VIBRATION TRANSMISSION THROUGH ROLLING ELEMENT BEARINGS, PART IV: STATISTICAL ENERGY ANALYSIS

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A theoretical coupling loss factor for the vibration transmission through a rolling element bearing problem has been developed for the statistical energy analysis (SEA) technique. This scheme includes the bearing stiffness matrix developed earlier in the companion paper Part I. The same problem has been analyzed previously by using deterministic vibration models in Parts II and III. In this paper, a new analytical procedure is developed to obtain the coupling loss factor between the shaft and casing which is based on the solution of the boundary value problem at the plate-bearing interface. Several example cases are chosen to demonstrate the salient features of this method: I, a plate-cantilevered beam problem; II, case I with circular shaft-bearing replacing the cantilevered beam; III, a shaft-bearing-plate system. Example case I is a revised version of the problem analyzed earlier by Lyon and Eichler (reference [6] of this paper). Our proposed calculation indicates an improvement in the coupling loss factor prediction for a thin plate-cantilevered beam system. Example case II deals with the development of a theoretical scheme to compute coupling loss factor for a bearing system. This development is extended further to predict mean-square vibratory response of a rectangular plate for example case III. Results of the statistical energy analysis compare well with the predictions yielded by deterministic model of Part II. Predictions for example cases I and III are found to be in good agreement with measurements.

1. INTRODUCTION

It has been shown in Parts I–III [1–3], by using classical lumped parameter and dynamic finite element (FEM) techniques, that the proposed bearing model is clearly superior to the existing simple models for predicting vibration transmission through bearings in a shaft-bearing-plate system. Although the proposed models have been shown to be reliable up to a moderately high frequency, it is conceivable that these models are inadequate at very high frequencies where the modal density tends to be high. Classical vibration models do not predict modes accurately in this frequency regime, and even if it is possible to do so by employing closely spaced nodal points, such models require a significantly large computational effort. Moreover, the vast amount of predicted response spectra at many spatial points would be difficult to interpret. Accordingly, asymptotic or statistical methods must be adopted; typical techniques include the statistical energy analysis [4–8], asymptotic modal analysis [9–11] and asymptotic analysis using infinite system impedances [12].

This study is concentrated on the development of a theoretical broadband coupling loss factor, which is a key parameter for the statistical energy analysis (SEA), of a shaft-bearing-plate system. The SEA method has been applied successfully to those structural and acoustic systems which exhibit high modal overlap and weak coupling

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between subsystems [4–8, 13–18]. However several unresolved research issues still exist [5,8–10, 19, 20]. The specific objectives of this study are (i) to develop a theoretical scheme to compute a coupling loss factor for a typical rolling element bearing system, (ii) to demonstrate the applicability of SEA to the bearing problem, (iii) to analyze the following three example cases: I, a plate–cantilevered rectangular beam; II, case I with circular shaft-bearing system replacing the cantilevered beam; III, a circular shaft-bearing–plate–mount system; and finally, (iv), to perform parametric studies to examine the characteristics of vibratory energy transfer through bearings. The first and second example cases are revised and extended versions of a study performed by Lyon and Eichler [4, 6]. The final example case examines an experimental set-up which was analyzed earlier in Part II [2] by using the deterministic finite element method. Predictions yielded by each method will be compared with the experimental results.

2. LITERATURE REVIEW

A majority of publications are on the application of SEA to dynamic systems with high modal overlap and weak coupling such as structural-acoustical interactions in a fuselage [4, 13, 21], interior noise predictions [13], and vibratory energy transmission in mechanical equipment [4]. Of interest here is analytical or experimental estimation of SEA parameters for simple structural systems described by a flat plate, a cylinder and/or a shaft [4, 6, 7, 14]. In these studies, structural connections are often assumed rigid such as in the ideally welded case.

Lyon and Eichler [4, 6] in 1964 and Lyon and Scharton [7] in 1965 developed analytical expressions for the coupling loss factor $\eta$ in several connected structures, such as a plate bonded to a cantilevered beam [4, 6]. Here $\eta$ was derived by assuming a semi-infinite beam attached to an infinite plate and by further assuming that only a dynamic moment coupling at the joint can describe the motion/force transmission phenomenon. This problem is re-examined in this study and is then extended to a circular shaft-bearing-plate-mount system. In addition, Lyon and Eichler [4, 6] also developed SEA models of two structures inter-connected through a single (scalar) stiffness element. Typical examples include two discrete masses coupled by a linear spring, and a plate attached to a single-degree-of-freedom resonator [4]. The longitudinal vibration of linearly coupled rods was analyzed by Remington and Manning [18] in 1975. Loss factors of typical line or point connected structures such as a plate welded to a cylinder, cross-beams and two perpendicular plates bolted or welded together have been calculated assuming ideal rigid joints [4, 6, 7, 22, 23]. But a compliant bearing support problem is yet to be analyzed.

3. EXAMPLE CASE I: COUPLING LOSS FACTOR OF PLATE-CANTILEVERED BEAM SYSTEM

First, we attempt to rework the plate-cantilevered beam problem of Lyon and Eichler [4, 6] as shown in Figure 1. Only flexural motions of the rectangular plate and rectangular beam are considered in this case. Accordingly, Lyon and Eichler [4, 6] developed an expression for the coupling loss factor $\eta_{sc}$ which describes the vibratory energy transfer between the beam(s) and the plate(c) due to a moment coupling at the joint,

$$\eta_{sc}(\omega) = \frac{(2\rho_l A_k \kappa_c c_e)^2}{\omega m_s} \frac{Z_c}{Z_c + Z_s} \left| \frac{1}{Z_c} \right|^2 \text{Re} \left( \frac{1}{Z_c} \right),$$

(1)

where $\kappa_c = \sqrt{I_c/A_c}$ is the radius of gyration, $c_e = \sqrt{E/\rho_s}$ is the wave speed, $m_s$ is the mass and $\text{Re} (\ )$ implies the real part of a complex variable (a list of symbols is given in the
Appendix) with the sign convention $e^{i\omega t}$, the deterministic junction moment impedances of the plate $Z_c$ for a moment applied to a small patch, and beam $Z_s$ are [6, 24]

$$Z_c(\omega) = \frac{16\rho_c h c^2 c_c^2}{\omega [1 - (4i/\pi) \ln (k_c h_s)]}, \quad Z_s(\omega) = \frac{\rho_s A_s \kappa^2 c_s^2 k_s}{\omega} (1 - i),$$  \tag{2}

where $k$ is the wavenumber and $h$ is the plate or beam thickness. Here, note that $Z_s$ of equation (2) differs from the Lyon and Eichler expression [4, 6] in the sign of the imaginary part, which is probably a typographical error. They assumed that $|Z_c| \gg |Z_s|$, $\kappa_s = \kappa_c$, $c_s = c_c$ and $\rho_s = \rho_c$, and simplified equation (1) to yield a frequency invariant expression for $\eta_{sc}$ as

$$\eta_{sc} = W_s/(4L_s),$$  \tag{3}

where $W_s$ and $L_s$ are the beam width and length, respectively, as shown in Figure 1. However, our calculations indicate that $|Z_c| > |Z_s|$ but not $|Z_c| \gg |Z_s|$ for the parameters used by Lyon and Eichler [6], as shown in Figure 2(a). By using equation (1), the frequency-averaged $\eta_{sc}$ has been recomputed and is compared with the approximate model given by equation (3) in Figure 2(b). Experimental results for $\eta_{sc}$ given by Lyon and Eichler [6] are also plotted in Figure 2(b). It seems that our calculation is better than the earlier prediction, even though an approximate impedance expression for a moment applied to a small patch is used. At low frequency, a discrepancy is observed between theory and experiment which is primarily due to the low modal density in this
regime. The presence of a low natural frequency may be due to the compliant epoxy bond between the beam and plate. However, above the threshold frequency where many modes participate, shown as a vertical line in Figure 2(b), the slope of the least squares straight line fit on the experimental data is nearly the same as the predicted $\eta_{sc}$.

4. EXAMPLE CASE II: COUPLING LOSS FACTOR OF CIRCULAR SHAFT-BEARING-PLATE SYSTEM

Next, we modify Figure 1 by inserting a ball bearing between the circular shaft (which replaces the rectangular beam in Figure 1) and the rectangular plate. Again, a semi-infinite shaft and an infinite plate are assumed. For SEA, we reduce the system to a plate subsystem and a shaft-bearing subsystem. The coupling loss factor $\eta_{sc}$ is still given by equation (1), but $Z_s$ must be modified to account for the compliant bearing.

Consider a shaft with boundary conditions shown in Figure 3. The bearing end is subjected to zero transverse velocity $v_{ssa}(t, z = 0^-) = 0$ and a sinusoidally varying moment $M_{ss}(t, z = 0^-) = M_{ssa}$.

Figure 3. Boundary conditions for example case II: semi-infinite shaft-bearing system.
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\[ M_{xsa}(t, z = 0^-) = M_{xsa} e^{iot}. \] Here, \( z = 0^- \) refers to the junction point between bearing and plate, and \( z = 0^+ \) is the junction point between the bearing and shaft. Next, we attempt to obtain a boundary condition at the bearing-shaft interface using the boundary condition at the plate-bearing interface. Substituting \( \nu_{yza}(\omega, z = 0^-) = 0 \) into the definition of bearing force vector \( \{f(\omega)\}_b = [K]_{bm} \{v(\omega)\}_b / i\omega \), where \( \{v(\omega)\}_b \) and \( [K]_{bm} \) are the bearing velocity vector and stiffness matrix respectively, yields

\[
F_{yza}(\omega, 0^+) = \left( \frac{k_{by\theta}}{k_{b\theta,\theta_x}} \right) M_{xsa} + \frac{v_{yza}(\omega, 0^+)}{i\omega} \left( \frac{k_{by\theta}}{k_{b\theta,\theta_x}} \right)^2 k_{b\theta,\theta_x} - k_{byy}. \tag{4}
\]

The governing equations for \( M_{xsa}(\omega, z) \) and \( F_{yza}(\omega, z) \) of the shaft in terms of \( v_{yza}(\omega, z) \) are

\[
M_{xsa} = -(EI_s/i\omega) d^2 v_{yza}/dz^2, \quad F_{yza} = (EI_s/i\omega) d^3 v_{yza}/dz^3. \tag{5}
\]

In general, the bounded solution for \( v_{yza}(\omega, z) \) is [25]

\[
v_{yza}(\omega, z) = \{b_1 e^{-i k_s z} + b_2 e^{-k_s z}\}. \tag{6}
\]

Using equations (5) and (6), we enforce the boundary conditions for the shaft at \( z = 0^+ \) to obtain the following closed form solution for the coefficient vector \( \{b\} = \{b_1, b_2\}^T \): \[ b_1 = \frac{i\omega M_{xsa} + (\omega k_{by\theta}/k_{b\theta,\theta_x}) - i\omega k_s - (\omega / EI k_s)^2 (k_{by\theta} - k_{by\theta}/k_{b\theta,\theta_x})}{EI k_s^2 (1 + i) - 2i k_{byy} + 2i k_{by\theta}/k_{b\theta,\theta_x}} M_{xsa}, \]

\[ b_2 = \frac{(\omega k_{by\theta}/k_{b\theta,\theta_x}) - i\omega k_s - (\omega / EI k_s^2 (k_{byy} - k_{by\theta}/k_{b\theta,\theta_x})}{EI k_s^2 (1 + i) - 2i k_{byy} + 2i k_{by\theta}/k_{b\theta,\theta_x}} M_{xsa}. \tag{7} \]

Hence, the bearing-shaft impedance \( Z_s \) is then given by

\[
Z_s(\omega) = \frac{M_{xsa}(\omega, 0)}{(dv_{yza}/dz)(\omega, 0^-)} = \frac{k_{b\theta,\theta_x}}{i\omega + \{k_{by\theta}, v_{yza}(\omega, 0^+) + k_{b\theta,\theta_x}(dv_{yza}/dz)(\omega, 0^+)\}/M_{xsa}}. \tag{8}
\]

Although \( M_{xsa} \) appears in equation (8), \( Z_s \) is actually independent of the magnitude of \( M_{xsa} \) since \( v_{yza}(\omega, 0^+) \) and \( (dv_{yza}/dz)(\omega, 0^+) \) are linearly proportional to \( M_{xsa} \) as well in equation (5).

By using equations (1), (7) and (8), the frequency-averaged \( \eta_{x\phi} \) has been computed for several bearings, the stiffness coefficients of which are tabulated in Table 1. These cases are compared in Figure 4 with the numerical values for shaft and plate properties such as \( k, \kappa, c \) and \( m \) being those used by Lyon and Eichler [4, 6] in example case I. For a

<table>
<thead>
<tr>
<th>Table 1</th>
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<tr>
<td>Typical bearing stiffness coefficients of Figure 4 for example case II</td>
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<table>
<thead>
<tr>
<th>Set</th>
<th>( k_{by\theta} ) (N/m)</th>
<th>( k_{byy} ) (N)</th>
<th>( k_{b\theta,\theta_x} ) (Nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Very compliant bearings ( (k_{bij} &lt; 1E3) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1E5</td>
<td>5E4</td>
<td>2E4</td>
</tr>
<tr>
<td>C</td>
<td>5E5</td>
<td>8E4</td>
<td>5E4</td>
</tr>
<tr>
<td>D</td>
<td>1E8</td>
<td>1E7</td>
<td>1E6</td>
</tr>
<tr>
<td>E</td>
<td>1E8</td>
<td>3E7</td>
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</table>
Figure 4. Coupling loss factor $\eta_{sc}$ used in example case II for various bearing stiffness coefficients listed in Table 1. ——, A; ----, B; --------, C; ----, D; ------, E.

very soft bearing, $Z_s$ is dominated by the bearing parameters, and hence $|Z_s| \gg |Z_e|$ is valid. This implies that $\eta_{sc}$ reduces to the frequency invariant expression given by equation (3) as shown by set A in Figure 4. On the other hand, $\eta_{sc}$ for a rigid bearing (say, set B or C) is typically smaller than $\eta_{sc}$ given by A. The extreme values of $\eta_{sc}$ as $k_{bij}$ or bearing preload becomes very large, depend on the relative magnitudes of the bearing stiffness coefficients, as shown in Figure 4 for sets D and E, although set E behavior occurs more frequently. Such deviation for a stiff bearing from set E is primarily due to the presence of off-diagonal stiffness coefficient such as $k_{by}e_x$.

5. EXAMPLE CASE III: A CIRCULAR SHAFT-BEARING-PLATE SYSTEM

5.1. THEORY

Consider another circular shaft-bearing-plate-mount system similar to example case II but with a shaft of finite length, as shown in Figure 5(a). Note that this system has been analyzed earlier in Part II [2] by using the deterministic finite element technique. Recall that the unconstrained end of the non-rotating circular shaft is subjected to a harmonically varying force $F_{yx}(t) = F_{yx} e^{i\omega t} + F_{ysm}$, where $F_{ysm}$ is the mean force. From Part II [2], we know that the longitudinal and torsional motions of the shaft and the in-plane vibration of the plate can be neglected. Hence, the shaft bending vibration and plate flexural motion are of interest here. Also, the previous deterministic vibration models indicate that the coupling between these two motions is mainly due to the dynamic moment at the bearing provided that the longitudinal shaft vibration is not excited. Accordingly, two subsystems which can be easily identified by using SEA are the transverse modes of the shaft-bearing system and the flexural modes of the plate-mount system, as shown in Figure 5(b). We now proceed to derive $\eta_{sc}$ governing the vibratory energy flow between these two subsystems, using the driving point junction impedance method suggested by Lyon [4].

Since the rectangular plate dimension is much larger than the bearing dimension, the plate is again assumed infinite. Consider the vibrational power flow $\Pi_{sc}$ from the shaft-bearing subsystem (subscript $s$) to the plate (subscript $c$), due to the uniform external
Figure 5. Example case III: (a) schematic of the circular shaft-bearing-plate-mount system described in references [2, 26] and (b) SEA model of the physical system.

Gaussian random force over a frequency bandwidth $\Delta \omega$ with center frequency $\omega$,

$$\Pi_{sc}(\omega) = \omega \eta_{sc}(\omega) n_s(\omega) \left( \frac{E_s(\omega)}{n_s(\omega)} - \frac{E_c(\omega)}{n_c(\omega)} \right),$$

(9)

where $n_j$ and $E_j$ ($j = s, c$) refer to subsystem modal density and total vibratory energy respectively. The modal densities per Hz of a shaft $n_s$ and rectangular plate $n_c$, given for bending motion with simply supported boundary conditions [4], are

$$n_s = L_s^4 \rho_s \pi d_s^2 / 4 E I_s \omega^2,$$

(10)

$$n_c = A_c / h_c \sqrt{3 \rho_c (1 - \mu^2)} / E,$$

where $\rho$ is the material density, $A_c$ is the plate surface area, $L_s$ is the shaft length, $E$ is the modulus of elasticity, $I_s$ is the area moment of inertia of the shaft, $\omega$ is the bandwidth center frequency, $h_c$ is the plate thickness, $d_s$ is the shaft diameter, $\mu$ is the Poisson ratio, and the subscripts $s$ and $c$ denote the shaft and the plate respectively. Since the plate is assumed to be reasonably well damped and geometrically large, equation (9) is approximated assuming $n_c \gg n_s$ or $E_c / n_c \ll E_s / n_s$ to yield

$$\eta_{sc}(\omega) = \Pi_{sc}(\omega) / \omega E_s(\omega).$$

(11)

For the shaft, $E_s = m_s \langle V_s^2 \rangle$, where $m_s$ is the shaft mass and $\langle V_s^2 \rangle$ is mean square shaft
transverse velocity averaged over $\Delta \omega$ and shaft length $L_s$. Using the expression for $\Pi_{sc}$ given by Lyon and Eichler [6] as discussed in the previous section gives a frequency-averaged $\eta_{sc}$ in terms of the deterministic $Z_c$ and $Z_e$ which is identical to that of equation (1), and which is rewritten here for the circular shaft case as

$$\eta_{sc}(\omega) = (4EI_s/\omega L_s) \text{Re} \left( \frac{1}{Z_c} \right) \frac{Z_e}{(Z_c + Z_e)}^2. \quad (12)$$

For an infinite steel plate of thickness $h_e$, the point moment impedance $Z_c$ is given in reference [25] as

$$Z_c(\omega) = \left\{ \frac{4Eh_e^2}{3\omega (1 - \mu^2)} \left[ 1 - (4i/\pi) \ln \left( \frac{\delta^2}{2k_d} \right) \right] \right\}^{-1}, \quad k_d \ll 1. \quad (13)$$

The shaft–bearing impedance $Z_e$ in equation (12) is derived next by solving the boundary value problem for the system shown in Figure 6(a). The free end of the shaft has a vanishing dynamic shear force $F_{yse}(\omega, z = L_s) = 0$ and bending moment $M_{yse}(\omega, z = L_s) = 0$. The bearing end is similar to that shown in Figure 3. Following the same argument used previously for example case II will lead to the two boundary conditions for the shaft described by $M_{yse}(\omega, z = 0^+) = M_{yse}$ and $F_{yse}(\omega, z = 0^+) = (k_{bbx}/k_{bby}) M_{yse} + v_{yse}(\omega, 0^+) \left( (k_{bbx}/k_{bby})^2 k_{bby} - k_{bbx} \right)/(i\omega)$. Governing equations are still given by equation (5), but the solution for $v_{yse}(\omega, z)$ is assumed to be of the following form [25]:

$$v_{yse}(t, z) = \{b_1 e^{-ik_1 z} + b_2 e^{ik_1 z} + b_3 e^{-ik_2 z} + b_4 e^{ik_2 z}\}. \quad (14)$$

Upon using equations (5) and (14), the boundary conditions yield the following algebraic problem

$$[B] \{b\} = M_{yse} \{k_{bye}/k_{bby}, i\omega/(EI_t^2), 0, 0\}^T. \quad (15)$$

![Figure 6](image)

Figure 6. Boundary conditions for example case III: finite shaft–bearing system. (a) Moment applied at the bearing end; (b) force applied to the unconstrained end.
The non-zero elements of coefficient matrix \([B]\) of dimension 4 are

\[
B_{11} = \frac{1}{\omega} \left( EIk_z^3 - ik_{byy} + ik_{byy}^2 \frac{k_{b\omega,\theta}}{k_{b\theta,\epsilon}} \right), \quad B_{12} = \frac{1}{\omega} \left( -EIk_z^3 - ik_{byy} + ik_{byy}^2 \frac{k_{b\omega,\theta}}{k_{b\theta,\epsilon}} \right),
\]

\[
B_{13} = \frac{i}{\omega} \left( EIk_z^3 - k_{byy} + k_{byy}^2 \frac{k_{b\omega,\theta}}{k_{b\theta,\epsilon}} \right), \quad B_{14} = \frac{i}{\omega} \left( -EIk_z^3 - k_{byy} + k_{byy}^2 \frac{k_{b\omega,\theta}}{k_{b\theta,\epsilon}} \right),
\]

\[
B_{21} = B_{22} = B_{23} = B_{24} = 1, \quad B_{31} = -iB_{41} = i e^{-ik_zL_z}, \quad B_{32} = iB_{42} = -i e^{ik_zL_z}, \quad B_{33} = -B_{43} = -e^{-k_zL_z}, \quad B_{34} = B_{44} = e^{k_zL_z}.
\]  

Both \([B]\) and \(\{b\} = \{b_1, b_2, b_3, b_4\}^T\) can be easily obtained numerically. The bearing–shaft impedance \(Z_z\) is still given by equation (8).

The same procedure may be applied to obtain the driving point force impedance for a harmonically varying transverse force \(F_{ysa}(t, z = 0)\). Note that the origin is redefined at the forcing point, as shown in Figure 6(b) for convenience. The boundary conditions are \(F_{ysa}(\omega, z = 0) = F_{ysa}, \quad M_{ysa}(\omega, z = 0) = 0, \quad v_{ysa}(\omega, z = -L_z) = 0\) and \((dv_{ysa}/dz)(\omega, -L_z) = 0\). In a manner similar to that of the previous analysis, one obtains

\[
M_{ysa}(\omega, z = -L_z) = -(k_{b\theta,\epsilon} \frac{dv_{ysa}}{dz}) (\omega, -L_z^+) + k_{byy,\epsilon} v_{ysa}(\omega, -L_z^+)/i(\omega),
\]

\[
F_{ysa}(\omega, z = -L_z) = -(k_{byy,\epsilon} \frac{dv_{ysa}}{dx}) (\omega, -L_z^+) + k_{byy} v_{ysa}(\omega, -L_z^+)/(\omega).
\] 

These prescribed boundary conditions again yield a set of algebraic problem similar to equation (15). The non-zero elements of the coefficient matrix \([B]\) of dimension 4 are

\[
B_{11} = -B_{21} = B_{12} = B_{22} = -B_{13} = iB_{23} = -B_{14} = -iB_{24} = -1, 0,
\]

\[
B_{31} = (ik_z k_{byy,\epsilon} + k_{byy} + EIk_z^2) e^{ik_zL_z}, \quad B_{32} = (ik_z k_{b\theta,\epsilon} + k_{byy,\epsilon} + EIk_z^2) e^{-ik_zL_z},
\]

\[
B_{33} = (-k_z k_{b\theta,\epsilon} + k_{byy,\epsilon} - EIk_z^3) e^{ik_zL_z}, \quad B_{34} = (k_z k_{b\theta,\epsilon} + k_{byy,\epsilon} - EIk_z^3) e^{-ik_zL_z},
\]

\[
B_{41} = (ik_z k_{byy,\epsilon} - k_{byy} - iEIk_z^3) e^{ik_zL_z}, \quad B_{42} = (-ik_z k_{byy,\epsilon} - k_{byy} + iEIk_z^3) e^{-ik_zL_z},
\]

\[
B_{43} = (k_z k_{byy,\epsilon} - k_{byy} + EIk_z^3) e^{ik_zL_z}, \quad B_{44} = (-k_z k_{byy,\epsilon} - k_{byy} + EIk_z^3) e^{-k_zL_z}.
\] 

The right-hand side vector \(\{b\}\) of the algebraic problem is \(\{0, F_{ysa}\omega/(EIk_z^3), 0, 0\}^T\). The force impedance at the driving point is then given by \(Z_f(\omega, z = 0) = F_{ysa}/v_{ysa}(\omega, 0)\). Accordingly, the input power is \(\Pi_z = (1/2)F_{ysa}^2 \text{Re}\{(1/Z_f)^*\}\) where \(\text{Re}\{\}\) is the real part of the complex variable and \(^*\) implies complex conjugation.

We can now compute the vibratory energy transfer \(\Pi_{sc}\) through the bearing and the steady state subsystem energy levels \(E_s\) and \(E_c\) by applying the energy balances to both subsystems shown in Figure 5(b); here \(\eta_{sc} = \eta_{sc}\eta_s/\eta_c\):

\[
\begin{bmatrix}
\eta_s + \eta_{sc}(\omega) & -\eta_{cs}(\omega) \\
-\eta_{sc}(\omega) & \eta_c + \eta_{cs}(\omega)
\end{bmatrix}
\begin{bmatrix}
E_s(\omega) \\
E_c(\omega)
\end{bmatrix} = \begin{bmatrix}
\Pi_z/\omega \\
0
\end{bmatrix},
\]  

\[
E_s(\omega) = \frac{\Pi_z(\eta_s + \eta_{cs})}{\omega(\eta_s \eta_c + \eta_c \eta_{sc} + \eta_s \eta_{cs})}, \quad E_c(\omega) = \frac{\Pi_z \eta_{sc}}{\omega(\eta_s \eta_c + \eta_c \eta_{sc} + \eta_s \eta_{cs})}.
\] 

Since \(E = m(V^2)\), the following velocity levels may be obtained at any center frequency
Figure 7. Comparison between theory (----) and experiment [26] (O) for example case III with a finite shaft. Here plate mobility level in dB is $10 \log_{10} \left( V_0^2 \right) \Re \left( V_0^2 \right) = 1 \cdot 0 \ (\text{m/s})^2$ for $F_{yss} = 1 \cdot 0 \ \text{N}$.

Figure 8. Comparison between theory (----) and experiment [26] (O) for example case III. Here theory is for a semi-infinite shaft.

Figure 9. Predicted coupling loss factor $\eta_{sc}$ for a semi-infinite (----) and a finite shaft (----) shaft in example case III.
\( \omega \) from either equation (19) or (20):

\[
\langle V_2^2 \rangle / \langle V_2^2 \rangle = (m_s / m_c) \left( \frac{\eta_{sc}(\omega)}{\eta_c + \eta_{sc}(\omega)} \right).
\]  

(21)

5.2. VALIDATION AND PARAMETRIC STUDIES

In order to validate our SEA formulation, we compare the mean square mobility level of the plate with experimental data provided by reference [26] as reported earlier in Part II [2]. Note that although all non-zero bearing stiffness coefficients \( k_{bij} \) were computed and given in Part II [2], only \( k_{byy} = 3.69 \times 10^8 \), \( k_{byy_c} = 3.52 \times 10^5 \) and \( k_{be_ye_c} = 4.19 \times 10^4 \) are used as they appear to be the most significant ones according to the proposed theory. By using equation (20), \( \langle V_2^2 \rangle \) has been computed, and is compared with experimental results in Figure 7. Theoretical predictions for three values of frequency-invariant dissipation loss factor \( \eta_s = \eta_c \) are given since the choice of structural damping seems to be critical for SEA. It can be seen from Figure 7 that the experimental data are approximately bounded by \( \eta_s = 0.0003 \) and \( \eta_s = 0.03 \), which are typical values for a lightly damped structure. It is possible that \( \eta_s \) and \( \eta_c \) vary with frequency in the experiment; this is not included in our calculations. Accordingly, the agreement between theory and experiment is deemed to be excellent.

Further comparison between theory and experiment can be made for the case of a semi-infinite shaft considered in the example case II. By using equations (1), (7) and (8) for \( \eta_{sc} \), the mobility levels were computed and found to be given by straight lines, as shown in Figure 8. These lines represent the asymptotic behavior of the system when the shaft is very long: i.e., \( L_s \to \infty \). Again, most experimental data are bounded within the range given by \( \eta_s = 0.0003 \) to 0.03. Next consider the finite shaft length \( L_s = 1.32 \text{ m} \) of high modal density \( \eta_s \). In Figure 9 the coupling loss factor \( \eta_{sc} \) for this compared with the result for a semi-infinite shaft. It can be seen that \( \eta_{sc} \) for the semi-infinite shaft follows the frequency-averaged values of the finite shaft. Finally, plate mobility levels predicted by SEA for these two cases are compared in Table 2 with the results yielded by the deterministic finite element model of Part II. Here, a nominal dissipation loss factor \( \eta_s = \eta_c = 0.003 \) was chosen for calculations. It is evident that SEA is indeed applicable for this physical system.

<table>
<thead>
<tr>
<th>One-third octave band center frequency (Hz)</th>
<th>Plate mobility level (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Finite shaft</td>
</tr>
<tr>
<td>400</td>
<td>-102</td>
</tr>
<tr>
<td>500</td>
<td>-92</td>
</tr>
<tr>
<td>630</td>
<td>-95</td>
</tr>
<tr>
<td>800</td>
<td>-88</td>
</tr>
<tr>
<td>1000</td>
<td>-87</td>
</tr>
<tr>
<td>1250</td>
<td>-97</td>
</tr>
<tr>
<td>1600</td>
<td>-108</td>
</tr>
<tr>
<td>2000</td>
<td>-106</td>
</tr>
</tbody>
</table>
6. CONCLUDING REMARKS

The vibration transmission through a bearing in a generic shaft-bearing-plate system has been analyzed by using the SEA technique. A new theoretical procedure has been developed to compute the frequency-averaged coupling loss factor which relies on the solution of the boundary value problem at the plate-bearing interface. This scheme incorporates the rolling element bearing stiffness matrix developed earlier as a part of the deterministic vibration models in the companion papers Parts I–III [1–3]. The proposed formulation has been applied to a physical set-up consisting of a shaft cantilevered to a plate through a bearing support. The vibratory response of the rectangular plate predicted by SEA has been found to be in good agreement with measurements and with the results of the deterministic models of Part II. There are still several unresolved issues which require further research. For example, a more precise analysis of the moment applied to a plate through a beam in example case I is required. Improved estimation procedures for damping associated with bearing systems are needed. Perhaps of more importance is the extension of proposed methodology to rotating machinery noise and vibration problems. In fact, work is in progress with application to the gearboxes.

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REFERENCES


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**APPENDIX: LIST OF SYMBOLS**

- $A_c$: plate surface area
- $A_s$: shaft cross-sectional area
- $c$: wave speed
- $EI$: flexural rigidity of the shaft
- $E_s$, $E_c$: total vibratory energy level in the shaft–bearing ($s$) subsystem and plate ($c$) subsystem
- $F_{pa}(t)$: applied alternating shear force on the shaft
- $F_{axm}$: applied mean shear force on the shaft
- $\{f\}_b$: bearing force vector
- $h_c$: plate thickness
- $[K]_{bm}$: proposed bearing stiffness matrix of dimension 6
- $k_{ij}$: bearing stiffness coefficient, $i,j = x, y, z, \theta_x, \theta_y, \theta_z$
- $k_s$, $k_c$: shaft ($s$) or plate ($c$) wavenumber
- $L_s$: shaft length
- $M_{xsa}$: alternating shaft bending moment about $x$ direction
- $m_c$: plate mass
- $m_s$: shaft mass
- $n_s$: plate modal density
- $n_c$: shaft modal density
- $Re$: real part of a complex number
- $t$: time
- $\langle V^2 \rangle$: spatially and frequency bandwidth averaged mean square mobility
- $\nu_{axa}$: alternating shaft transverse velocity
- $\{v\}_b$: bearing velocity vector
- $Z_c$: driving point plate impedance at bearing junction
- $Z_s$: driving point shaft impedance at bearing junction
- $Z_f$: driving point shaft impedance at forcing location
- $\eta_s$, $\eta_c$: dissipation loss factor for the shaft–bearing ($s$) or plate ($c$) subsystems
- $\eta_{hc}$: coupling loss factor
\( \kappa \) radius of gyration
\( \Pi_s \) external power input to the shaft-bearing \((s)\) subsystem
\( \Pi_{sc} \) net power transfer from the shaft-bearing \((s)\) subsystem to plate \((c)\) subsystems
\( \rho \) material density
\( \omega \) bandwidth center frequency
\( (\quad)^* \) complex conjugate