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Application of Floquet Theory to On-Off Valve Controlled Pneumatic Actuators

A periodic linear time varying (LTV) model for on-off valve controlled pneumatic actuation systems, was described in an earlier paper by the authors. Based on this LTV model formulation, Floquet's Theorem is invoked to characterize dynamic response of the system. A new computational technique called the expanded state space method is developed to calculate the frequency response of the LTV system with staircase coefficient variations. This technique is computationally superior to the straightforward solution scheme. Floquet Theory is also used to assess the nature of transient response. A single acting cylinder system controlled by an on-off valve is considered to illustrate the stability and transient response issues. Computer simulation based on the nonlinear model is used to obtain detailed results. It is shown that application of the Floquet Theory provides valuable insight into the dynamic response of the class of actuators considered.

1 Introduction

This paper deals with the analytical characterization of the dynamic response of on-off valve controlled pneumatic actuators. Important applications of these systems are automated manufacturing operations, assembly of parts, and accurate position control of robotic manipulators (Lai et al., 1990). In an earlier paper (Kunt and Singh, 1990) a new linear time varying (LTV) model was developed for open loop, on-off valve controlled pneumatic actuation systems. This formulation was based on a periodic profile description for variable operating points and directional control valve flow area variations. The dynamic behavior of the example case, a single acting cylinder controlled by a two way-two port rotary valve, under the cyclic pressure loading was obtained using the proposed LTV model. Experimental evidence and digital simulation predictions based on the nonlinear mathematical equations validated the analytical formulation. The proposed LTV model was found to be better and more applicable than linear time invariant (LTI) models used previously by many investigators as discussed in Kunt and Singh (1990). The present study is an extension of this previous work with focus on the application of Floquet's Theory (Richards, 1983) for the prediction of the dynamic response and stability characteristics of the pneumatic actuation system. First, related theoretical models are outlined. Second, a new solution scheme based on the Floquet Theory is developed. Finally, stability and transient response issues are addressed and illustrated through an example case.

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2 Problem Statement

The actuation system considered in this study consists of an on-off directional control valve (DCV) and a linear actuator with an inertial load as shown in Fig. 1. Controlled reciprocating motion $x_p(t)$ along with pressures within the system comprise the system dynamic response to command DCV spool displacement $\theta(t)$. Of interest is the characterization of the system dynamic response under cyclic pressure excitation provided by the DCV switching the flow at a rate Ω_p, i.e., by maintaining a periodic $\theta(t)$ with period $T_p = 1/\Omega_p$.

Some of the relevant equations given earlier are presented here briefly for the sake of completeness. Thermofluid processes of the system of Fig. 1 are considered isothermal and modeled through control volumes defined for the actuator chambers and the DCV internal volume. For the kth control volume, the ideal gas equation is

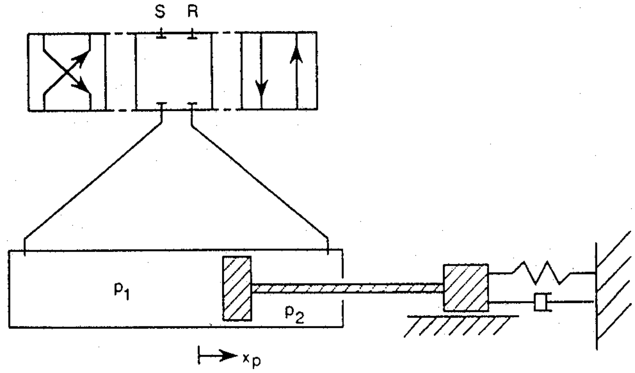

Fig. 1 Generic on-off DCV valve controlled actuator in the open loop mode. Dashes in the DCV symbol indicate that number of valve operating positions and flow ports are not restricted.

$$p_k = R_a m_k T / V_k \tag{1}$$

where R_a is the gas constant, T is the absolute temperature, and p_k, m_k, and V_k are time dependent pressure, mass, and volume of the air within the kth control volume, respectively. Mass flow rate is modeled using a one dimensional compressible mass flow rate function $M_f\,(i, k)$ between the ith and the kth pressure points (Andersen, 1967; Kunt and Singh, 1990). Conservation of mass for the kth control volume requires

$$\dot{m}_f = \sum_{i=1}^{P} \begin{cases} C_{ik} A_{ik} M_f(i,k), & p_i \geq p_k \\ -C_{ik} A_{ik} M_f(k,i), & p_i \leq p_k \end{cases} \tag{2}$$

where C_{ik} is a discharge coefficient and A_{ik} denotes the minimum flow area between the two pressure points. This is in general a function of time due to DCV operation (see Kunt and Singh, 1990, for details). For the system of Fig. 1, there are four pressure points, actuator left and right chambers, and supply and return ports. Supply and return pressures are assumed to be constants.

The equation of motion for the single degree of freedom actuator-load system is written as

$$\ddot{x}_p = [F_e(t) - F_f(t)]/M_t \tag{3}$$

where M_t is the total moving mass. The external force F_e and the total friction force F_f are given by

$$F_e(t) = (A_{p1} - C_p)p_1(t) - (A_{p2} - C_p)p_2(t) - K_s x_p(t) \tag{4a}$$

$$F_f(t) = \begin{cases} F_d \operatorname{sign}[\dot{x}_p(t)] + C_v \dot{x}_p(t), & \dot{x}_p(t) \neq 0 \\ F_e(t), & \dot{x}_p(t) = 0,\ F_e(t) \leq F_s \\ F_s \operatorname{sign}[F_e(t)] & \dot{x}_p(t) = 0,\ F_e(t) \geq F_s \end{cases} \tag{4b}$$

where A_{p1} and A_{p2} are the areas of the piston right and left-side pressure faces, respectively. Stiction force is denoted by F_s and dry friction by F_d. And, K_s, C_v, and C_p denote the spring stiffness, viscous damping coefficient, and pressure differential friction coefficient, respectively. Equations (1)–(4) constitute a nonlinear mathematical model and were used for simulation of the system dynamic response based on a 4th order Runge-Kutta method (Kunt, 1988). Authors focus on the linearized version of this model here.

3 LTV Model

The operating points of the system variables $p_1(t)$, $p_2(t)$, and $x_p(t)$ are defined for periodic excitation and Eqs. (1)–(4) are linearized to yield the following model:

$$\dot{\mathbf{z}}(t) = \mathbf{G}(t)\mathbf{z}(t) + \mathbf{f}(t) \tag{5}$$

where

$$\mathbf{G}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_n^2 & -2\xi\omega_n & K_{p1} & -K_{p2} \\ 0 & -K_{v1} & \dfrac{-1}{\tau_1(t)} & 0 \\ 0 & -K_{v2} & 0 & \dfrac{-1}{\tau_2(t)} \end{bmatrix},$$

$$\mathbf{z}(t) = \begin{Bmatrix} x_p(t) \\ \dot{x}_p(t) \\ p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \mathbf{f}(t) = \begin{Bmatrix} 0 \\ 0 \\ f_{p1}(t) \\ f_{p2}(t) \end{Bmatrix} \tag{6a,b,c}$$

System parameters are given by:

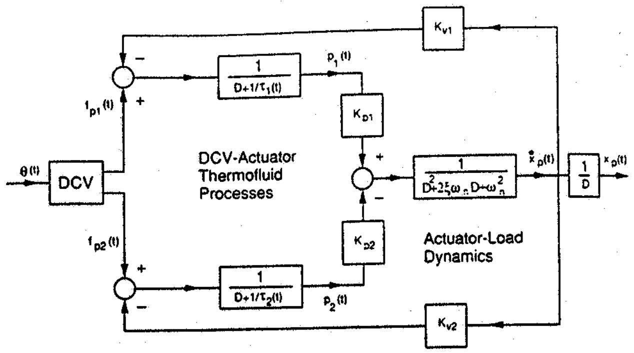

Fig. 2 Linear time varying (LTV) model block diagram for the system of Fig. 1

$$\tau_1(t) = \frac{1}{R_a T_a \left(\sum_{3,4} C_{i1} A_{i1}(t) \beta_{i1} \right) / (V_{o1} + A_{p1}\bar{x}_p)}$$

$$\tau_2(t) = \frac{1}{R_a T_a \left(\sum_{3,4} C_{i2} A_{i2}(t) \beta_{i2} \right) / (V_{o2} - A_{p2}\bar{x}_p)}$$

$$f_{p1}(t) = \frac{R_a T_a}{V_{o1} + A_{p1}\bar{x}_p} \sum_{3,4} [C_{i1} A_{i1}(t)(\alpha_{i1} + \beta_{i1}\bar{p}_1)]$$

$$f_{p2}(t) = \frac{R_a T_a}{V_{o2} - A_{p2}\bar{x}_p} \sum_{3,4} [C_{i2} A_{i2}(t)(\alpha_{i2} + \beta_{i2}\bar{p}_2)]$$

$$K_{p1} = (A_{p1} - C_p)/M_t, \quad K_{p2} = (A_{p2} - C_p)/M_t$$

$$K_{v1} = \frac{A_{p1}\bar{p}_1}{V_{o1} + A_{p1}\bar{x}_p}, \quad K_{v2} = \frac{A_{p2}\bar{p}_2}{V_{o2} - A_{p2}\bar{x}_p}$$

$$\omega_n = (K_s/M_t)^{1/2}, \quad \xi = C_v/[2(K_s M_t)^{1/2}] \tag{7a-j}$$

where overhead bars indicate operating points. Actuator left and right chamber volumes corresponding to zero piston displacement are denoted by V_{o1} and V_{o2}, respectively. Mass flow rate linearization constants are denoted by α_{ik} and β_{ik}; k = 1, 2, i = 3, 4 as in Kunt and Singh (1990). Equations (5)–(7) constitute a periodic LTV model due to the presence of variable coefficients $\tau_1(t)$ and $\tau_2(t)$ of period T_p. Figure 2 shows the corresponding block diagram. For fixed operating points, time varying blocks correspond to the actuator chamber dynamics. However, this model can accommodate variable operating points, in which case the gains K_{v1} and K_{v2} will also be time varying. On-off valve flow area variations can be approximated by piecewise constant or staircase profiles as illustrated in Fig. 3 to simplify the LTV model. Then, the following characterization of the system matrix and the forcing vector are obtained: $\mathbf{G}(t) = \mathbf{G}_m$ and $\mathbf{f}(t) = \mathbf{f}_m$, $\theta_{m-1} \leq \theta \leq \theta_m$, where $m = 1, 2, \ldots, N_p$. Here $\mathbf{G}_m$ and $\mathbf{f}_m$ are constant matrices and vectors, respectively. Number of steps in the profile is denoted by N_p, which is equal to the number of DCV operating positions. The spool position at which the spool moves from position m to $m + 1$ is indicated by θ_m. Denoting the duration of step m by T_m, $\sum_{m=1}^{N_p} T_m = T_p$. This simplification reduces the LTV model to a set of constant coefficient ordinary differential equations, which are valid over consecutive intervals. Hence a straightforward analytical solution is possible as outlined by Kunt and Singh (1990) by forcing the initial conditions of an interval to be equal to the final conditions of the previous interval. In Section 5.1 of this paper an alternate solution method will be developed based on Floquet Theory.

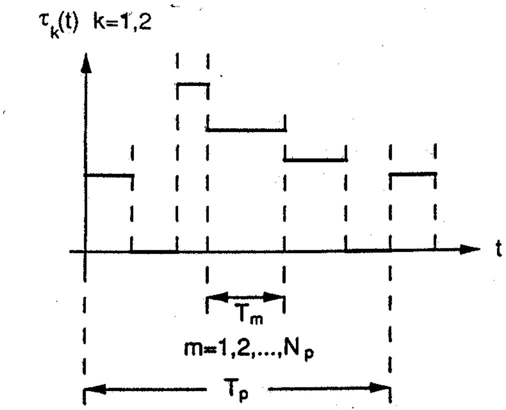

Fig. 3 Typical staircase profile used for the variable system coefficients

4 Dynamic Response

This study focuses on the characterization of the system dynamic response due to a periodic excitation i.e., $\theta(t) = \theta(t + T_p)$. This allows definition of operating points for the system variables provided that the system response is periodic. Then, the LTV model becomes a viable alternative to the nonlinear model. Periodic LTV system analysis has been extensively applied to the study of electrical networks and circuits. In mechanical systems, chief applications to date involve the analysis of rotating structures such as helicopter blades (Friedmann and Hammond, 1977; Dugundji and Wendell, 1983). No reference could be found on the application of the LTV formulation to fluid power actuation systems prior to Kunt and Singh (1990).

The general solution to Eq. (5) with initial conditions at $t = 0$ can be written as (De Russo, 1965)

$$\mathbf{z}(t) = \mathbf{W}(t)\mathbf{W}^{-1}(0)\mathbf{z}(0) + \int_0^t \mathbf{W}(t)\mathbf{W}^{-1}(\tau)\mathbf{f}(\tau)d\tau \qquad (8)$$

where $W(t)$ is the Wronskian matrix, the columns of which are the basis solutions to the homogeneous system equation. Unlike the LTI System, it is not in general possible to determine all the basis solutions for the periodic LTV system and construct an analytical solution.

5 Formulation

5.1 Floquet Theory. Equation (8) can be rewritten as

$$\mathbf{z}(t) = \phi(t,0)\mathbf{z}(0) + \int_0^t \phi(t,\tau)\mathbf{f}(\tau)d\tau \qquad (9)$$

where $\phi(t,0) = \mathbf{W}(t)\mathbf{W}^{-1}(0)$ is the state transition matrix over the interval $(t,0)$. An important property of the state transition matrix is that if it is determined over one period of the coefficients, it can be determined for all $t \geq 0$: $\phi(t,0) = \phi(t,kT_p)\,\phi^k(T_p,0)$ where $k = 0, 1, 2, \ldots$ and $kT_p \leq t \leq (k+1)T_p$. It is customary to define a discrete transition matrix $\mathbf{C} = \phi(T_p,0)$ that describes system behavior over one full period of the coefficients. In view of the foregoing, $\mathbf{z}(kT_p) = \mathbf{C}^k z(0)$, $k = 0, 1, 2, \ldots$. This is an expression of the Floquet's Theorem stating that the unforced response of a periodic LTV system is related to the response one full period away by a complex constant matrix $\mathbf{C}$ (Richards, 1983).

5.2 The Expanded State Space Method. Choosing the state variables as the piston displacement $x_p(t)$ and its first three derivatives, i.e., $\mathbf{z}(t) = \{x_p(t)\ \dot{x}_p(t)\ \ddot{x}_p(t)\ \dddot{x}_p(t)\}^T$, the state matrix and the forcing vector assume the following forms:

$$\mathbf{G}_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_{0m} & -a_{1m} & -a_{2m} & -a_{3m} \end{bmatrix}, \quad \mathbf{f}_m(t) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ b_m \end{Bmatrix} \qquad (10a,b)$$

where the constants a_{km}, $k = 1, 2, 3, 4$ and b_m are related to the original system parameters through

$$a_{0m} = \frac{\omega_n^2}{\tau_{1m}\tau_{2m}}$$

$$a_{1m} = \frac{2\xi\omega_n}{\tau_{1m}\tau_{2m}} + \frac{\omega_n^2}{\tau_{1m}} + \frac{\omega_n^2}{\tau_{2m}} + \frac{K_{p1}K_{v1}}{\tau_{2m}} - \frac{K_{p2}K_{v2}}{\tau_{1m}}$$

$$a_{2m} = \omega_n^2 + \frac{2\xi\omega_n}{\tau_{1m}} + \frac{2\xi\omega_n}{\tau_{2m}} + \frac{1}{\tau_{1m}\tau_{2m}} + K_{p1}K_{v1} - K_{p2}K_{v2}$$

$$a_{3m} = 2\xi\omega_n + \frac{1}{\tau_{1m}} + \frac{1}{\tau_{2m}}$$

$$b_m = \frac{K_{p1}f_{p1m}}{\tau_{2m}} - \frac{K_{p2}f_{p2m}}{\tau_{1m}} \qquad (11a\text{–}e)$$

Here τ_{1m} and τ_{2m} denote the values of $\tau_1(t)$ and $\tau_2(t)$ during the interval m. The case of $N_p = 2$ ($m = 1, 2$) will be considered for illustration purposes. The governing equations can be cast into

$$\frac{d^4x_p}{dt^4} + a_{31}\frac{d^3x_p}{dt^3} + a_{21}\frac{d^2x_p}{dt^2} + a_{11}\frac{dx_p}{dt} + a_{01}(x_p - x_{\infty,1}) = 0,\ \theta_0 \leq \theta < \theta_1 \qquad (12a)$$

$$\frac{d^4x_p}{dt^4} + a_{32}\frac{d^3x_p}{dt^3} + a_{22}\frac{d^2x_p}{dt^2} + a_{12}\frac{dx_p}{dt} + a_{02}(x_p - x_{\infty,2}) = 0,\ \theta_1 \leq \theta < \theta_2 \qquad (12b)$$

where

$$x_{\infty,1} = b_1/a_{01},\ x_{\infty,2} = b_2/a_{02} \qquad (12c,d)$$

This form of the equations suggests that by defining two new state variables $(x_p(t) - x_{\infty,1})$ and $(x_p(t) - x_{\infty,2})$ instead of $x_p(t)$, an "apparently" unforced system description can be obtained as

$$\dot{\mathbf{z}}_e(t) = \mathbf{G}_e(t)\mathbf{z}_e(t) \qquad (13a)$$

where

$$\mathbf{z}_e(t) = \{x_p(t) - x_{\infty,1}\ x_p(t) - x_{\infty,2}\ \dot{x}_p(t)\ \ddot{x}_p(t)\ \dddot{x}_p(t)\}^T$$

and

$$\mathbf{G}_e(t) = \begin{cases} \mathbf{G}_{e,1}, & \theta_0 \leq \theta < \theta_1 \\ \mathbf{G}_{e,2}, & \theta_1 \leq \theta < \theta_2 \end{cases} \qquad (13b)$$

with

$$\mathbf{G}_{e,1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_{01} & 0 & -a_{11} & -a_{21} & -a_{31} \end{bmatrix},$$

$$\mathbf{G}_{e,2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -a_{02} & -a_{12} & -a_{22} & -a_{32} \end{bmatrix} \qquad (13c,d)$$

Note that a general periodic LTV system of order ν with piecewise constant coefficients and forcing function can be formulated into a homogeneous system of order $\nu_e = \nu + N_p$

− 1, hence the name expanded state space approach has been used here. Although it increases the order of the system by $N_p - 1$, this approach leads to a simplified solution scheme as it eliminates the superposition integral of Eq. (9). The solution of Eq. (13a) is given by $\mathbf{z}_e(t) = \mathbf{\Phi}_e(t,0)\, \mathbf{z}_e(0)$, where the expanded state transition matrix $\mathbf{\Phi}_e(t,0)$ can be determined for $N_p = 2$ from

$$\mathbf{\Phi}_e(t,0) = \begin{cases} \exp(\mathbf{G}_{e,1}t)\mathbf{\Phi}_e^k(T_p,0), & \theta_0 \le \theta < \theta_1 \\ \exp[\mathbf{G}_{e,2}(t-T_1)]\exp(\mathbf{G}_{e,1}T_1)\mathbf{\Phi}_e^k(T_p,0), & \theta_1 \le \theta < \theta_2 \end{cases} \quad (14)$$

The expanded discrete transition matrix $\mathbf{C}_e = \mathbf{\Phi}_e(T_p,0)$ embodies the homogeneous system information. Therefore, for the fourth order LTV system $\lambda_{e,i} = \lambda_i$, $i = 1, 2, 3, 4$; where $\lambda_{e,i}$, $i = 1, 2, \ldots, 5$ are the eigenvalues of $\mathbf{C}_e$. The extra eigenvalue $\lambda_{e,5}$ is due to the forcing function, which is partly incorporated into $\mathbf{C}_e$. Invoking Floquet's Theorem

$$\mathbf{z}_e(kT_p) = \mathbf{\Phi}_e^k(T_p,0)\mathbf{z}_e(0),\ k = 0,1,2,\ldots \quad (15)$$

It is inferred that $\lambda_{e,5} = 1$. Otherwise $\mathbf{z}_e(t)$ would die out or grow without bounds neither of which is possible for a periodically forced linear stable system.

The constant matrix which $\mathbf{C}_e^k$ approaches as $k \to \infty$ can be found from

$$\lim_{k\to\infty} \mathbf{C}_e^k = \sum_{i=1}^{5} \lambda_i^k \mathbf{Z}_{\lambda,i} = \mathbf{Z}_{\lambda,5} \quad (16a)$$

where using Sylvester's Theorem (Zadeh, 1963)

$$\mathbf{Z}_{\lambda,5} = \frac{\sum_{k=1}^{4}(\mathbf{C}_e - \lambda_k \mathbf{I})}{\sum_{k=1}^{4}(\lambda_k - 1)} \quad (16b)$$

Thus from Eqs. (14)–(16), the steady state forced response can be written as

$$\mathbf{z}_{fe}(t) = \begin{cases} \exp(\mathbf{G}_{e,1}t)\mathbf{Z}_{\lambda,5}\mathbf{z}_e(0), & \theta_0 \le \theta < \theta_1 \\ \exp[\mathbf{G}_{e,2}(t-T_1)]\exp(\mathbf{G}_{e,1}T_1)\mathbf{Z}_{\lambda,5}\mathbf{z}_e(0), & \theta_1 \le \theta < \theta_2 \end{cases} \quad (17)$$

Note that the characteristic polynomial for $\mathbf{G}_{e,m}$, $m = 1, 2, \ldots, N_p$ is $(\lambda - \lambda_{1m})(\lambda - \lambda_{2m})(\lambda - \lambda_{3m})(\lambda - \lambda_{4m})\lambda^{N_p - 1}$ where λ_{im}, $i = 1, 2, 3, 4$ are the nonzero eigenvalues of $\mathbf{G}_{e,m}$ and the zero eigenvalue ($\lambda = 0$) has a multiplicity of $N_p - 1$ due to the $N_p - 1$ columns of zeros in $\mathbf{G}_{e,m}$. It can be shown that rank $(\mathbf{G}_{e,m}) = 4$, and the degeneracy associated with $\lambda = 0$ is equal to $N_p - 1$, i.e., full degeneracy (Brogan, 1985). Consequently, the minimum polynomial of $\mathbf{G}_{e,m}$ is given by $(\lambda - \lambda_{1m})(\lambda - \lambda_{2m})(\lambda - \lambda_{3m})(\lambda - \lambda_{4m})\lambda$. Also, since

$$\mathbf{C}_e = \prod_{m=1}^{N_p} \exp(\mathbf{G}_{e,m}T_m) \quad (18)$$

rank $(\mathbf{C}_e) \le 4$, but $\lambda_{e,i} = \lambda_i$, $i = 1, 2, 3, 4$ as before; therefore rank $(\mathbf{C}_e) = 4$ (provided λ_i are distinct). The extra eigenvalues $\lambda_{e,i} = 1$, $i = 5, 6, \ldots, N_p + 3$ and have a degeneracy of $N_p - 1$. The minimum polynomial of $\mathbf{C}_e$ is given by $(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)(\lambda - 1)$. Then, using Sylvester's Theorem

$$\mathbf{C}_e = \sum_{i=1}^{4} \lambda_i \mathbf{Z}_{\lambda,i} + \mathbf{Z}_{\lambda,5} \quad (19)$$

where $\mathbf{Z}_{\lambda,5}$ is as given by Eq. (21). Then $\lim_{k\to\infty} \mathbf{C}_e^k = \mathbf{Z}_{\lambda,5}$ and the solution is given by

$$\mathbf{z}_{fe}(t) = \exp(\mathbf{G}_{e,m}t_m)\prod_{i=1}^{m-1}(\mathbf{G}_{e,i}T_i)\mathbf{Z}_{\lambda,5}\mathbf{z}(0),\quad \theta_m \le \theta < \theta_{m+1} \quad (20)$$

where $0 \le t_m \le T_m$ is a local time variable. The steady state forced response of a linear system is independent of initial conditions, which suggests that

$$\mathbf{Z}_{\lambda,5} = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & c_{1N_p} & 0 & 0 & 0 \\ c_{21} & c_{22} & \cdot & \cdot & c_{2N_p} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{v_e1} & c_{v_e2} & \cdot & \cdot & c_{v_eN_p} & 0 & 0 & 0 \end{bmatrix} \quad (21)$$

where c_{mk} are constants with $\sum_{k=1}^{N_p} c_{mk} = 0$, and Eq. (20) becomes

$$\mathbf{z}_{fe}(t) = \exp(\mathbf{G}_{e,m}t_m)\prod_{i=1}^{m-1}(\mathbf{G}_{e,i}T_i)\begin{bmatrix} c_{11}x_{\infty,1} & \cdot & \cdot & c_{1N_p}x_{\infty,N_p} & 0 & 0 & 0 \\ c_{21}x_{\infty,1} & \cdot & \cdot & c_{2N_p}x_{\infty,N_p} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{v_e1}x_{\infty,1} & \cdot & \cdot & c_{v_eN_p}x_{\infty,N_p} & 0 & 0 & 0 \end{bmatrix}.$$

Compared with the straightforward solution method used by Kunt and Singh (1990), this approach avoids the inversion of the matrix of size $4N_p$; hence, it is a more efficient scheme to calculate the system forced response. Determination of the matrix $\mathbf{Z}_{\lambda,5}$ is simplified in view of the above results concerning the eigenvalues $\lambda_{e,i}$ of the expanded discrete transition matrix $\mathbf{C}_e$. Also, the sparsity of the matrices $\mathbf{G}_{e,m}$ can be exploited to calculate their exponentials in an efficient manner.

6 Stability and Natural Modes of Response

Statement of the Floquet Theory (Section 5.1) hints that the stability characteristics of the periodic LTV system are determined by the discrete transition matrix $\mathbf{C}$. Denoting the eigenvalues of $\mathbf{C}$ by λ_i, $i = 1, 2, \ldots, v$ LTV system is asymptotically stable if $|\lambda_i| < 1$.

It is convenient to introduce a time invariant matrix $\mathbf{\Gamma}$ such that $\mathbf{C} = \exp(\mathbf{\Gamma}\, T_p)$. Then corresponding to the formulation of Section 5.1, $\mathbf{z}(kT_p) = [\exp(\mathbf{\Gamma}\, T_p)]^k\, \mathbf{z}(0)$ is obtained. This can also be written as (Richards, 1983)

$$\mathbf{z}(kT_p) = \left[\sum_{i=1}^{v} \exp(\mu_i kT_p)\mathbf{z}_{\mu,i}\right]\mathbf{z}(0) \quad (22a)$$

where

$$\mathbf{Z}_{\mu,i} = \frac{\prod_{k=1, k\ne i}^{v}(\mathbf{C} - \lambda_k \mathbf{I})}{\prod_{k=1, k\ne i}^{v}(\lambda_i - \lambda_k)} \quad (22b)$$

and μ_i denote the eigenvalues of the matrix $\mathbf{\Gamma}$. Therefore, the linear LTV system is stable if all the eigenvalues of the matrix $\mathbf{\Gamma}$ have negative real parts.

The eigenvalues λ_i of $\mathbf{C}$ are termed as the characteristic roots

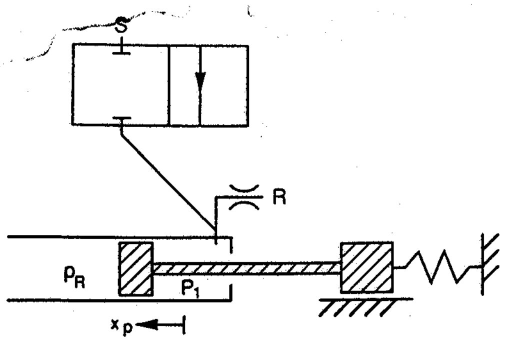

Fig. 4 Schematic of the two-way DCV controlled single acting cylinder

and the eigenvalues μ_i of $\mathbf{\Gamma}$ as the characteristic exponents. The characteristic roots are related to the characteristic exponents through $\lambda_i = \exp(\mu_i T_p)$. Letting $\mathbf{P}(t) = \mathbf{\Phi}(t,0)\exp(-\mathbf{\Gamma}t)$, the natural or unforced response of the system is given by $\mathbf{z}(t) = \mathbf{P}(t)\exp(\mathbf{\Gamma}t)\mathbf{z}(0)$. The significance of this form lies in the fact that $\mathbf{P}(t)$ is periodic with period T_p as shown in the following.

$$\begin{aligned} \mathbf{P}(t+T_p) &= \mathbf{\Phi}(t+T_p,0)\exp[-\mathbf{\Gamma}(T_p+t)] \\ &= \mathbf{\Phi}(t,0)\mathbf{C}(0)\exp(-\mathbf{\Gamma}T_p)\exp(-\mathbf{\Gamma}t) \qquad (23) \\ &= \mathbf{\Phi}(t,0)\exp(-\mathbf{\Gamma}t) \\ &= \mathbf{P}(t) \end{aligned}$$

Considering the scalar equation $\mathbf{z}(t) = \sum_{i=1}^{\nu} \exp(\mu_i t)\,\rho_i(t)$ where $\rho_i(t+T_p) = \rho_i(t)$. Its υ entries are the linearly independent basis solutions. From this formulation it is seen that the nature of the basis solutions is determined by the associated characteristic root or exponent. For example, the basis solutions corresponding to a real μ_i will be similar in shape to $\rho_i(t)$ with exponential modulation of amplitude. Therefore, the natural response of a periodic system can be classified according to its eigenvalues. Depending on whether μ_i is positive, negative, or complex, the corresponding solution is referred to as P type, N type, or C type, respectively (Richards, 1983).

For a P type solution the characteristic exponent will be of the form $\mu_i = \alpha_i + j\dfrac{2\pi k}{T_p}$, $k = 0, 1, 2, \ldots$ where α_i is a real constant and $j = \sqrt{-1}$. Typical P type solution is therefore given by

$$\exp(\alpha_i t)\exp\left(j\frac{2\pi k}{T_p}t\right)\rho_i(t) = \exp(\alpha_i t)\vartheta(t) \qquad (24a)$$

and

$$\begin{aligned} \vartheta(t+T_p) &= \exp\left(j\frac{2\pi k}{T_p}t\right)\exp\left(j\frac{2\pi T_p}{T_p}\right)\rho_i(t+T_p) \\ &= \exp\left(j\frac{2\pi kt}{T_p}\right)\rho_i(t) \qquad (24b) \\ &= \vartheta(t) \end{aligned}$$

Thus P type solutions are exponentially modulated oscillatory functions of period T_p.

N type solutions have characteristic exponents of the form $\mu_i = \alpha_i + j(2k+1)\pi/T_p$. For the simple case of $k = 0$, the N type solution is

$$\exp(\alpha_i t)\exp(j\pi t/T_p)\rho_i(t) = \exp(\alpha_i t)\Xi(t) \qquad (25a)$$

Note that

$$\begin{aligned} \Xi(t+T_p) &= \exp(j\pi t/T_p)\exp(j\pi)\rho_i(t+T_p) \\ &= -\exp(j\pi T_p)\rho_i(t) = -\Xi(t) \end{aligned} \qquad (25b)$$

and

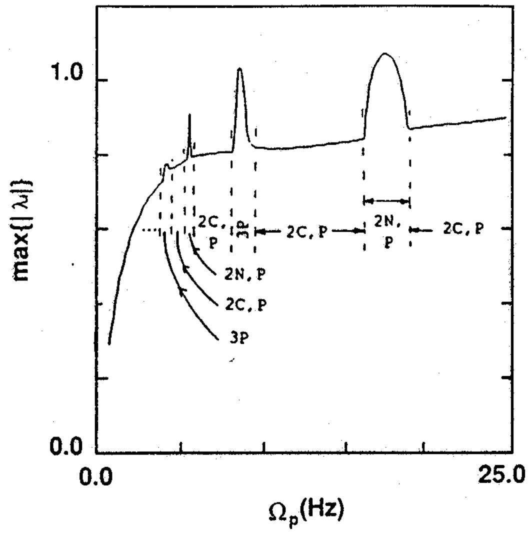

Fig. 5 Variation of the maximum eigenvalue of the discrete transition matrix with Ω_p for the system described in Section 6. Types of basis solutions are indicated for Ω_p > 4Hz.

$$\Xi(t+2T_p) = \Xi(t) \qquad (25c)$$

Therefore N type solutions are subharmonic to the frequency of parameter variation. Since

$$\prod_{i=1}^{n} \lambda_i = \exp\left\{\int_0^{T_p} \text{trace}[\mathbf{G}(t)]dt\right\},$$

negative eigenvalues occur in pairs to maintain an overall positive product. Therefore, N type solutions also occur in pairs.

C type solutions arise from complex λ_i and have the form $\exp[(\alpha_i + j\beta_i)t]\rho_i(t)$, i.e., they are an exponentially modulated product of a pair of periodic functions one of frequency β_i and the other Ω_p. Note that C type solutions also occur in pairs.

7 Results

A two way—two position DCV controlled single acting cylinder system, schematically shown in Fig. 4, was analyzed. Numerical values of system parameters listed below are chosen to cause stability problems for illustration purposes: $A_{p,1}$ = 0.92 in.^{2}, $A_{p,2}$ = 0.99 in.^{2}, V_{01} = 2.2 in.^{3}, M_t = 0.002 lb-s^2/in., K_s = 2.0 lb/in., $C_{S1} = C_{1R} = 0.3$, $F_d = F_s = C_v = C_p = 0$, P_s = 60 psi, T_s = 68 F, P_R = 14.7 psi and T_R = 68 F. In this system, an orifice of area 0.017 in.^{2} is used to discharge air and DCV on time is chosen to be 13 percent.

Stability assessment using the LTV model is based on the eigenvalues λ_i of the discrete transition matrix $\mathbf{C}$. Figure 5 shows the variation of the maximum eigenvalue (max $\{|\lambda_i|\}$) of $\mathbf{C}$ with DCV operating speed Ω_p and a number of peaks are observed. Two of these corresponding to $\Omega_p \cong$ 17.5Hz and 8.5Hz have magnitudes greater than unity signifying instability at these two speeds.

In general, a passive (time invariant) stable system of natural frequency Ω_N will become unstable if parameters are varied (pumped) at a rate Ω_p such that $\Omega_p = 2\Omega_N/k$, $k = 1, 2, \ldots$. For small amplitudes of parameter variation, this is true only for a slightly damped system. With increased damping instability will initiate only for larger amplitudes of pumping. For the system considered here $\Omega_N \cong$ 8.3Hz (see Kunt, 1988); hence, instability is predicted at Ω_p = 16.6, 8.3, 5.4, 4.2Hz, etc., which closely follow the frequencies at which peaks are seen in Fig. 5.

Figure 5 also identifies the frequency ranges corresponding to different combinations of basis solutions. For example, the peak around Ω_p = 17.5Hz is associated with $2N$ type solutions

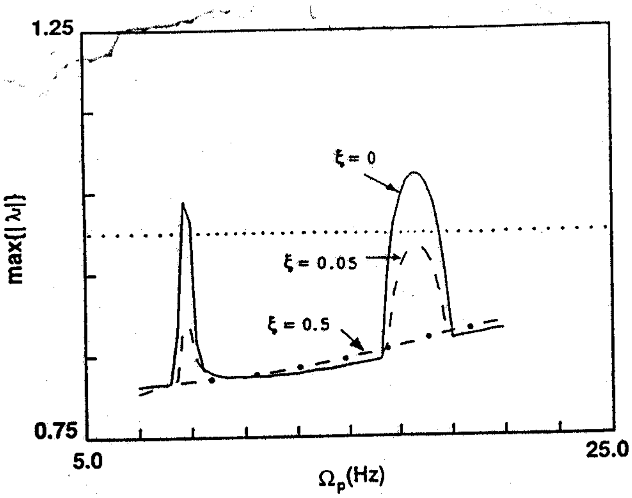

Fig. 6 Effect of increasing the damping ratio on the unstable peaks of Fig. 5. Dotted line corresponds to $|\lambda| = 1$.

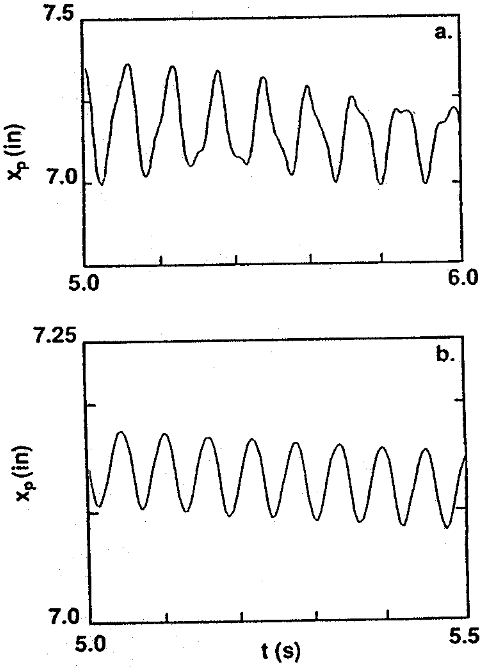

Fig. 7 Computer simulation results for the single acting cylinder piston displacement x_p for a speed Ω_p of 17.5Hz. (*a*) Case of damping ratio ξ = 0 leading to a subharmonic transient response at 8.75Hz, (*b*) Case of ξ = 0.5 with response at the excitation frequency of 17.5Hz.

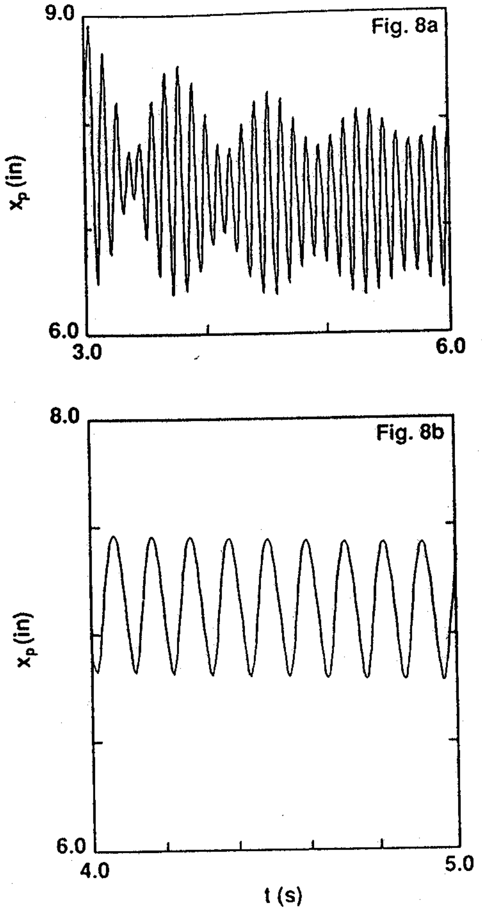

Fig. 8 Computer simulation results for the single acting cylinder piston displacement x_p for an excitation speed Ω_p of 9.5Hz. (*a*) Case of damping ratio ξ = 0 leading to amplitude modulated transient response, (*b*) Case of ξ = 0.5 with amplitude modulation suppressed.

and a P type solution. This instability is due to one of the N type solutions corresponding to a negative λ_i of magnitude greater than unity. The peak at Ω_p = 9Hz involves $3P$ type solutions. Figure 6 shows that increasing the damping ratio ξ stabilizes the system by lowering the eigenvalue peaks.

Next, computer simulation is used to examine this phenomenon in the corresponding nonlinear system. Figure 7(*a*) displays load displacement x_p time history for Ω_p = 17.5Hz, which is in the neighborhood of the first parametric frequency. It is seen that a steady state is yet to be reached after 5 seconds of operation, a considerably long duration compared with the period of DCV flow area variations. The response is almost periodic with a repetition rate of approximately 8.75Hz, which is a subharmonic of the order 1/2 to the excitation frequency Ω_p. This is in accordance with the LTV model prediction of instability due to an N type basis solution.

Computer simulation results for Ω_p = 9.5Hz, which is in the neighborhood of the second parametric frequency, are shown in Fig. 8(*a*). The response is now governed by a persistent transient leading to an amplitude modulated history with a carrier frequency equal to Ω_p. The LTV model indicates an instability due to a P type solution in this region. Due to the saturation effects not included in the LTV model the response is bounded, but still undesirable for most applications.

Computer simulations with increased damping are repeated for Ω_p = 17.5 and 9.5Hz as shown in Figs. 7(*b*) and 8(*b*), respectively. It is seen that with a sufficiently high ξ, the steady state is reached faster, and subharmonic and amplitude modulation effects are eliminated.

These results indicate that the LTV model predictions concerning stability reveal the transient characteristics of the corresponding open loop nonlinear system. For the actuation system under study it can be used to design against undesirable transient phenomena such as shown in Figs. 7 and 8. It provides an efficient design tool especially when it is necessary to assess the effects of the system parameters over a wide range of possible values. The LTV model successfully captures the parametric behavior due to the on-off flow area variations. However, caution should be exercised because in some cases instability may be due to the nonlinearities that are not taken into account by the LTV model.

8 Conclusion

An experimentally validated periodic LTV model is used along with the Floquet Theory to study the dynamic response of on-off valve controlled pneumatic actuators. A new solution technique called the expanded state space method is developed for the LTV model with staircase coefficient profiles based on Floquet's Theorem. It is noteworthy that a periodic LTV model with staircase coefficient variations and forcing term can be reformulated as a homogeneous LTV model of higher order. This observation forms the basis of the expanded state space method which is computationally superior to the straightforward solution technique. Stability and transient response issues are addressed based on the LTV model and the eigenvalues of the discrete transition matrix. For the example case considered, instability is predicted at a frequency of $\Omega_p = \Omega_N$ and $2\Omega_N$ where Ω_N is the natural frequency of the corresponding LTI system. This is in accordance with the general result concerning parametric instability; $\Omega_p = 2\Omega_N/k$, $k = 1, 2, \ldots$. Low frequencies ($k > 2$) are not critical due to the increased effect of damping at those frequencies. Computer simulation for $\Omega_p = \Omega_N$ and $2\Omega_N$ shows that the nonlinear model response is bounded due to the saturation effects but highly degraded by persistent transients. The LTV model predicts the nature of the transient response correctly based on the type of the unstable basis solution associated with the excitation speed. For example, a subharmonic of order 1/2 is obtained for an N type unstable solution, and an amplitude modulated response with carrier frequency equal to the excitation frequency for a P type unstable solution. Thus, it has been shown that the periodic LTV model can be used to assess the transient characteristics of the actuation system as well as its frequency response.

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