations themselves will be different.

3.3. IMPACT CASE: SINGLE- AND DOUBLE-SIDED

Impacts will occur for a given oscillator whenever the displacement repeatedly exceeds the stiffness transitions at one or both sides. For single-sided impacts,

$$\text{type 1: } \bar{\delta}_{mj}+\bar{\delta}_{pj}>\bar{b}_j \quad \text{and} \quad \bar{\delta}_{mj}-\bar{\delta}_{pj}>-\bar{b}_j; \tag{14a}$$

$$\text{type 2: } \bar{\delta}_{mj}-\bar{\delta}_{pj}>-\bar{b}_j \quad \text{and} \quad \bar{\delta}_{mj}-\bar{\delta}_{pj}<\bar{b}_j; \tag{14b}$$

and for two-sided impacts,

$$\bar{\delta}_{pj}>|\bar{b}_j-\bar{\delta}_{mj}| \qquad \text{and} \qquad \bar{\delta}_{pj}>|\bar{b}_j+\bar{\delta}_{mj}|. \tag{14c}$$

The describing functions take the same form as for the SDOF oscillator [28] and are as follows: for single-sided impacts,

$$N_{fmj}=1+\frac{\bar{\delta}_{pj}(1-\alpha_j)}{2\bar{\delta}_{mj}}\left[\frac{2}{\pi}\left(1+\left(\frac{\pi-2}{2}\right)\left(\frac{\bar{b}_j-\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)^2\right)-\left(\frac{\bar{b}_j+\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)\right], \tag{15a}$$

$$N_{fpj}=\frac{(1+\alpha_j)}{2}+\frac{2}{\pi\bar{\delta}_{pj}}(\bar{b}_j-\bar{\delta}_{mj})(1-\alpha_j)\left[1-\left(\frac{4-\pi}{4}\right)\left(\frac{\bar{b}_j-\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)^2\right], \tag{15b}$$

and for two-sided impacts,

$$N_{fmj}=1-\frac{2(\pi-2)}{\pi}(1-\alpha_j)\left(\frac{\bar{b}_j}{\bar{\delta}_{pj}}\right), \tag{15c}$$

$$N_{fpj}=1-\frac{4}{\pi}(1-\alpha_j)\left(\frac{\bar{b}_j}{\bar{\delta}_{pj}}\right)\left[1-\left(\frac{4-\pi}{4}\right)\left(\frac{\bar{b}_j^2+3\bar{\delta}_{mj}^2}{\bar{\delta}_{pj}^2}\right)\right]. \tag{15d}$$

The equation for the mean component $\bar{\delta}_{mj}$ is obtained by substituting equation (15a) into equation (7a) and is, for single-sided impacts,

$$\bar{\delta}_{mj}=\left\{\bar{F}_{mj}+\frac{(1-\alpha_j)}{2}\left(\bar{b}_j-\frac{2\bar{\delta}_{pj}}{\pi}-\left(\frac{\pi-2}{\pi\bar{\delta}_{pj}}\right)(\bar{b}_j-\bar{\delta}_{mj})^2\right)\right\}\Big/\alpha_{Aj}, \qquad \alpha_{Aj}=\frac{1+\alpha_j}{2}, \tag{16a, b}$$

and for two-sided impacts,

$$\bar{\delta}_{mj}=\frac{\bar{F}_{mj}}{1-\bar{F}_{tsmj}/\bar{\delta}_{pj}}, \qquad \bar{F}_{tsmj}=\frac{2(\pi-2)}{\pi}(1-\alpha_j)\bar{b}_j. \tag{16c, d}$$

Comparing equation (16) with the corresponding equations for the SDOF oscillator [28] shows that the behavior of the mean component of an oscillator in a MDOF system is the same as it is for the SDOF system.

The equation for the alternating component $\bar{\delta}_{pj}$ is obtained by substituting equation (15b) into equation (7) and is

$$\bar{\delta}_{pj}=(\pm\bar{F}'_{pj}+\bar{F}'_{isj})/\Lambda_j, \qquad i=t,s, \quad j=1,2,3, \tag{17a}$$

where

$$\bar{F}'_{p1}=\frac{\bar{F}_p}{\bar{\Omega}_{11}^2}\left[\left(N_{fp2}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)\left(N_{fp3}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}N_{fp2}N_{fp3}\right], \tag{17b}$$

$$\bar{F}'_{is1}=\bar{F}_{is1}\left[\left(\left(1-\frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\right)N_{fp2}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)-\left(N_{fp3}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}N_{fp2}N_{fp3}\right], \tag{17c}$$

$$\bar{F}'_{p2}=\frac{\bar{F}_p\bar{\Omega}^2_{21}}{\bar{\Omega}^2_{11}\bar{\Omega}^2_{22}}N_{fp1}\left(N_{fp3}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{33}}\right), \tag{17d}$$

$$\bar{F}'_{is2}=\bar{F}_{is2}\left[\left(\left(1-\frac{\bar{\Omega}^2_{12}\bar{\Omega}^2_{21}}{\bar{\Omega}^2_{11}\bar{\Omega}^2_{22}}\right)N_{fp1}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{11}}\right)\left(N_{fp3}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{33}}\right)-\frac{\bar{\Omega}^2_{23}\bar{\Omega}^2_{32}}{\bar{\Omega}^2_{22}\bar{\Omega}^2_{33}}N_{fp3}\left(N_{fp1}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{11}}\right)\right], \tag{17e}$$

$$\bar{F}'_{p3}=\frac{\bar{F}_p\bar{\Omega}^2_{32}\bar{\Omega}^2_{21}}{\bar{\Omega}^2_{11}\bar{\Omega}^2_{22}\bar{\Omega}^2_{33}}N_{fp1}N_{fp2}, \tag{17f}$$

$$\bar{F}'_{is3}=\bar{F}_{is3}\left[\left(N_{fp1}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{11}}\right)\left(N_{fp2}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{22}}\right)-\frac{\bar{\Omega}^2_{12}\bar{\Omega}^2_{21}}{\bar{\Omega}^2_{11}\bar{\Omega}^2_{22}}N_{fp1}N_{fp2}-\frac{\bar{\Omega}^2_{23}\bar{\Omega}^2_{32}}{\bar{\Omega}^2_{22}\bar{\Omega}^2_{33}}N_{fp2}\left(N_{fp1}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{11}}\right)\right], \tag{17g}$$

$$\Lambda=\left\{\left[\left(N_{fp1}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{11}}\right)\left(N_{fp2}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{22}}\right)-\frac{\bar{\Omega}^2_{12}\bar{\Omega}^2_{21}}{\bar{\Omega}^2_{11}\bar{\Omega}^2_{22}}N_{fp1}N_{fp2}\right]\left(N_{fp3}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{33}}\right)-\frac{\bar{\Omega}^2_{23}\bar{\Omega}^2_{32}}{\bar{\Omega}^2_{22}\bar{\Omega}^2_{33}}N_{fp2}N_{fp3}\left(N_{fp1}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{11}}\right)\right\}, \tag{17h}$$

$$\bar{F}_{ssj}=\frac{2}{\pi}(\bar{b}_j-\bar{\delta}_{mj})(1-\alpha_j)\left[1-\left(\frac{4-\pi}{\pi}\right)\left(\frac{\bar{b}_j-\bar{\delta}_{mj}}{\bar{\delta}_{pj}}\right)^2\right], \tag{17i}$$

$$\bar{F}_{tsj}=\frac{4}{\pi}(1-\alpha_j)\bar{b}_j\left[1-\left(\frac{4-\pi}{4}\right)\left(\frac{\bar{b}^2_j+3\bar{\delta}^2_{mj}}{\bar{\delta}^2_{pj}}\right)\right]. \tag{17j}$$

Here Λ_j in equation (17a) is given by equation (17h) when the appropriate describing function N_{fpj} is replaced by one for two-sided impacts or by α_A for single-sided impacts.

Equation (17a) is similar to the equation for the undamped response of the SDOF oscillator [28] and the existence of solutions is governed by the relative magnitude of $\bar{F}'_{isj}$ and $\bar{F}'_{pj}$. Specifically, when $|\bar{F}'_{pj}|\geq|\bar{F}'_{isj}|$ equation (17a) is single-valued and solutions will always exist. If $|\bar{F}'_{pj}|<|\bar{F}'_{isj}|$, however, the question of the existence of solutions becomes more complicated. For a hardening oscillator, solutions exist when the displacement is in phase with the force and do not exist when the two are out of phase (the opposite occurs for a softening oscillator). For the MDOF system a phase change occurs whenever $\bar{\Omega}_p$ passes through a resonance (given by the zeros of equation (17h)) and will manifest itself as a change in sign of the denominator of equation (17a). Hence, there will be two solutions when the denominator of equation (17a) and $\bar{F}'_{isj}$ have the same sign and no solutions when the signs are different.

To understand better the type of frequency response behavior which might be expected, consider the following special cases.

If the off diagonal terms in equation (3a) are small relative to the diagonal terms the coupling between the modes will be small and each will respond like a single-degree-of-freedom system. For example, examine equations (17c) and (17h) for two-sided impacts and negligible coupling terms:

$$\bar{F}'_{is1}=\bar{F}_{is1}\left[\left(N_{fp2}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{22}}\right)\left(N_{fp3}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{33}}\right)\right], \tag{18a}$$

$$\Lambda=\left(1-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{11}}\right)\left(N_{fp2}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{22}}\right)\left(N_{fp3}-\frac{\bar{\Omega}^2_p}{\bar{\Omega}^2_{33}}\right). \tag{18b}$$

The only time that Λ and $\bar{F}'_{is1}$ will have the same sign is when, for a hardening system $\bar{\Omega}^2_p<1$ and for a softening system $\bar{\Omega}^2_p>1$. Therefore, for this case a jump transition will only occur at $\bar{\Omega}^2_p=1$ and the oscillator in the MDOF system behaves like the SDOF oscillator with respect to the existence of solutions. Similar behavior is also exhibited by the other two oscillators ($j = 2, 3$) and is shown graphically in Figure 5 for an idealized hardening system.

Now consider a system where the coupling terms are small but finite and the resonances are widely spaced. From the bracketed terms in equations (17b) and (17c), $\bar{F}'_{is1}$ is initially positive (or negative) and undergoes a sign change near the zeros of $\bar{F}'_{p1}$. The zeros of $\bar{F}'_{p1}$ will occur between the zeros of Λ (resonances of the frequency response function) and so for a hardening (softening) system $\bar{F}'_{is1}$ and Λ will have the same sign just below (above) each resonance and a different sign just above (below) each one. Therefore, for the first oscillator, a jump transition may occur at each of the resonant peaks in the frequency response function. For $\bar{F}'_{is2}$, the first zero of the bracketed term occurs at the first resonant frequency, and the second zero occurs at the anti-resonance between the second and third resonances. The signs for $\bar{F}'_{is2}$ and Λ will be the same on either side of the first resonance but different on either side of the second two. Hence for this case a jump transition can only occur at the second two resonances. Finally, for $\bar{F}'_{is3}$ the zeros will occur at the first two resonant frequencies and hence a jump transition can occur only at the third. This case is illustrated for a hardening system in Figure 6. For the previous cases it has been assumed that $\bar{F}'_{isj}$ is either hardening or softening. For a type 2 system undergoing single-sided impacts, $\bar{F}'_{ssj}$ can be both hardening and softening and so a transition can occur near any of the single-sided impact resonant frequencies.

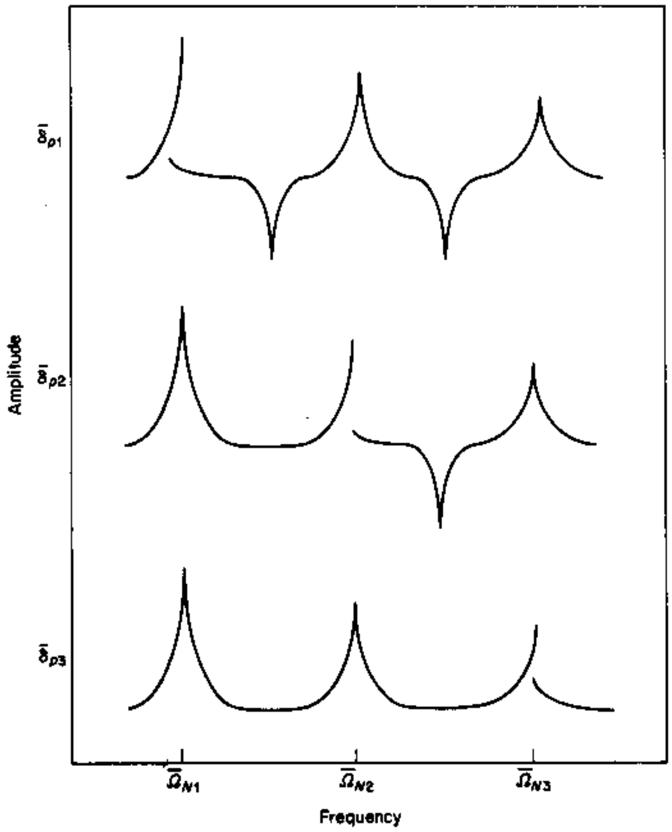

Figure 5. Existence of impact solutions for an idealized MDOF hardening system undergoing two-sided impacts with negligible coupling terms.

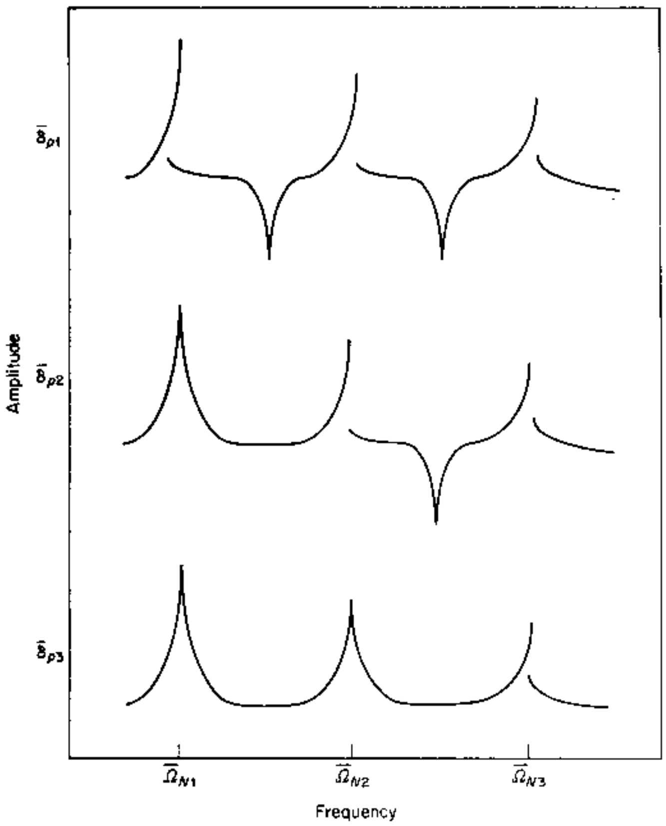

Figure 6. Existence of impact solutions for an idealized MDOF hardening system undergoing two-sided impacts with finite coupling terms and widely spaced modes.

The most general case is when the coupling terms are large and the modes closely spaced. Analytically it is no longer possible to predict, *a priori*, where the jumps will occur without considering a specific set of numerical parameters.

For the impact case, two types of transitions exist: one associated with a decrease in $\bar{\delta}_{pj}$ as $\bar{\Omega}_p$ moves away from a resonance and one associated with the jump phenomenon. The frequency at which the first type of transition occurs is calculated by using equation (17) when $\bar{\delta}_{pj} = |\bar{\delta}_{mj}| + \bar{b}_j$ and again the characteristic frequency equation has the following general form:

$$a_3\bar{\Omega}_p^6 + a_2\bar{\Omega}_p^4 + a_1\bar{\Omega}_p^2 + a_0 = 0. \tag{19}$$

Like the no-impact case, the coefficients will depend on the motion of the other two impact pairs and equation (19) can again have a maximum of six positive roots, one above and one below each resonant peak (see Figure 4). The jump transitions occur at the resonant frequencies which are found by solving for the zeros of equation (17h).

4. STABILITY OF STEADY STATE SOLUTIONS

The local stability of the solutions discussed in the previous section is determined by perturbing the steady state solution and studying the resulting motion. If the perturbed oscillating motion decays with time the solution is considered dynamically stable and if it grows it is dynamically unstable. It should be pointed out that in this analysis only the first harmonic of the response is considered, and hence it is a local stability analysis. A study of the global stability of the system requires a consideration of more complicated types of resonances, which is beyond the scope of this study.

The solutions of the governing equations are perturbed such that:

$$\bar{\delta}'_{mj} = \bar{\delta}_{mj} + \Delta\bar{\delta}_{mj}, \qquad \bar{\delta}'_{pj} = \bar{\delta}_{pj} + \Delta\bar{\delta}_{pj}, \qquad \phi'_{pj} = \phi_{pj} + \Delta\phi_{pj}. \tag{20a-c}$$

The perturbed solution is substituted into the original equations and expanded for small perturbations by using a Taylor series with only the linear terms in the perturbed variables retained. This results in a set of nine equations in the nine perturbed variables. Noting that for zero damping, $\sin(\phi_{pi} - \phi_{pj}) = 0$, the simplified equations are manipulated to reduce the number of coupling terms yielding the following equations in matrix form:

$$\begin{bmatrix} A_{11} & 0 & 0 & A_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & A_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & A_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{54} & A_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{65} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \Delta\bar{\delta}_{m1} \\ \Delta\bar{\delta}_{m2} \\ \Delta\bar{\delta}_{m3} \\ \Delta\bar{\delta}_{p1} \\ \Delta\bar{\delta}_{p2} \\ \Delta\bar{\delta}_{p3} \\ \Delta\bar{\phi}_{p1} \\ \Delta\bar{\phi}_{p2} \\ \Delta\bar{\phi}_{p3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{21a}$$

Here

$$A_{jj} = -\left(N_{fmj} + \frac{\partial N_{fmj}}{\partial\bar{\delta}_{mj}}\bar{\delta}_{mj}\right), \qquad A_{j(j+3)} = -\left(\frac{\partial N_{fmj}}{\partial\bar{\delta}_{pj}}\bar{\delta}_{mj}\right), \tag{21b, c}$$

$$A_{44} = -\left(N_{fp1} - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left\{\left[\left(\Xi_1 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left(\Xi_2 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right) - \frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\Xi_1\Xi_2\right]\left(\Xi_3 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)\right.$$
$$\left. - \frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\Xi_2\Xi_3\left(\Xi_1 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\right\}\cos(\phi_{F_p} - \phi_{p1}), \tag{21d}$$

$$A_{54} = \frac{\bar{\Omega}_{21}^2}{\bar{\Omega}_{22}^2}\Xi_1\left(\Xi_3 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)\cos(\phi_{p1} - \phi_{p2}), \tag{21e}$$

$$A_{55} = -\left(N_{fp2} - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)\left[\left(\Xi_2 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right)\left(\Xi_3 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right) - \frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\Xi_2\Xi_3\right]\cos(\phi_{F_p} - \phi_{p1}), \tag{21f}$$

$$A_{65} = \frac{\bar{\Omega}_{32}^2}{\bar{\Omega}_{33}^2}\Xi_2\cos(\phi_{p2} - \phi_{p3}), \tag{21g}$$

$$A_{66} = -\left(N_{fp3} - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)\left(\Xi_3 - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{33}^2}\right)\cos(\phi_{F_p} - \phi_{p1}), \tag{21h}$$

$$A_{77} = -\frac{\bar{F}_p}{\bar{\Omega}_{11}^2}, \qquad \Xi_j = N_{fpj} + \frac{\partial N_{fpj}}{\partial\bar{\delta}_{pj}}\bar{\delta}_{pj}. \tag{21i, j}$$

The solutions for the perturbations have the general form

$$\Delta_j(t) = \Delta_{j0}\,\mathrm{e}^{\lambda t}, \tag{22}$$

where $j = 1, 2, 3$. Here $\Delta_j(t)$ represents the perturbed variable ($\Delta\bar{\delta}_{mj}$, $\Delta\bar{\delta}_{pj}$, or $\Delta\phi_{pj}$), Δ_{j0} represents the magnitude of the initial perturbation at $t = 0$, and λ represents the eigenvalues of equation (21).

The stability of the solutions, therefore, depends on the eigenvalues λ of equation (21a). For this case the characteristic equation involves only the diagonal terms and is

$$(A_{11}-\lambda)(A_{22}-\lambda)(A_{33}-\lambda)(A_{44}-\lambda)(A_{55}-\lambda)(A_{66}-\lambda)(A_{77}-\lambda)(-1-\lambda)(-1-\lambda)=0. \tag{23}$$

Since A_{11}, A_{22}, A_{33}, and A_{77} are always negative and Ξ_j is always positive, the stability is governed by the sign of A_{44}, A_{55}, and A_{66}. An examination of the terms in equation (21) reveals that the stability is determined by considering only the terms like $N_{fpj}-(\bar{\Omega}_p^2/\bar{\Omega}_{jj}^2)$ as the net sign on the remaining terms is always positive when the appropriate phase changes are included. Accordingly, the requirement for stability is

$$\bar{\delta}_{pj} > \frac{\bar{F}_{isj}}{(\alpha_{sj}-\bar{\Omega}_p^2/\bar{\Omega}_{jj}^2)}, \tag{24a}$$

where

$$\bar{F}_{sj}=\begin{Bmatrix} 0 & \text{no impacts} \\ \bar{F}_{ssj} & \text{single-sided impacts} \\ \bar{F}_{tsj} & \text{two-sided impacts} \end{Bmatrix}, \qquad \alpha_{sj}=\begin{Bmatrix} \alpha_j & \text{no impacts (type 1)} \\ 1 & \text{no impacts (type 2)} \\ \alpha_{Aj} & \text{single-sided impacts} \\ 1 & \text{two-sided impacts} \end{Bmatrix}. \tag{24b, c}$$

For the no-impact case solutions are always stable as would be expected. For the impact cases (single- and two-sided), equation (24a) is the same as equations (33b) and (35) of reference [28], with zero damping and, therefore, the condition for stability in the MDOF takes the same form as for the SDOF.

5. ANALYSIS OF TWO SPECIAL CASES

5.1. THE PHYSICAL MODEL

Since the MDOF non-linear system can exhibit a wide range of dynamic behavior, general observations and conclusions usually cannot be made without considering special cases. Further, a general non-linear analysis can be very complicated and expensive so it is useful to know when a non-linear analysis is actually required and when a linear analysis might be acceptable. Although the specific results from analyses of the special cases considered here cannot be directly applied to a wider class of problems, the general methodology introduced should be applicable. These points are illustrated by examining two special but practical cases. Specifically, consider a non-linear oscillator connected to a linear MDOF system, where the resonances associated witht the linear part of the system are well separated from the resonance regime associated with the non-linear oscillator. The two example cases considered here are described below and the relevant numerical values are given in Table 1.

(a) Case 1: the resonance associated with the non-linear oscillator is assumed to be the lowest resonance ($\bar{\Omega}_{N1}$) in the system and the linear resonances ($\bar{\Omega}_{N2}$ and $\bar{\Omega}_{N3}$) are assumed to be much higher. The alternating load F_{pl} is applied to the non-linear oscillator and the frequency response of the system is found for an operating frequency range around the first resonance. The system is shown schematically in Figure 7 along with a typical frequency response function.

(b) Case 2: the resonance associated with the non-linear oscillator is assumed to be the highest resonance ($\bar{\Omega}_{N3}$) in the system and hence the linear resonances ($\bar{\Omega}_{N1}$ and $\bar{\Omega}_{N2}$) are assumed to be much lower. The alternating load F_{pl} is applied to the linear part of the system and the frequency response of the system is found for an operating

TABLE 1

Numerical data sets chosen for the two special cases

Parameter	Strongly non-linear case 1	Weakly non-linear case 2
$\bar{\Omega}_{11}^2$	1·0	1·0
$\bar{\Omega}_{12}^2$	3·6	0·57
$\bar{\Omega}_{21}^2$	0·57	0·57
$\bar{\Omega}_{22}^2$	9·0	1·43
$\bar{\Omega}_{23}^2$	6·0	3·30
$\bar{\Omega}_{32}^2$	5·4	0·855
$\bar{\Omega}_{33}^2$	10·0	10·0
α_1	0	1
α_2	1	1
α_3	1	0
$\bar{b}_1$	1·0	N/A
$\bar{b}_2$	N/A	N/A
$\bar{b}_3$	N/A	0·10
$\bar{F}_{m1}$	0·25	0·05
$\bar{F}_{m2}$	0·04	0·05
$\bar{F}_{m3}$	0·036	0·0125
$\bar{F}_p$ (lower)	0·0625	0·050
$\bar{F}_p$ (higher)	0·09375	0·075

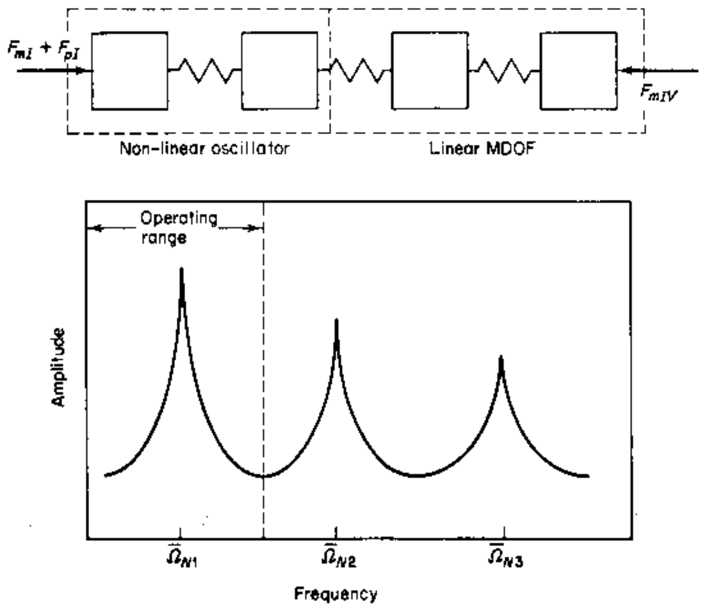

Figure 7. Special case 1: (a) physical model; (b) typical driving point frequency response function.

frequency range around the linear resonances. This system is shown schematically in Figure 8, along with a typical frequency response function.

From a qualitative point of view, case 1 is strongly non-linear over the operating range because the dynamic behavior of the overall system is dominated by the behavior of the non-linear oscillator. Conversely, case 2 is weakly non-linear because the behavior of the system is dominated by the linear system response in the operating range.

5.2. CASE 1: STRONGLY NON-LINEAR SYSTEM

Typical frequency response functions for the alternating components $\bar{\delta}_{p1}$, $\bar{\delta}_{p2}$, and $\bar{\delta}_{p3}$ are given in Figures 9 and 10. A typical comparison of the harmonic balance results and analog simulation results is shown in Figure 11. A comparison of the frequency ranges of the different impact regimes is given in Table 2. To facilitate a comparison with the single impact pair system all frequencies have been scaled so that the linear resonant frequency occurs a 1·0 instead of 0·80.

From an analysis of Figures 9 and 10, several conclusions can be drawn. First, by comparing the frequency response functions for the strongly non-linear case (case 1) to those given by Comparin and Singh [28] it is clear that the non-linear oscillator in the MDOF system behaves like the SDOF case except for a frequency shift associated with the increased inertia of the coupled system. Second, over the frequency range of interest, the shape of the frequency response functions for $\bar{\delta}_{p2}$ and $\bar{\delta}_{p3}$ are similar to $\bar{\delta}_{p1}$ except that the amplitudes are smaller. Finally, the amplitude ratios $\bar{\delta}_{p2}/\bar{\delta}_{p1}$ and $\bar{\delta}_{p3}/\bar{\delta}_{p1}$ are dependent on the motion of $\bar{\delta}_{p1}$ as they differ from the linear case ($\alpha_1 = 1$) whenever impacts occur.

Additional insight into the problem can be gained by simplifying the solutions given by equations (7). For case 1 it is assumed that $\bar{\Omega}_{22} \gg \bar{\Omega}_p$ and $\bar{\Omega}_{33} \gg \bar{\Omega}_p$; therefore, equations (7) can be simplified to yield

$$\bar{\delta}_{p1} = \pm\left\{\frac{\bar{F}_p}{\bar{\Omega}_{11}^2}\right\}\Bigg/\left\{\left(1 - \frac{\bar{\Omega}_{33}^2\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2(\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2 - \bar{\Omega}_{23}^2\bar{\Omega}_{32}^2)}\right)N_{fp1} - \frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right\}, \tag{25a}$$

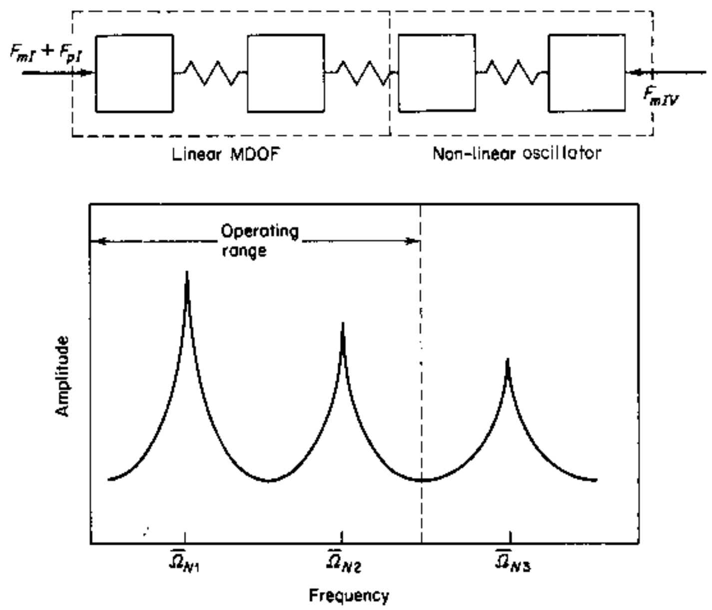

Figure 8. Special case 2: (a) physical model; (b) typical driving point frequency response function.

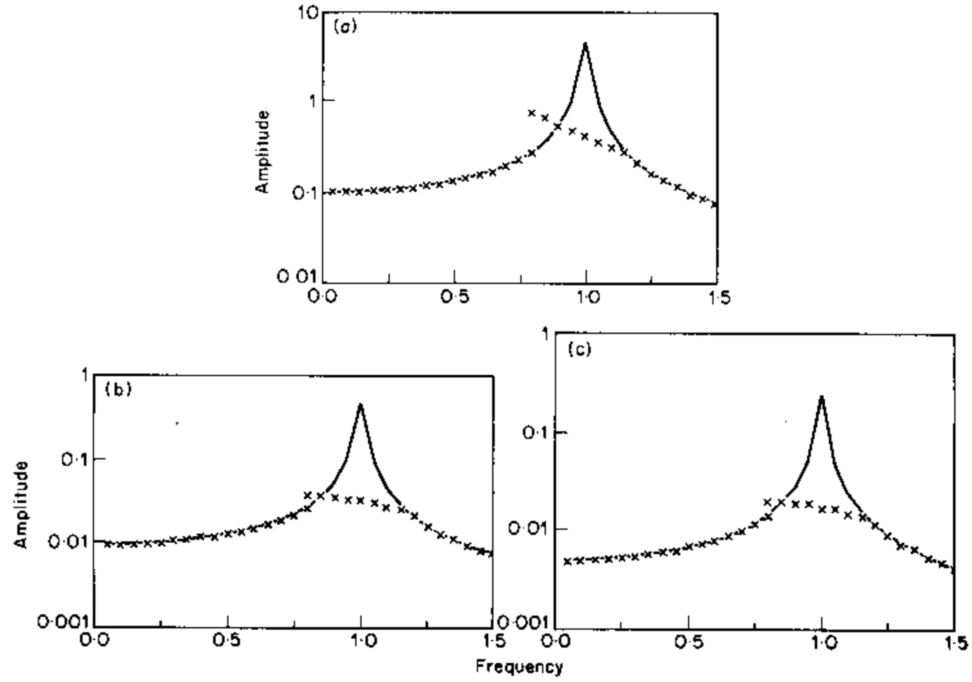

Figure 9. Frequency response $\bar{\delta}_p$ for case 1 and $\bar{F}_p = 0{\cdot}0625$: (a) $\bar{\delta}_{p1}$, (b) $\bar{\delta}_{p2}$, (c) $\bar{\delta}_{p3}$; ——, linear ($\alpha = 1$); *, non-linear.

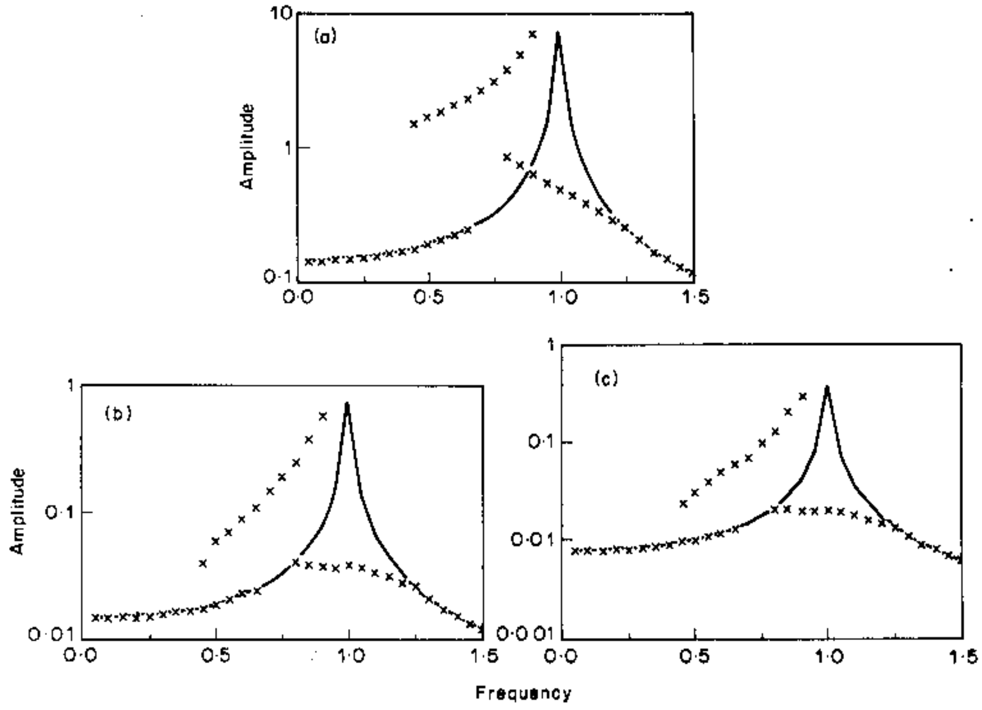

Figure 10. Frequency response $\bar{\delta}_p$ for case 1 and $\bar{F}_p = 0{\cdot}09375$; (a) $\bar{\delta}_{p1}$, (b) $\bar{\delta}_{p2}$, (c) $\bar{\delta}_{p3}$; ——, linear ($\alpha = 1$); *, non-linear.

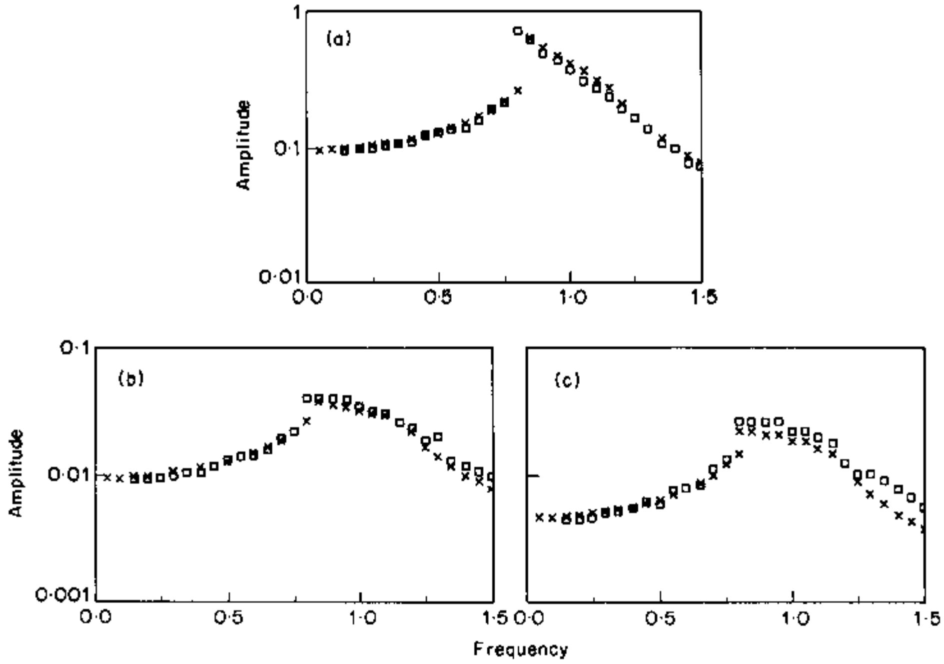

Figure 11. Frequency response $\bar{\delta}_p$ for case 1 when $\alpha_1 = 0$: (a) $\bar{\delta}_{p1}$, (b) $\bar{\delta}_{p2}$, (c) $\bar{\delta}_{p3}$; *, harmonic balance; □, analog simulation.

TABLE 2

Comparison of the frequency ranges of impact regions for the strongly non-linear case (case 1) between harmonic balance and analog simulation

Impact region	Harmonic balance	Analog simulation
No impact	$\bar{\Omega}_p < 0{\cdot}782$	$\bar{\Omega}_p < 0{\cdot}750$
	$\bar{\Omega}_p > 1{\cdot}178$	$\bar{\Omega}_p > 1{\cdot}150$
Single-sided impact	$0{\cdot}782 < \bar{\Omega}_p < 0{\cdot}800$	$0{\cdot}750 < \bar{\Omega}_p < 0{\cdot}800$
	$0{\cdot}800 < \bar{\Omega}_p < 1{\cdot}178$	$0{\cdot}800 < \bar{\Omega}_p < 1{\cdot}150$

$$\bar{\delta}_{p2} = \pm\left\{\frac{\bar{F}_p\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\left(\frac{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2-\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}\right)N_{fp1}\right\}\Big/\left\{\left(1-\frac{\bar{\Omega}_{33}^2\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2(\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2-\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2)}\right)N_{fp1}-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right\}, \tag{25b}$$

$$\bar{\delta}_{p3} = \frac{\pm\left\{\dfrac{\bar{F}_p\bar{\Omega}_{32}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\left(\dfrac{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2-\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}\right)N_{fp1}\right\}}{\left\{\left(1-\dfrac{\bar{\Omega}_{33}^2\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2(\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2-\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2)}\right)N_{fp1}-\dfrac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right\}}. \tag{25c}$$

The behavior observed in Figures 9 and 10 is clearly reflected in equations (25). Equation (25a) has the same form as for a SDOF system but with a shift in the resonance frequency because of the increased inertia of the coupled system. The frequency response functions for $\bar{\delta}_{p2}$ and $\bar{\delta}_{p3}$ have the same form as for $\bar{\delta}_{p1}$ and can be expressed in the general form

$$\bar{\delta}_{p2} = C_1\bar{\delta}_{p1}, \qquad \bar{\delta}_{p3} = C_2\bar{\delta}_{p1}, \tag{26a, b}$$

where

$$C_1 = \frac{\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2 \bar{\Omega}_{22}^2} \left(\frac{\bar{\Omega}_{22}^2 \bar{\Omega}_{33}^2}{\bar{\Omega}_{22}^2 \bar{\Omega}_{33}^2 - \bar{\Omega}_{23}^2 \bar{\Omega}_{32}^2} \right) N_{fp1}, \tag{26c}$$

$$C_2 = \frac{\bar{\Omega}_{32}^2 \bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2 \bar{\Omega}_{22}^2 \bar{\Omega}_{33}^2} \left(\frac{\bar{\Omega}_{22}^2 \bar{\Omega}_{33}^2}{\bar{\Omega}_{22}^2 \bar{\Omega}_{33}^2 - \bar{\Omega}_{23}^2 \bar{\Omega}_{32}^2} \right) N_{fp1}. \tag{26d}$$

For a linear system, $N_{fp1} = 1$, the values of C_1 and C_2 are constant. When the system is non-linear, $\bar{\delta}_{p2}$ and $\bar{\delta}_{p3}$ are still similar to $\bar{\delta}_{p1}$ but C_1 and C_2 are no longer constant, as N_{fp1} is now a function of $\bar{\delta}_{p1}$. Therefore, N_{fp1} is not only a quantitative but also a qualitative measure of the effect of the non-linear oscillator on the rest of the system. This information can be used to further simplify the analysis if desired. Consider the case for a hardening system undergoing two-sided impacts. As $\bar{\delta}_{p1}$ increases N_{fp1} approaches 1 from below and hence $\bar{\delta}_{p2}$ and $\bar{\delta}_{p3}$ will also approach the linear system response ($N_{fp1} = 1$) from below. A conservative estimate of the response could be obtained by analyzing a non-linear SDOF oscillator for $\bar{\delta}_{p1}$ and approximating the amplitudes $\bar{\delta}_{p2}$ and $\bar{\delta}_{p3}$ by using the corresponding linear system constants for C_1 and C_2. The error involved in the approximation can be estimated for the maximum deviation of N_{fp1} from 1.

5.3. CASE 2: WEAKLY NON-LINEAR SYSTEM

Typical frequency response functions for the alternating components $\bar{\delta}_{p1}$, $\bar{\delta}_{p2}$, and $\bar{\delta}_{p3}$ are given in Figures 12 and 13. A typical comparison of the harmonic balance results

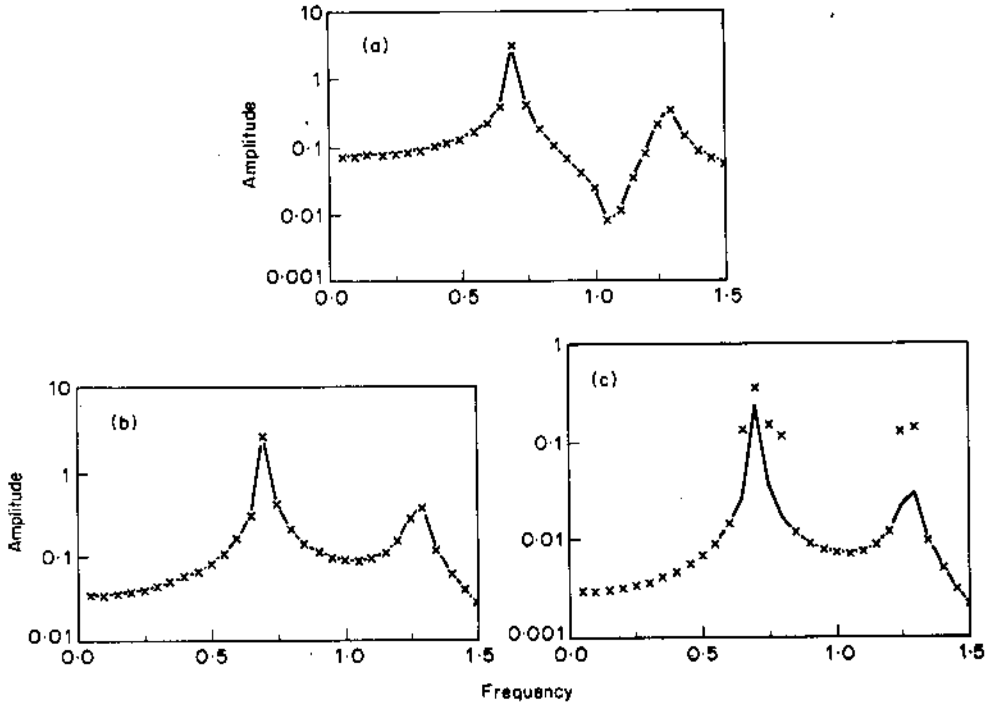

Figure 12. Frequency response $\bar{\delta}_p$ for case 2 and $\bar{F}_p = 0{\cdot}050$: (a) $\bar{\delta}_{p1}$, (b) $\bar{\delta}_{p2}$, (c) $\bar{\delta}_{p3}$; ——, linear ($\alpha = 1$); *, non-linear.

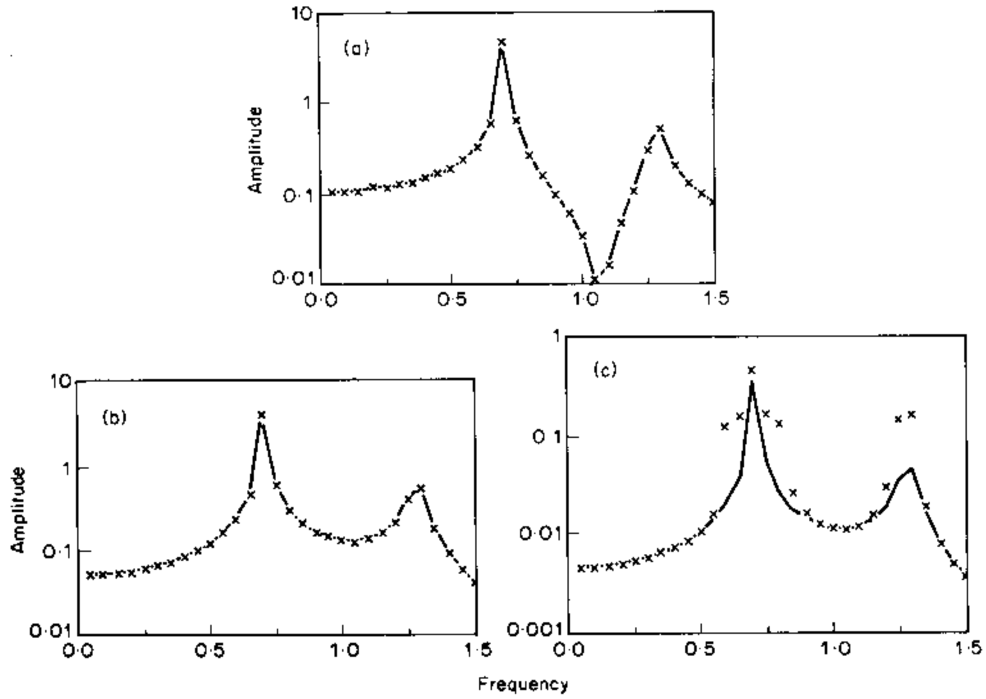

Figure 13. Frequency response $\bar{\delta}_p$ for case 2 and $\bar{F}_p = 0{\cdot}075$: (a) $\bar{\delta}_{p1}$, (b) $\bar{\delta}_{p2}$, (c) $\bar{\delta}_{p3}$; ——, linear ($\alpha = 1$); *, non-linear.

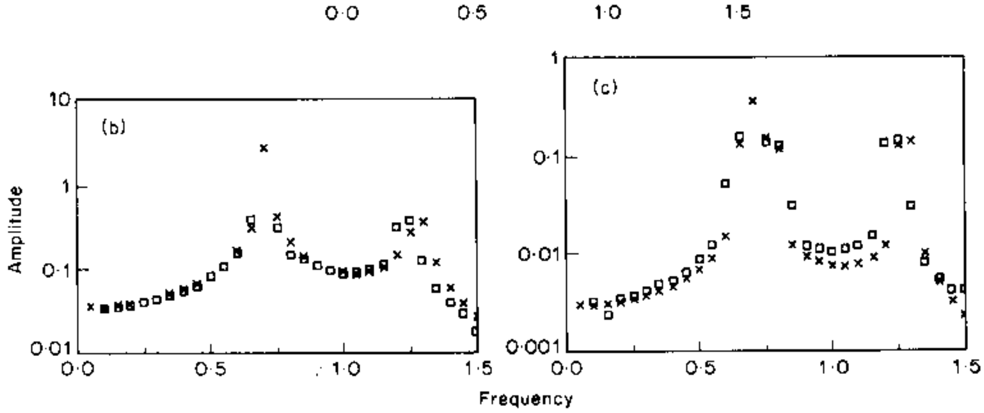

Figure 14. Frequency response $\bar{\delta}_p$ for case 2 when $\alpha_3 = 0$: (a) $\bar{\delta}_{p1}$, (b) $\bar{\delta}_{p2}$, (c) $\bar{\delta}_{p3}$; *, harmonic balance; □, analog simulation.

and analog simulation results is shown in Figure 14. A comparison of the frequency ranges of the different impact regimes is given in Table 3.

For this case, the motion of $\bar{\delta}_{p1}$ and $\bar{\delta}_{p2}$ is independent of the motion of $\bar{\delta}_{p3}$ and is given strictly by the linear case ($\alpha_3 = 1$). Over the frequency range of interest, the shape of the frequency response function for $\bar{\delta}_{p3}$ is similar to that of $\bar{\delta}_{p2}$. The amplitude ratio $\bar{\delta}_{p3}/\bar{\delta}_{p2}$ is not constant and will be larger, when impacts occur, than the corresponding linear case ($\alpha_3 = 1$). Unlike the strongly non-linear case (case 1), over the frequency range of interest, the non-linear oscillator in this MDOF system does not behave like the SDOF system [28] as its motion is governed by the motion of $\bar{\delta}_{p2}$. If $\bar{\delta}_{p2}$ is small enough, impacts do not occur but if $\bar{\delta}_{p2}$ becomes sufficiently large, impacts will occur. Also solutions exist over the entire impact regime and jump transitions do not occur.

It should be pointed out that the frequency response function for $\bar{\delta}_{p3}$ will appear somewhat distorted because the clearance space is the same order of magnitude as the elastic deformation. This leads to very large slopes in the transition regions which appear as jumps in the frequency response function. These "apparent" jumps are different from the jump transitions which occur in the strongly non-linear case (case 1). The jump transitions in case 1 occur at resonance because of the phase change between the applied force and the displacement. The apparent jumps in the response $\bar{\delta}_{p3}$ are not related to phase changes as they do not necessarily occur at resonance. Instead they are simply a function of the magnitude of $\bar{\delta}_{p2}$.

For case 2, it is assumed that $\bar{\Omega}_{33} \gg \bar{\Omega}_p$, and equations (7) can be simplified to yield

$$\bar{\delta}_{p1} = \pm\left\{\frac{\bar{F}_p}{\bar{\Omega}_{11}^2}\left[\left(1-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right]\right\}\Big/\left\{\left(1-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left[\left(1-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right]-\frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\right\}, \tag{27a}$$

$$\bar{\delta}_{p2} = \pm\left\{\frac{\bar{F}_p\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\right\}\Big/\left\{\left(1-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left[\left(1-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right]-\frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\right\}, \tag{27b}$$

$$\bar{\delta}_{p3} = \pm\left\{\frac{\bar{F}_p\bar{\Omega}_{32}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\left(\frac{1}{N_{fp3}}\right)\right\}\Big/\left\{\left(1-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{11}^2}\right)\left[\left(1-\frac{\bar{\Omega}_{23}^2\bar{\Omega}_{32}^2}{\bar{\Omega}_{22}^2\bar{\Omega}_{33}^2}\right)-\frac{\bar{\Omega}_p^2}{\bar{\Omega}_{22}^2}\right]-\frac{\bar{\Omega}_{12}^2\bar{\Omega}_{21}^2}{\bar{\Omega}_{11}^2\bar{\Omega}_{22}^2}\right\}. \tag{27c}$$

TABLE 3

Comparison of the frequency ranges of impact regions for the weakly non-linear case (case 2) between harmonic balance and analog simulation

Impact region	Harmonic balance	Analog simulation
No impact	$\bar{\Omega}_p < 0{\cdot}590$	$\bar{\Omega}_p < 0{\cdot}600$
	$0{\cdot}840 < \bar{\Omega}_p < 1{\cdot}200$	$0{\cdot}850 < \bar{\Omega}_p < 1{\cdot}150$
	$\bar{\Omega}_p > 1{\cdot}350$	$\bar{\Omega}_p > 1{\cdot}350$
Single-sided impact	$0{\cdot}590 < \bar{\Omega}_p < 0{\cdot}640$	$0{\cdot}600 < \bar{\Omega}_p < 0{\cdot}650$
	$0{\cdot}780 < \bar{\Omega}_p < 0{\cdot}840$	$0{\cdot}800 < \bar{\Omega}_p < 0{\cdot}850$
	$1{\cdot}200 < \bar{\Omega}_p < 1{\cdot}240$	$1{\cdot}150 < \bar{\Omega}_p < 1{\cdot}200$
	$1{\cdot}320 < \bar{\Omega}_p < 1{\cdot}350$	$1{\cdot}300 < \bar{\Omega}_p < 1{\cdot}350$
Two-sided impact	$0{\cdot}640 < \bar{\Omega}_p < 0{\cdot}780$	$0{\cdot}650 < \bar{\Omega}_p < 0{\cdot}800$
	$1{\cdot}240 < \bar{\Omega}_p < 1{\cdot}320$	$1{\cdot}200 < \bar{\Omega}_p < 1{\cdot}300$

The behavior shown in Figures 10 and 11 is clearly reflected in equations (27). $\bar{\delta}_{p1}$ and $\bar{\delta}_{p2}$ represent the response of a linear 2DOF linear system and $\bar{\delta}_{p3}$ has the general form

$$\bar{\delta}_{p3} = C\bar{\delta}_{p2}, \tag{28a}$$

where

$$C = (\bar{\Omega}_{32}^2/\bar{\Omega}_{33}^2)(1/N_{fp3}). \tag{28b}$$

C is constant for the linear system as $N_{fp3} = 1$ and will be a function of $\bar{\delta}_{p3}$ when the system is non-linear. Equation (28) would suggest that a weakly non-linear system, such as the one given in case 2, could be analyzed as a linear system with the response of the non-linear oscillator ($\bar{\delta}_{p3}$) calculated separately.

6. CONCLUDING REMARKS

This study represents an initial investigation into the dynamic behavior of MDOF systems composed of coupled non-linear oscillators. The method of harmonic balance was used to develop approximate analytical solutions of the undamped equations of motion of a multi-degree-of-freedom system composed of three coupled non-linear oscillators. For primary resonances and a harmonic excitation, general formulations were presented which can be used to study the existence and the stability of the solutions as well as modal spacing and modal coupling issues. These formulations provided an analytical basis for future research in the areas of multi-degree-of-freedom systems with multiple non-linearities) not necessarily just the clearance type) and the development of experimental programs for model validation.

In addition to the analytical formulations, an analysis methodology for MDOF systems was also presented and illustrated by two special but practical cases: a strongly non-linear system and a weakly non-linear system. An analysis of the special cases of the general MDOF non-linear formulation provided not only an improved understanding of the dynamic behavior of the non-linear systems, but also formed the basis for the development of simplified approximate solutions.

Because this represents an initial study, the scope of the work has been limited. Some of these limitations which are currently being addressed by the authors are as follows. First, an undamped formulation was chosen deliberately over a damped formulation in an effort to simplify the mathematics so that general observations about the system dynamic behavior could be made easily. Damping is easily added to the formulation, at the expense of the complexity of the equations and the resulting solutions, using the complex mass and stiffness concept in the frequency domain.

Second, this analysis was limited to the study of the response due to a harmonic excitation. Although this type of excitation does not necessarily represent a realistic operating environment it is, nonetheless, an important type of excitation and should be considered for several reasons. First, in order to understand the response for more complicated types of excitation, it is usually helpful and sometimes necessary to understand the response to a very simple type of excitation first. More importantly, a harmonic excitation is the preferred type of excitation for experimental studies of the general characteristics of a non-linearity and for estimation of system parameters. Therefore, to relate analytical and experimental studies, a clear understanding of the response for a simple harmonic excitation is required.

Finally, for the MDOF analysis presented here it is assumed that the forced response is given by the first harmonic only and not only are the higher harmonics neglected but also superharmonic, subharmonic, combination and internal resonances. For the special cases considered in section 3, the single frequency approximation is probably valid because the problem of combination and internal resonances is reduced by the requirement that the non-linear and linear resonances be widely separated. This requirement also reduces the possibility of superharmonic responses appearing in the operating frequency range for case 2. For case 1, superharmonic and subharmonic resonances may occur for different excitation levels as discussed by Comparin and Singh [28]. A more general analysis of the MDOF non-linear system, including an analysis of modal coupling, will require a consideration of the more complicated types of resonances as discussed by Gelb and Vander Velde [29] and Nayfeh and Mook [30]. Although the analysis involved is beyond the scope of this work the method of harmonic balance can be extended to treat these cases and is the subject of a current research effort.

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APPENDIX: LIST OF SYMBOLS

b	break points for stages in a general clearance non-linearity
$f(\ )$	general non-linear function
F	force
K	general stiffness term
M	mass
N_{fm}	describing function for mean component
N_{fp}	describing function for alternating component
t	time
α	stiffness ratio
α_A	average stiffness of first and second stage of non-linearity
$\delta, \dot{\delta}, \ddot{\delta}$	relative displacement, velocity and acceleration
ϕ	phase angle
μ	generalized argument of describing function
φ	generalized phase angle ($\varphi = \Omega t + \phi$)
ω	angular frequency (rad/s)
ω_n	natural frequency (rad/s)
Ω	angular frequency of external excitation (rad/s)
Ω_{ij}	stiffness coefficient for multi-degree-of-freedom case

Subscripts

F_p excitation force
m mean
p alternating component
ss single-sided
ts two-sided

Superscripts

$(\dot{\ })$ first derivative with respect to time
$(\ddot{\ })$ second derivative with respect to time
$(\bar{\ })$ non-dimensional variable