Journal of Sound and Vibration (1993) **168**(1), 71–92

FREQUENCY RESPONSE OF LINEAR SYSTEMS WITH PARAMETER UNCERTAINTIES

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(*Received* 27 *January* 1992, *and in final form* 29 *May* 1992)

The primary objective of the present study is to develop a new analytical method for estimating the frequency response characteristics of a linear time-variant, discrete vibratory system when the mass, stiffness and damping parameters are uncertain. The excitation amplitude is also considered to be random but the frequency is deterministic. Given several simplifying assumptions, a direct product technique is proposed to estimate the mean and standard deviation of the steady state displacement response at the excitation frequency. Application of this theory is demonstrated by several single- and multi-degree-of-freedom examples. The distinction between off- and near-resonance regime characteristics is clearly pointed out. In order to verify the proposed analytical technique, predictions are compared with the results obtained by the Monte-Carlo simulation and/or perturbation methods. It is seen that, unlike the perturbation method, the proposed technique works well in the vicinity of a resonance. Our methodology can be implemented easily, and is reasonably accurate and computationally inexpensive, when compared with the existing methods.

1. INTRODUCTION

Analysis of parameter uncertainties remains a viable area of research, as evident from recent review articles by Ibrahim [1] and Benaroya and Rehak [2]. In particular, forced response characteristics of linear dynamic systems must be understood in order to solve design problems associated with random fluctuations in parameters due to manufacturing variations, measurement uncertainties or modeling inaccuracies. One of the unresolved issues is the estimation of response moments under harmonic excitation [1–6]. In this study, we address this specific issue by developing a new analytical technique for discrete vibratory system with uncertain inertial, elastic and damping properties. The excitation amplitude is also considered to be random but the frequency is deterministic.

The first order perturbation methods have been used commonly to determine the frequency response characteristics with limited success [1–6]. Previous investigators [4–6] have pointed out that in the neighborhood of resonant frequencies the standard deviation of the displacement response becomes quite large, indicating greater uncertainty in such regions. This is obviously due to the existence of secular and near secular terms. Stochastic finite element methods have also been developed, but these are based on the theory of first order perturbation [7–9]. Additionally, the Monte-Carlo simulation method [10] can always be employed. However, such a simulation is computationally intensive since a large number of iterations is needed to estimate the probability distributions. Based on this literature review and earlier assessments by prior investigators [1–3], it is clear that new analytical techniques are definitely needed to overcome the deficiencies of existing methods. The method proposed here is a step in this direction.

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2. PROBLEM FORMULATION

The scope of this paper is limited to a linear time invariant, viscously damped vibratory system of dimension N with uncertain parameter matrices **M**, **C** and **K**; each matrix is assumed to be symmetric and positive definitive. For harmonic excitation at a given deterministic frequency Ω but with a randomly varying complex amplitude vector $\mathbf{F}(\Omega)$ of dimension N, the random differential equations can be given in matrix form as follows (see Appendix B for the identification of symbols):

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) = \mathbf{F}(\Omega)\,e^{j\Omega t}. \tag{1}$$

Since only the steady state harmonic response $\mathbf{x}(t) = \mathbf{X}(\Omega)\,e^{j\Omega t}$ is of interest, initial conditions are $\mathbf{x}(0) = \mathbf{0}$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$. Here, $\mathbf{X}(\Omega)$ is the complex valued response. In terms of the dynamics stiffness matrix $\mathbf{G}(\Omega)$,

$$\mathbf{X}(\Omega) = \mathbf{G}(\Omega)^{-1}\mathbf{F}(\Omega), \qquad \mathbf{G}(\Omega) = (-\Omega^2\mathbf{M} + j\Omega\mathbf{C} + \mathbf{K}). \tag{2a, b}$$

The primary objective of this paper is to determine the frequency response characteristics of single- and multi-degree-of-freedom systems. Specifically, mean $\langle|X_i(\Omega)|\rangle$ and standard deviation $\sigma(|X_i(\Omega)|)$ of displacement magnitude $|X_i(\Omega)|$ are examined and compared with the deterministic response $\langle|\bar{X}_i|\rangle$ over the applicable frequency range of interest. A new analytical technique is proposed and several numerical examples are taken to illustrate the theory. Predictions are compared with the results yielded by the commonly used first order perturbation technique [4, 5] and the Monte-Carlo simulation [10]. For the Monte-Carlo method, equation (2) is used to simulate the probabilistic behavior given random fluctuation in **M**, **C**, **K** and **F**. Given the sample size, a random ensemble of parameters is generated by a digital computer. Its distribution must be transformed appropriately; for instance, numerical simulation of a uniform distribution assumes a range from 0 to 1. The simulation is then executed to yield $\langle|X_i(\Omega)|\rangle$ and $\sigma(|X_i(\Omega)|)$ at a given value of Ω. Even though the Monte-Carlo simulation is considered here as the benchmark method, caution must be exercised in interpreting its results, since accuracy depends on the finite ensemble size and prediction may vary from trial to trial [10].

The following simplifying assumptions are made to develop the new solution methdology: (i) random matrices and vectors of equation (1) or (2) can be given by the sum of deterministic or mean (identified by bar) and random or fluctuating components (identified by tilde), i.e., $\mathbf{M} = \bar{\mathbf{M}} + \tilde{\mathbf{M}}$, $\mathbf{C} = \bar{\mathbf{C}} + \tilde{\mathbf{C}}$, $\mathbf{K} = \bar{\mathbf{K}} + \tilde{\mathbf{K}}$, $\mathbf{F} = \bar{\mathbf{F}} + \tilde{\mathbf{F}}$ and $\mathbf{X} = \bar{\mathbf{X}} + \tilde{\mathbf{X}}$; (ii) expected means of system matrices $\bar{\mathbf{M}} = \langle\mathbf{M}\rangle$, $\bar{\mathbf{C}} = \langle\mathbf{C}\rangle$, $\bar{\mathbf{K}} = \langle\mathbf{K}\rangle$ and excitation amplitude $\langle\mathbf{F}\rangle = \bar{\mathbf{F}}$ are known; (iii) probability distributions of **M**, **C**, **K** and **F** are of the same type and are known; (iv) means of a random parameter matrix and $\tilde{\mathbf{F}}$ are equal to null, e.g. $\langle\tilde{\mathbf{M}}\rangle = \mathbf{0}$, $\langle\tilde{\mathbf{F}}\rangle = \mathbf{0}$; (v) parameter fluctuations are much smaller compared to the deterministic values, i.e. $\|\tilde{\mathbf{M}}\| \ll \|\bar{\mathbf{M}}\|$; (vi) covariances of parameter fluctuations are known in the form of cross-correlation matrices such as $\mathbf{R}_{MM} = \langle\tilde{\mathbf{M}}\otimes\tilde{\mathbf{M}}\rangle = \mathrm{Var}(\mathbf{M})$ and $\mathbf{R}_{M,K} = \langle\tilde{\mathbf{M}}\otimes\tilde{\mathbf{K}}\rangle = \mathrm{Cov}(\mathbf{M}, \mathbf{K})$; and (vii) matrix $\mathbf{G}(\Omega)$ can be inverted numerically.

3. SINGLE-DEGREE-OF-FREEDOM SYSTEM

3.1. THEORY

The frequency response of a one-dimensional system with parameter uncertainties, as given by equation (2), is

$$X(\Omega) = \bar{X}(\Omega) + \tilde{X}(\Omega) = G^{-1}(\Omega)F(\Omega) = (\bar{G}(\Omega) + \tilde{G}(\Omega))^{-1}(\bar{F}(\Omega) + \tilde{F}(\Omega))$$
$$= (-\Omega^2(\bar{M} + \tilde{M}) + j\Omega(\bar{C} + \tilde{C}) + \bar{K} + \tilde{K})^{-1}(\bar{F}(\Omega) + \tilde{F}(\Omega)), \tag{3}$$

and the response of the corresponding deterministic system of natural frequency $\bar{\omega}_1 = (\bar{K}/\bar{M})^{0.5}$ and damping ratio $\bar{\zeta}_1 = 0{\cdot}5\bar{C}/(\bar{K}\bar{M})^{0.5}$ is given by the following well known expression:

$$\bar{X}(\Omega) = \bar{G}^{-1}(\Omega)\bar{F}(\Omega), \qquad \bar{G}^{-1} = (-\Omega^2\bar{M} + \mathrm{j}\Omega\bar{C} + \bar{K})^{-1}. \tag{4a, b}$$

From this point on we drop (Ω) for the sake of brevity. The magnitude of deterministic response is

$$|\bar{X}| = |\bar{G}|^{-1}\bar{F} = [(\bar{K} - \Omega^2\bar{M})^2 + \Omega^2\bar{C}^2]^{-0.5}\bar{F}. \tag{5}$$

Before proceeding further, we need to examine the first and second moments of an arbitrary function $h(y) = h(\bar{y} + \tilde{y})$ of the random variable y. The expected values of $h(y)$ is approximated by using the Taylor series expansion as

$$\langle h(y)\rangle \approx \left\langle \bar{h} + \frac{\mathrm{d}\bar{h}}{\mathrm{d}y}\tilde{y} + \frac{1}{2!}\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}y^2}\tilde{y}^2 + \frac{1}{3!}\frac{\mathrm{d}^3\bar{h}}{\mathrm{d}y^3}\tilde{y}^3\ldots\right\rangle$$
$$= \bar{h} + \frac{1}{2!}\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}y^2}\langle\tilde{y}^2\rangle + \frac{1}{3!}\frac{\mathrm{d}^3\bar{h}}{\mathrm{d}y^3}\langle\tilde{y}^3\rangle,$$

and variance is given as follows by considering first four central moment terms:

$$\mathrm{Var}\,(h(y)) = \langle(h - \langle h\rangle)^2\rangle$$
$$\approx \left\langle\left(\frac{\mathrm{d}\bar{h}}{\mathrm{d}y}\tilde{y} + \frac{1}{2!}\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}y^2}\tilde{y}^2 + \frac{1}{3!}\frac{\mathrm{d}^3\bar{h}}{\mathrm{d}y^3}\tilde{y}^3\right)^2 - \left(\frac{1}{2!}\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}y^2}\langle\tilde{y}^2\rangle\frac{1}{2!} + \frac{1}{3!}\frac{\mathrm{d}^3\bar{h}}{\mathrm{d}y^3}\langle\tilde{y}^3\rangle\right)^2\right\rangle$$
$$\approx \left\langle\left(\frac{\mathrm{d}\bar{h}}{\mathrm{d}y}\right)^2 R_{y,y} + \frac{\mathrm{d}\bar{h}}{\mathrm{d}y}\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}y^2}\tilde{y}^3 + \frac{1}{4}\left(\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}y^2}\right)^2(\tilde{y}^4 - \langle\tilde{y}^2\rangle^2) + \frac{2}{3!}\frac{\mathrm{d}\bar{h}}{\mathrm{d}y}\frac{\mathrm{d}^3\bar{h}}{\mathrm{d}y^3}\tilde{y}^4\ldots\right\rangle. \tag{6a, b}$$

Suppose that a deterministic variable $u = \bar{u}$ is given such that $h = y^{-1}u = (\bar{y} + \tilde{y})^{-1}\bar{u}$. Equation (6b) can be rewritten as

$$\mathrm{Var}\,(h(y, u)) \approx [(\bar{y}^2 - (R_{y,y} + 2y^{-1}\langle\tilde{y}^3\rangle + \bar{y}^{-2}\langle 3\tilde{y}^4 - 2\langle\tilde{y}^2\rangle^2\rangle))^{-1} - \bar{y}^{-2}]\bar{u}^2. \tag{7}$$

If we retain only the first and second central moment terms, equations (6a) and (7) are then reduced to

$$\langle h(y, u)\rangle \approx (y - R_{y,y})^{-1}\bar{u}, \qquad \mathrm{Var}\,(h(y), u) \approx [(\bar{y}^2 - R_{y,y})^{-1} - \bar{y}^{-2}]\bar{u}^2. \tag{8a, b}$$

Taking the moments of equation (3) and using equation (8b), the variance terms of response amplitude X, associated with random fluctuations individually in M, C, K or F, are found to be as follows; here it is assumed that various covariance terms such as $R_{M,K}$ and $R_{M,F}$ are zero:

$$P_{XM} = |\langle X^2\rangle - \langle X\rangle^2| \approx |[(\bar{G}^2 - R_{G,G})^{-1} - \bar{G}^{-2}]\bar{F}^2|, \qquad R_{G,G} = \Omega^4 R_{M,M},$$
$$P_{XC} = |\langle X^2\rangle - \langle X\rangle^2| \approx |[(\bar{G}^2 - (\mathrm{j}\Omega)^2 R_{C,C})^{-1} - \bar{G}^{-2}]\bar{F}^2|,$$
$$P_{XK} = |\langle X^2\rangle - \langle X\rangle^2| \approx |[(\bar{G}^2 - R_{K,K})^{-1} - \bar{G}^{-2}]\bar{F}^2|,$$
$$P_{XF} = |\langle X^2\rangle - \langle X\rangle^2| \approx |\bar{G}^{-2}R_{F,F}|. \tag{9a–e}$$

The total standard deviation of the response amplitude is

$$\sigma(|X|) = [\mathrm{Var}\,(|X|)]^{0.5} = [P_{XM} + P_{XC} + P_{XK} + P_{XF}]^{0.5}. \tag{10}$$

The corresponding expected mean from (8a) is given as follows (note that $\langle|X|\rangle \neq |\bar{X}|$):

$$\langle|X|\rangle = \langle|\bar{X} + \tilde{X}|\rangle = |(\bar{G} - \bar{G}^{-1}\,\mathrm{Var}\,(G))^{-1}|F. \tag{11}$$

All expressions can be normalized by taking the corresponding deterministic system response as a reference. Accordingly, equations (10) and (11) are normalized:

$$\sigma(|X|)/|\bar{X}| = [P_{XM} + P_{XC} + P_{XK} + P_{XF}]^{0\cdot5}/|\bar{X}|,$$
$$\langle|X|\rangle/|\bar{X}| = |(\bar{G} - \bar{G}^{-1}\,\mathrm{Var}\,(G))^{-1}|/|\bar{G}|. \qquad (12a, b)$$

3.2. LIMITATIONS AND MODIFICATIONS

Four cases in which the theory of section 3.1 must be modified are discussed next. First, equation (7) must be used when the third and fourth central moment terms are not relatively smaller than the second central moment term. Such a case arises for a virtually undamped system, especially near the resonance. Second, when $\Omega^2\bar{C}^2$ is close to $\Omega^4 R_{M,M}$ or $R_{K,K}$, then equation (10) will not yield accurate answers at the resonance. This will be illustrated later through a numerical example. Third, equation (11) may not yield an accurate answer for a lightly damped system, say when $\zeta = o(0\cdot01)$, within the resonance regime because of numerical inversion problems. Such a regime can be given by $\bar{\omega}_{di} - \epsilon\bar{\omega}_{di} \leqslant \Omega \leqslant \bar{\omega}_{di} + \epsilon\bar{\omega}_{di}$, where ϵ is a random variable because of the statistical nature of the system. For the sake of convenience, define $\epsilon = \sigma_{\omega di}/\bar{\omega}_{di}$, where the standard deviation $\sigma_{\omega di}$ can be found from the random eigensolution [11]. The standard deviation of damped natural frequency ω_{di} associated with random fluctuations M and K is approximated as follows, provided that $\mathrm{var}\,(\zeta) \ll 1$ [11]:

$$\sigma_{\omega di} \approx 0\cdot5\bar{\omega}_{di}(R_{M,M}/\bar{M}^2 - 2R_{M,K}/(\bar{M}\bar{K}) + R_{K,K}/\bar{K}^2)^{0\cdot5}. \qquad (13)$$

In the resonance regime given by $\bar{\omega}_{di} - \epsilon\bar{\omega}_{di} \leqslant \Omega \leqslant \bar{\omega}_{di} + \epsilon\bar{\omega}_{di}$, the following expressions for $\bar{G}(\Omega)$ and $\mathrm{Var}\,(G)$, must be used when computing $\langle|X|\rangle$ by using equation (11);

$$\bar{G}(\Omega) = \bar{G}(\bar{\omega}_{di} - \sigma_{\omega di}) = (\bar{K} - (\bar{\omega}_{di}^2 - 2\bar{\omega}_{di}\sigma_{\omega di} + \sigma_{\omega di}^2)\bar{M}) + \mathrm{j}(\bar{\omega}_{di} - \sigma_{\omega di})\bar{C},$$
$$\mathrm{Var}\,(G) = \mathrm{Var}\,(G(\bar{\omega}_{di} - \sigma_{\omega di})). \qquad (14a, b)$$

Fourth, if system parameters and/or force amplitude are correlated with each other, then the covariance terms such as $R_{M,K}$ and $R_{M,F}$ must be considered. For instance, consider the case in which M and K are fully correlated. Equation (9b) now is modified as follows:

$$R_{G,G} = \Omega^4(R_{M,M} - 2\Omega^2(R_{M,M}R_{K,K})^{0\cdot5} + R_{K,K}. \qquad (15)$$

3.3. FIRST ORDER PERTURBATION ANALYSIS

If $|\tilde{G}| \ll |\bar{G}|$, equation (9) may be approximated by using the first order perturbation analysis to yield the individual response variances identified below by the Q terms; in each case only one randomness such as equation (9) is considered:

$$Q_{XM} = \Omega^4|\bar{G}^{-2}R_{M,M}\bar{X}^2| = \Omega^4|\bar{G}^{-2}\delta_M^2\bar{M}^2\bar{X}^2|, \qquad R_{M,M} = \langle\tilde{M}^2\rangle = \delta_M^2\bar{M}^2,$$
$$Q_{XC} = \Omega^4|\bar{G}^{-2}\delta_C^2\bar{C}^2\bar{X}^2|, \qquad \delta_C^2 = R_{C,C}/\bar{C}^2,$$
$$Q_{XK} = \Omega^4|\bar{G}^{-2}\delta_K^2\bar{K}^2\bar{X}^2|, \qquad \delta_K^2 = R_{K,K}/\bar{K}^2,$$
$$Q_{XF} = |\bar{G}|^{-2}\delta_F^2\bar{F}^2, \qquad \delta_F^2 = R_{F,F}/\bar{F}^2. \qquad (16a\text{–}h)$$

Thus, the total standard deviation of response amplitude is

$$\sigma(|X|) = [\mathrm{Var}\,(|X|)]^{0\cdot5} = [Q_{XM} + Q_{XC} + Q_{XK} + Q_{XF}]^{0\cdot5}. \qquad (17)$$

The expected mean of X, as defined by equation (11), is approximated as

$$|\langle X\rangle| = |\bar{X} + \langle\tilde{X}\rangle| \approx |(\bar{G}^{-1} + \bar{G}^{-3}[\mathrm{Var}\,(G)])F|. \qquad (18)$$

Again, equations (17) and (18) are normalized as

$$\sigma(|X|)/|\bar{X}| = [Q_{XM} + Q_{XC} + Q_{XK} + Q_{XF}]^{0\cdot5}/|\bar{X}|, \qquad |\langle X\rangle|/|\bar{X}| \approx |1 + \bar{G}^{-2}[\mathrm{Var}\,(G)]|. \tag{19a, b}$$

Equations (16)–(19) yield results identical to those given by the Taylor series expansion or perturbation method [4, 5]. Errors associated with the perturbation analysis can be determined by comparing P terms of equation (9) with corresponding Q terms of equation (16). The difference between any P and Q term is expected to be large near and at the resonance, provided that $\bar{\zeta}_1$ is very small.

3.4. EXAMPLE I: VISCOELASTIC SYSTEM

The deterministic parameters of the system shown in Figure 1(a) are chosen to be $\bar{C} = 1$, $\bar{K} = 1$ and $\bar{F} = 1$. Random variation is considered only in the damper and uniformly distributed, i.e., $R_{C,C} = \langle \tilde{C}^2\rangle = 0{\cdot}01$ and $R_{K,K} = R_{F,F} = R_{F,K} = 0$. Excitation is considered from $\Omega = 0$ to 2 rad/s and the corresponding results for the $\langle|X|\rangle$ and $\sigma(|X|)/|\bar{X}|$ spectra are shown in Figure 2. It is seen that the predictions yielded by the proposed method (equations (11) and (12a)) are in virtual agreement with the results of existing methods, namely the perturbation method (equations (18) and (19a)) and the Monte-Carlo

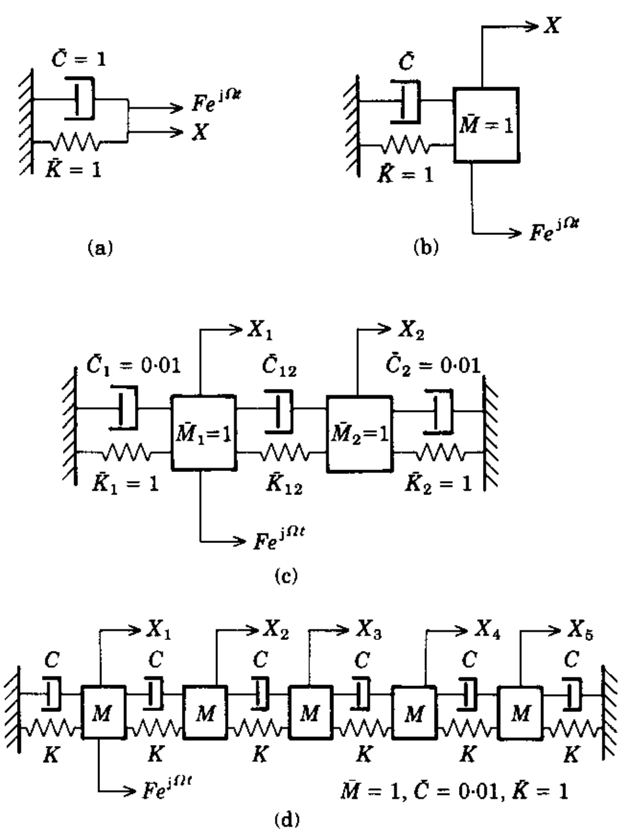

Figure 1. Physical systems used to illustrate and validate the proposed method: (a) viscoelastic system (example I); (b) single-degree-of-freedom system (examples II–IV); (c) two-degree-of-freedom system (examples V–VIII); and (d) five-degree-of-freedom periodic system (example IX). Deterministic system parameters are given here. Also the locations of harmonic force are shown.

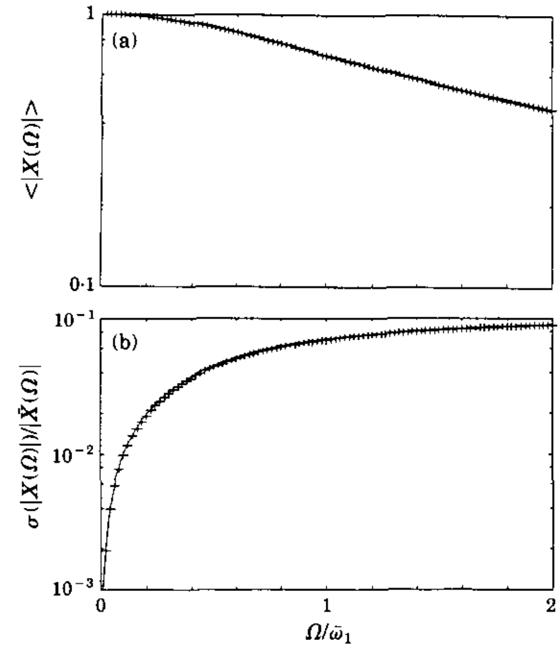

Figure 2. Comparison of frequency response spectra for example 1 shown in Figure 1(a): (a) mean $\langle|X(\Omega)|\rangle$; (b) standard deviation $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$. ——, Proposed technique; ----, first order perturbation method; + + +, Monte-Carlo simulation.

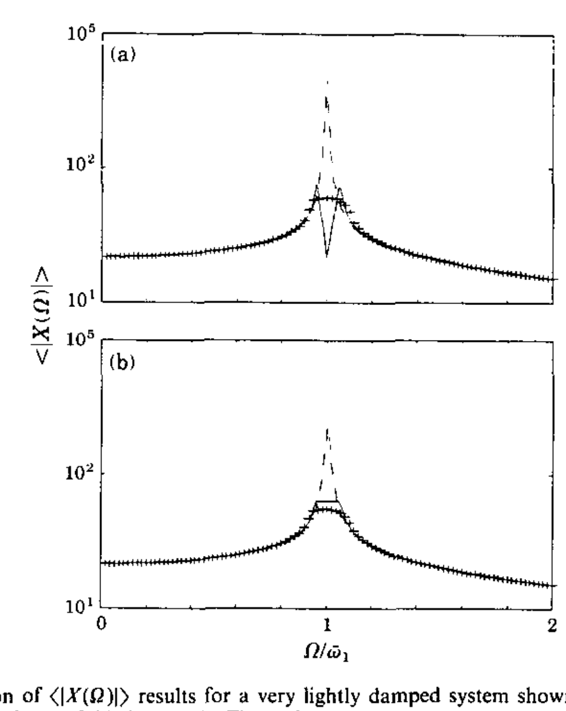

Figure 3. Comparison of $\langle|X(\Omega)|\rangle$ results for a very lightly damped system shown in Figure 1(b) (example II) with $\bar{\zeta}_1 = 0{\cdot}005$ and $R_{M,M} = 0{\cdot}01$. Key as in Figure 2.

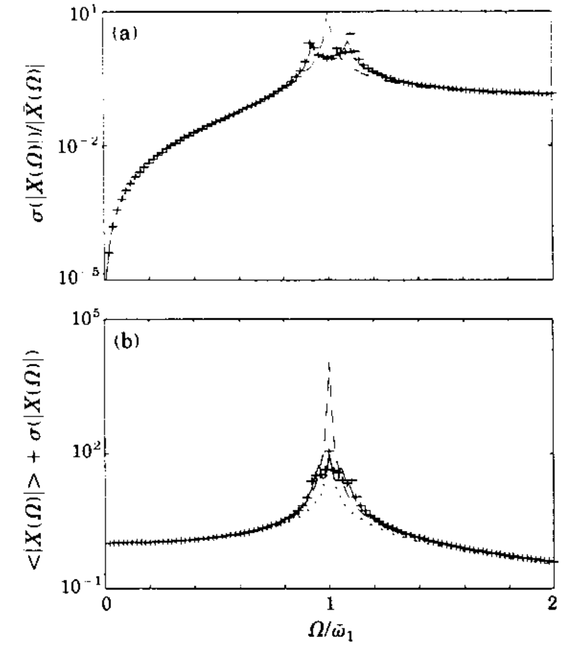

Figure 4. Frequency response spectra for example II: (a) $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$; (b) $|\bar{X}(\Omega)|$ *vs.* probabilistic $\langle|X(\Omega)|\rangle + \sigma(|X(\Omega)|)$. Key as in Figure 2, except; $\cdots$, deterministic response $|\bar{X}(\Omega)|$.

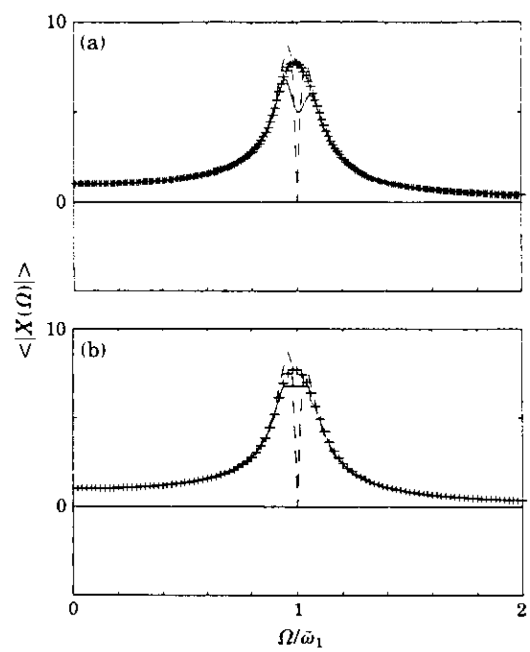

Figure 5. Comparison of $\langle|X(\Omega)|\rangle$ results for a lightly damped system (example III) with $\bar{\zeta}_1 = 0{\cdot}05$. Key as in Figure 2.

simulation (sample size = 400). This is expected since no resonance-like behavior leading to large displacement is seen in this case.

3.5. EXAMPLE II: VERY LIGHTLY DAMPED SYSTEM

Consider the physical system shown in Figure 1(b) with $\bar{M}=1$, $\bar{K}=1$, $\bar{C}=0{\cdot}01$ (or $\zeta_1=0{\cdot}005$), $\bar{F}=1$ and $\Omega=0$ to 2 rad/s. For illustration purposes, only the randomness in mass with uniform distribution is included, i.e., $R_{M,M}=\langle\tilde{M}^2\rangle=0{\cdot}01$ and $R_{C,C}=R_{K,K}=R_{F,F}=R_{M,C}=R_{M,K}=R_{M,F}=R_{C,K}=R_{C,F}=0$. Equations (11), (12a) and (14) are used to predict the $\langle|X|\rangle$, $\sigma(|X|)/|\bar{X}|$ and $\langle|X|\rangle+\sigma(|X|)$ spectra, as shown in Figures 3 and 4. Third and fourth central moment terms are concluded in equation (12a). In each case, results obtained by the perturbation method (equations (18) and (19a)) and the Monte-Carlo simulation (sample size = 400) are also plotted on the same graphs. It is seen that the perturbation method deviates from the Monte-Carlo simulation within the resonance regime because of secular or near secular terms. However, our method overcomes this deficiency and is in reasonable agreement with the numerical solutions except for the $\langle|X|\rangle$ value at $\Omega=\bar{\omega}_1$ in Figure 3(a). This is due to the numerical inversion problem associated with $G(\Omega)$. But this discrepancy seems to disappear when $\langle|X|\rangle$ is predicted by equations (11) and (14) within the resonance regime and by equation (11) outside this regime, as shown in Figure 3(b). It should also be pointed out that there is some uncertainty associated with the Monte-Carlo simulation as well since the sample size is finite.

3.6. EXAMPLE III: EFFECT OF DAMPING

Now we increase $\bar{C}$ from 0·01 to 0·1 ($\zeta_1=0{\cdot}05$) and 0·5 ($\zeta_1=0{\cdot}25$) and repeat example III. Figure 5(a) shows the $\langle|X|\rangle$ results for $\bar{C}=0{\cdot}1$ for the three methods, and Figure 5(b) shows the modified expected mean $\langle|X|\rangle$. The perturbation method indicates a singularity at $\Omega=\bar{\omega}_1$, but this is again due to the numerical inversion of $G(\Omega)$. The results of $\sigma(|X|)/|\bar{X}|$, where all methods are in close agreement except for the Monte-Carlo curve at the resonance, are shown in Figure 6; the precise reason for this dip is not obvious. Such discrepancies between the three methods seem to vanish for a highly damped system with $\zeta_1=0{\cdot}25$, as illustrated in Figure 7. Because of high damping, random fluctuations in the displacement response are indeed reduced by about one order of magnitude as we compare Figures 3–7 for both $\langle|X|\rangle$ and $\sigma(|X|)/|\bar{X}|$ spectra.

3.7. EXAMPLE IV: EFFECT OF LARGE RANDOM FLUCTUATIONS

We go beyond the limits of our assumption of samll random fluctuations and repeat example III with $\bar{C}=0{\cdot}1$, but increase $R_{M,M}$ from 0·01 to 0·25. Such a fluctuation is indeed large as $\delta_M=0{\cdot}5$. Results for $\sigma(|X|)/|\bar{X}|$ are shown in Figure 8(a), and we observe that the Monte-Carlo curve is quite different from the perturbation method curve, but our technique is closer to the Monte-Carlo simulation. This demonstrates that our method is capable of handling second and higher order perturbations, even though earlier we had assumed that $\tilde{M}\ll\bar{M}$. This is seen, from Figure 8(b) as well, where $\langle|X|\rangle+\sigma(|X|)$ spectra are compared; observe that our method predicts results in reasonable proximity of numerical solutions even though minor peaks and valleys do not match.

3.8. EXAMPLE V: CORRELATED MASS AND STIFFNESS

Next, we reconsider example II for the case in which M and K are fully correlated, i.e., $R_{M,M}=(0{\cdot}2)^2$, $R_{K,K}=(0{\cdot}1)^2$ and $R_{M,K}=(R_{M,M}R_{K,K})^{0{\cdot}5}$. Using the analytical expression given by equation (15), response spectra are predicted and compared with two existing methods

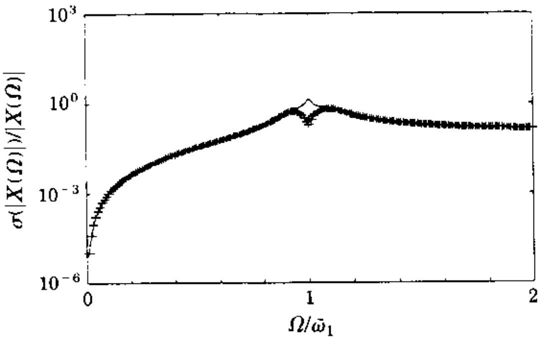

Figure 6. Frequency response spectra of $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$ for example III. Key as in Figure 2.

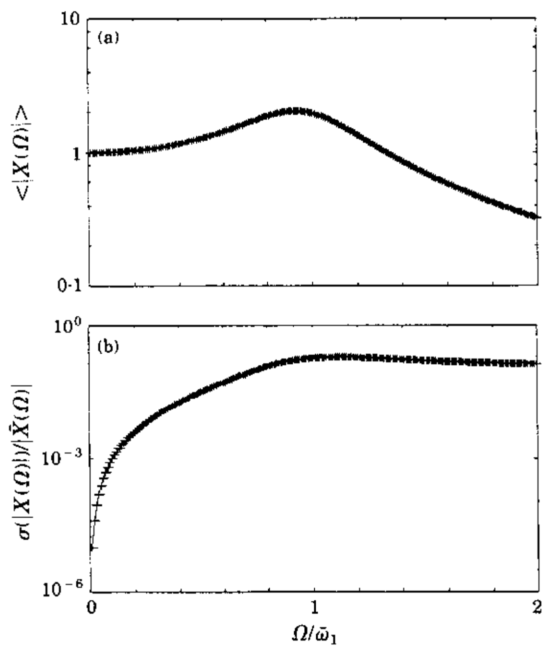

Figure 7. Comparison of frequency response spectra for a damped system (example III) with $\bar{\zeta}_1 = 0{\cdot}25$: (a) mean $\langle|X(\Omega)|\rangle$; (b) standard deviation $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$. Key as in Figure 2.

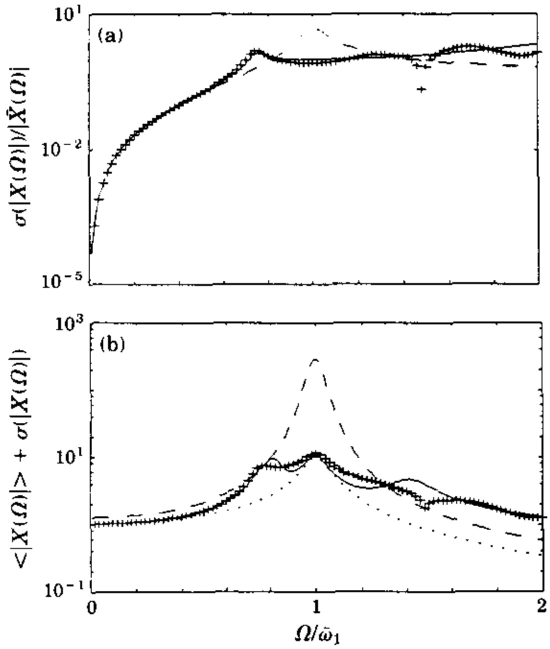

Figure 8. Effect of large random fluctuations examined by example IV with $R_{M,M} = 0{\cdot}25$ and $\bar{\zeta}_1 = 0{\cdot}05$: (a) standard deviation $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$; (b) probabilistic $\langle|X(\Omega)|\rangle + \sigma(|X(\Omega)|)$ *vs.* deterministic $|\bar{X}(\Omega)|$ response characteristics. Key as in Figure 2, except; $\cdots$, deterministic system.

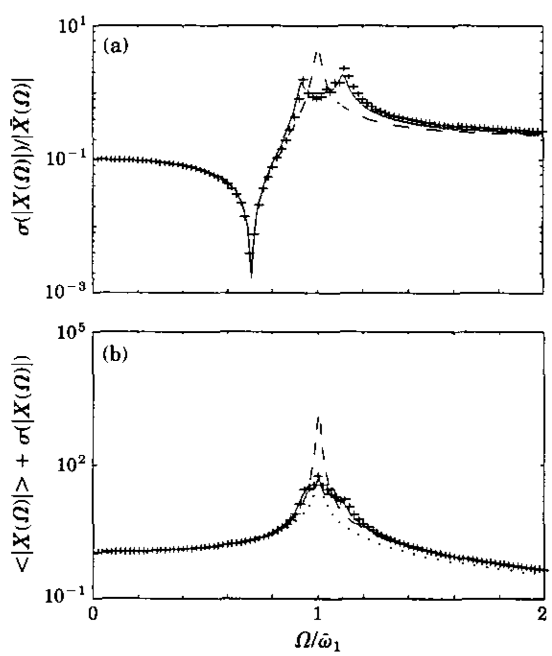

Figure 9. Comparison of results for a fully correlated case (example V) with $R_{M,K} = (R_{M,M}R_{K,K})^{0{\cdot}5}$: (a) Standard deviation $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$; (b) probabilistic $\langle|X(\Omega)|\rangle + \sigma(|X(\Omega)|)$ *vs.* deterministic response $|\bar{X}(\Omega)|$. Key as in Figure 2, except ——, $|\bar{X}(\Omega)|$.

in Figure 9. Observe that the $\sigma(|\bar{X}(\Omega)|)/\langle|\bar{X}(\Omega)|\rangle$ curve is different from previously discussed results, but predictions again match with the Monte-Carlo simulation.

4. MULTI-DEGREE-OF-FREEDOM-SYSTEMS

4.1. THEORY

The mathematical expressions presented in section 3.1 will now be generalized for a discrete system of dimension N. The frequency response of equation (2) is given as follows (again, (Ω) is dropped):

$$\mathbf{X} = \bar{\mathbf{X}} + \tilde{\mathbf{X}} = \mathbf{G}^{-1}\mathbf{F} = (-\Omega^2(\bar{\mathbf{M}} + \tilde{\mathbf{M}}) + \mathrm{j}\Omega(\bar{\mathbf{C}} + \tilde{\mathbf{C}}) + \bar{\mathbf{K}} + \tilde{\mathbf{K}})^{-1}(\bar{\mathbf{F}} + \tilde{\mathbf{F}}), \quad (20)$$

and the response of the corresponding deterministic system is

$$\bar{\mathbf{X}} = \bar{\mathbf{G}}^{-1}\bar{\mathbf{F}} = (-\Omega^2\bar{\mathbf{M}} + \mathrm{j}\Omega\bar{\mathbf{C}} + \bar{\mathbf{K}})^{-1}\bar{\mathbf{F}}. \quad (21)$$

Recall equation (6), which was used to develop moments of a random variable. Equation (6) is now generalized by taking an arbitrary random vector $\mathbf{h}(\mathbf{y}, \bar{\mathbf{u}}) = \mathbf{y}^{-1}\bar{\mathbf{u}}$, where $\mathbf{y} = \bar{\mathbf{y}} + \tilde{\mathbf{y}}$ is a symmetric random matrix of dimension N and $\mathbf{u} = \bar{\mathbf{u}}$ is a deterministic vector of dimension N. The Taylor matrix expansion yields the following; see also Appendix A for a few direct product calculations:

$$\langle\mathbf{h}\rangle = \left\langle \bar{\mathbf{h}} + \mathscr{D}_{\mathrm{cs(y)^T}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})] + \frac{1}{2!}\mathscr{D}^2_{\mathrm{cs(y)^{\otimes 2T}}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})^{\otimes 2}] + \frac{1}{3!}\mathscr{D}^3_{\mathrm{cs(y)^{\otimes 3T}}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})^{\otimes 3}]\dots\right\rangle. \quad (22)$$

After some manipulation, equation (22) is approximated up to third order terms as

$$\langle\mathbf{h}(\mathbf{y})\rangle \approx \langle\bar{\mathbf{h}} + \mathbf{y}^{-1}\mathbf{y}_1\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}} + \bar{\mathbf{y}}^{-1}\mathbf{y}_2\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}} + \bar{\mathbf{y}}^{-1}\mathbf{y}_3\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}}\rangle = \bar{\mathbf{h}} + \bar{\mathbf{y}}^{-1}\langle\mathbf{y}_2\rangle\bar{\mathbf{h}} + \bar{\mathbf{y}}^{-1}\langle\mathbf{y}_3\rangle\bar{\mathbf{h}}, \quad (23a)$$

and the variance expression given below includes up to fourth central moment terms:

$$\mathrm{Var}(\mathbf{h}) = \langle\mathbf{h}\otimes\mathbf{h}\rangle - \langle\mathbf{h}\rangle\otimes\langle\mathbf{h}\rangle$$
$$\approx \mathbf{D}_{y,y}^{-1}\begin{bmatrix}\mathbf{R}_{y,y} - \langle\mathbf{y}_2\rangle\otimes\langle\mathbf{y}_2\rangle + \langle\mathbf{y}_1\otimes\mathbf{y}_2\rangle + \langle\mathbf{y}_2\otimes\mathbf{y}_1\rangle \\ + \langle\mathbf{y}_2\otimes\mathbf{y}_2\rangle + \langle\mathbf{y}_1\otimes\mathbf{y}_3\rangle + \langle\mathbf{y}_3\otimes\mathbf{y}_1\rangle\end{bmatrix}\mathbf{D}_{y,y}^{-1}\mathbf{D}_{u,u}, \quad (23b)$$

where $\mathbf{D}$ is the direct product of two deterministic matrices, e.g., $\mathbf{D}_{y,y} = \bar{\mathbf{y}}\otimes\bar{\mathbf{y}}$. See Appendix A for the definitions of $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$. Equation (23) is now rewritten by retaining only the first and second central moment terms:

$$\langle\mathbf{h}\rangle \approx (\bar{\mathbf{y}} - \mathbf{y}_2)^{-1}\bar{\mathbf{u}},$$
$$\mathrm{Var}(\mathbf{h}) \approx [(\mathbf{D}_{y,y} - \mathbf{R}_{y,y})^{-1} - \mathbf{D}_{y,y}^{-1}]\mathbf{D}_{u,u}. \quad (24a\text{–}b)$$

Using equations (24a), (20) and (21), the expected mean of the response vector is obtained as

$$\langle|\mathbf{X}|\rangle = |\bar{\mathbf{X}} + \langle\tilde{\mathbf{X}}\rangle| \approx |(\bar{\mathbf{G}} - \mathbf{G}_2)^{-1}\bar{\mathbf{F}}|. \quad (25)$$

For the sake of simplicity, as in section 3.1, all cross-covariance matrices such as $\mathbf{R}_{M,K}$ and $\mathbf{R}_{M,F}$ are assumed to be null. Taking first and second moments of equation (20), the individual variance of the response amplitude associated individually with $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$ or $\mathbf{F}$

can be determined by

$$\mathbf{P}_{XM} = |\langle \mathbf{X}\otimes\mathbf{X}\rangle - \mathbf{D}_{X,X}| = |[(\mathbf{D}_{G,G} - \mathbf{R}_{G,G})^{-1} - \mathbf{D}_{G,G}^{-1}]\mathbf{D}_{F,F}|, \qquad \mathbf{R}_{G,G} = \Omega^4\mathbf{R}_{M,M},$$
$$\mathbf{P}_{XC} = |\langle \mathbf{X}\otimes\mathbf{X}\rangle - \mathbf{D}_{X,X}| = |[(\mathbf{D}_{G,G} - (\mathrm{j}\Omega)^2\mathbf{R}_{C,C})^{-1} - \mathbf{D}_{G,G}^{-1}]\mathbf{D}_{F,F}|,$$
$$\mathbf{P}_{XK} = |\langle \mathbf{X}\otimes\mathbf{X}\rangle - \mathbf{D}_{X,X}| = |[(\mathbf{D}_{G,G} - \mathbf{R}_{K,K})^{-1} - \mathbf{D}_{G,G}^{-1}]\mathbf{D}_{F,F}|,$$
$$\mathbf{P}_{XF} = |\langle \mathbf{X}\otimes\mathbf{X}\rangle - \mathbf{D}_{X,X}| = |\mathbf{D}_{G,G}^{-1}\mathbf{R}_{F,F}|. \qquad (26a\text{–}e)$$

The total variance and standard deviation of the displacement amplitude is defined as:

$$\mathrm{Var}\,(|\mathbf{X}|) = \mathbf{P}_{XM} + \mathbf{P}_{XC} + \mathbf{P}_{XK} + \mathbf{P}_{XF},$$
$$\sigma(|X_i|) = |\langle X_i\otimes X_i\rangle - \langle X_i\rangle\otimes\langle X_i\rangle|^{0\cdot5} = [\mathrm{Var}\,(|X_i|)]^{0\cdot5}, \qquad i = 1, 2, \ldots, N. \quad (27a\text{–}b)$$

The effect of cross-correlation matrices such as $\mathbf{R}_{M,K}$ and $\mathbf{R}_{M,C}$ can be included in the analysis in a similar manner; see sections 3.2 and 3.8. When **M** and **K** are assumed to be fully correlated, use the following expression instead of equation (26b):

$$\mathbf{R}_{G,G} = \Omega^4\mathbf{R}_{M,M} - \Omega^2(\mathbf{R}_{M,K} + \mathbf{R}_{K,M}) + \mathbf{R}_{K,K}. \qquad (28)$$

4.2. FIRST ORDER PERTURBATION ANALYSIS

Analytical expressions given by equations (25) and (26) are approximated by using the first order expansion as follows, provided that $\|\tilde{G}\| \ll \|\bar{G}\|$:

$$\langle|\mathbf{X}|\rangle = |\bar{\mathbf{X}} + \langle\tilde{\mathbf{X}}\rangle| \approx |(\bar{\mathbf{G}}^{-1} + \bar{\mathbf{G}}^{-1}\mathbf{G}_2\bar{\mathbf{G}}^{-1})\mathbf{F}|, \qquad (29)$$

$$\mathbf{Q}_{XM} = \Omega^4|\mathbf{D}_{G,G}^{-1}\mathbf{R}_{M,M}\mathbf{D}_{X,X}|, \qquad \mathbf{Q}_{XC} = \Omega^2|\mathbf{D}_{G,G}^{-1}\mathbf{R}_{C,C}\mathbf{D}_{X,X}|,$$
$$\mathbf{Q}_{XK} = |\mathbf{D}_{G,G}^{-1}\mathbf{R}_{K,K}\mathbf{D}_{X,X}|, \qquad \mathbf{Q}_{XF} = |\mathbf{D}_{G,G}^{-1}\mathbf{R}_{F,F}|. \qquad (30a\text{–}d)$$

Like for the single-degree-of-freedom system theory, here **Q** is the variance of the response amplitude given only one uncertainty, e.g., $\mathbf{Q}_{XM}$ considers only the effect of randomness in inertial properties. The total variance and standard deviation are given as

$$\mathrm{Var}\,(|\mathbf{X}|) = \mathbf{Q}_{XM} + \mathbf{Q}_{XC} + \mathbf{Q}_{XK} + \mathbf{Q}_{XF}, \qquad \sigma(|X_i|) = [\mathrm{Var}\,(|X_i|)]^{0\cdot5}, \quad i = 1, 2, \ldots, N. \qquad (31a, b)$$

Equations (29) and (30) yield the same results as those given by the Taylor matrix expansion or perturbation method [4, 5]; it will be illustrated later via a numerical example. The limitations of equations (29) and (30) are exactly the same as those associated with the existing perturbation methods [4, 5]. As for the single-degree-of-freedom system, errors committed by the first order perturbation can be quantified by examining appropriate **P** and **Q** terms.

4.3. EXAMPLE VI: TWO-DEGREE-OF-FREEDOM SYSTEM

Consider the physical system shown in Figure 1(c) with $\bar{K}_{12} = 2$ and a proportionally damped case ($\bar{\mathbf{C}} = 0{\cdot}01\bar{\mathbf{K}}$). The deterministic eigenvalue problem yields the following modal database: $\bar{\omega}_1 = 1{\cdot}0$, $\bar{\omega}_2 = 2{\cdot}236$, $\zeta_1 = 0{\cdot}005$, $\zeta_2 = 0{\cdot}011$, $\mathbf{\Phi}_1^{\mathrm{T}} = [1\ 1]$ and $\mathbf{\Phi}_2^{\mathrm{T}} = [1\ -1]$. As in previous examples, we consider randomness in masses only and set it as $\mathbf{R}_{M,M} = 0{\cdot}01\mathbf{D}_{M,M}$. First, examine selected standard deviation results yielded by a conventional Monte-Carlo simulation, as listed in Table 1. All results are for the case in which the excitation frequency is fixed as 2·4 rad/s, which places it in the vicinity of second resonance. We observe uncertainties, which, as expected, depend on the sample size and the trial index. Accordingly, numerical answers should be considered valid within a certain confidence interval, say about ±5% to 10% about any predicted value. For the sake of computational

TABLE 1

Selected Monte-Carlo results for example VI

Sample size	Trial number	$\sigma(\|X_1\|)/\|\bar{X}_1\|$	$\sigma(\|X_2\|)/\|\bar{X}_2\|$
400	1	1·21	1·69
800	1	1·077	1·51
1200	1	1·232	1·734
1200	2	1·12	1·577
1600	1	1·14	1·59
1600	2	1·13	1·58
1600	3	1·24	1·74

convenience, we choose a sample size of 400 and run only one trial, as in a previous example, and execute simulation from $\Omega = 0$ to 3·45 rad/s. Analogous results obtained by using the proposed direct product technique (equations (23), (25) and (27)) and the first order perturbation method (equations (29) and (30)) are compared with numerical answers in Figures 10 and 11. All three methods match in the off-resonance regimes about $\bar{\omega}_1$ and $\bar{\omega}_2$. The proposed method seems to track all of the trends followed by the numerical simulation, even within the resonance regimes which are obviously well separated.

4.4. EXAMPLE VII: SPECTRAL COUPLING ISSUES

For a deterministic system it is well known that spectral coupling is dictated by the separation between natural frequencies and the damping ratios. Now this concept is analyzed for a two-degree-of-freedom system with random parameters by bringing two natural frequencies closer. Three cases used for this study are listed in Table 2. We note a strong coupling between two resonances as $\bar{\omega}_2 \to \bar{\omega}_1$. Also, two spectral shapes change. However, unlike the proposed method, first order perturbation does not predict the broader and combined resonance very well. This example demonstrates that random fluctuations in **M** and **K** can also couple modes strongly, provided that they are moderately coupled for the corresponding deterministic problem.

4.5. EXAMPLE VIII: CORRELATED MASS AND STIFFNESS

Now we consider the example VII (case c from Table 2) but with non-proportionally damping matrix $\bar{\mathbf{C}} = [0{\cdot}0205\ \ -0{\cdot}0005;\ -0{\cdot}0005\ \ 0{\cdot}0005]$. Here mass and stiffness matrices are assumed to be fully correlated and chosen as $\mathbf{R}_{M,M} = 0{\cdot}05^2\mathbf{D}_{M,M}$, $\mathbf{R}_{K,K} = 0{\cdot}1^2\mathbf{D}_{M,M}$, and $\mathbf{R}_{M,K} = 0{\cdot}005\mathbf{D}_{M,M}$. Using equation (28), response spectra are predicted and compared with two existing methods in Figures 16 and 17. Predictions again match well with the Monte-Carlo simulation.

TABLE 2

Data set used for spectral coupling study

Case	Example	$\bar{K}_{12}$	$\bar{\omega}_2/\bar{\omega}_1$	Results in Figures
a	VI	2	2·23	10, 11
b	VII	0·6	1·5	12, 13
c	VIII	0·05	1·05	14, 15

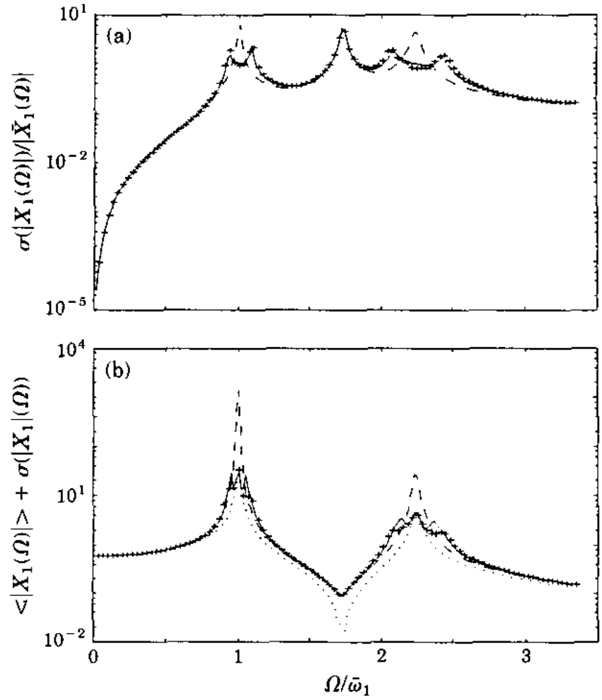

Figure 10. Comparison of $|X_1(\Omega)|$ spectra for example VI shown in Figure 1(c) with $\bar{\omega}_2/\omega_1 = 2{\cdot}23$: (a) standard deviation $\sigma(|X_1(\Omega)|)/|\bar{X}_1(\Omega)|$; (b) probabilistic $\langle|X_1(\Omega)|\rangle + \sigma(|X_1|(\Omega))$ *vs.* deterministic $|\bar{X}_1(\Omega)|$. ——, Proposed technique; ----, first order perturbation method, + + +. Monte-Carlo simulation; · · · ·, deterministic system.

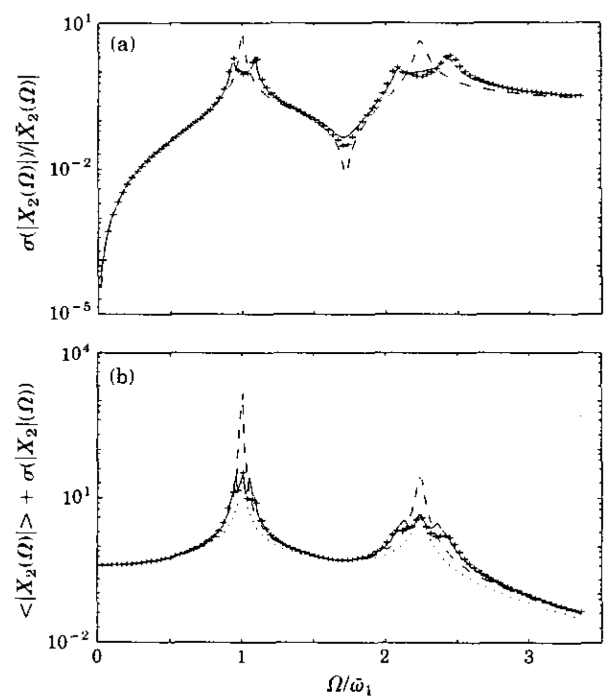

Figure 11. Comparison of $|X_2(\Omega)|$ spectra for example VI shown in Figure 1(c). Key and format as in Figure 10.

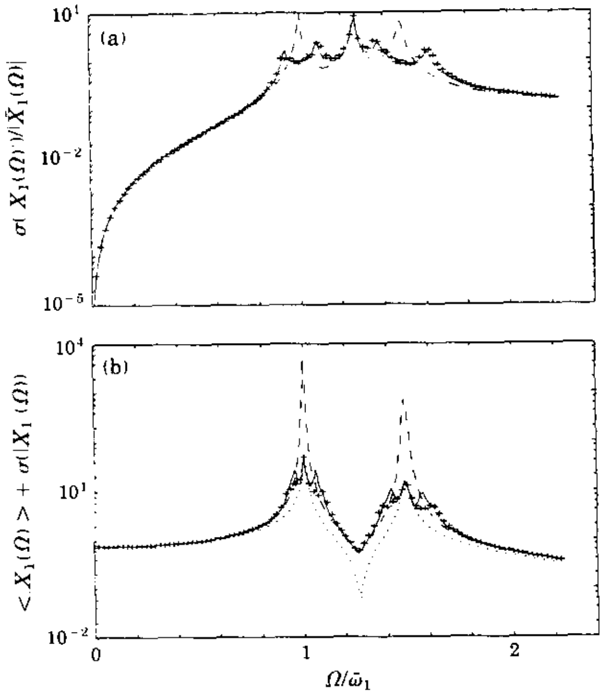

Figure 12. Comparison of $|X_1(\Omega)|$ spectra for example VI shown in Figure 1(c) with $\bar{\omega}_2/\bar{\omega}_1 = 1{\cdot}5$. Key and format as in Figure 10.

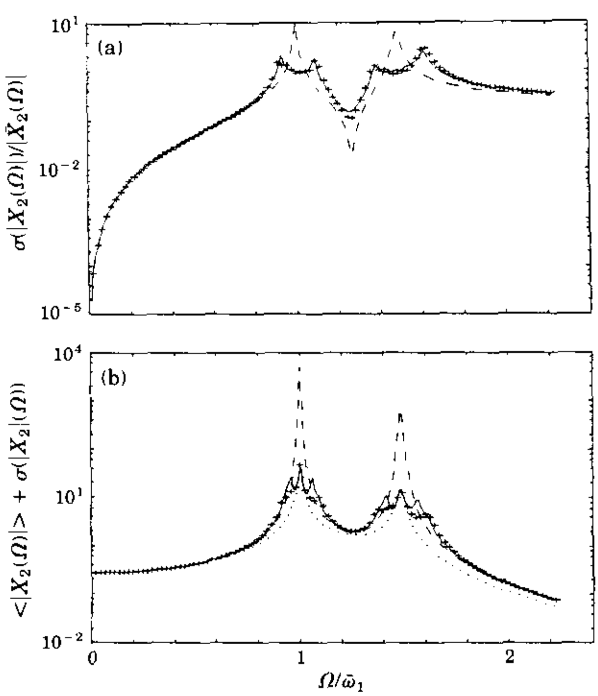

Figure 13. Comparison of $|X_2(\Omega)|$ spectra for example VI shown in Figure 1(c) $\omega_2/\omega_1 = 1{\cdot}5$. Key and format as in Figure 10.

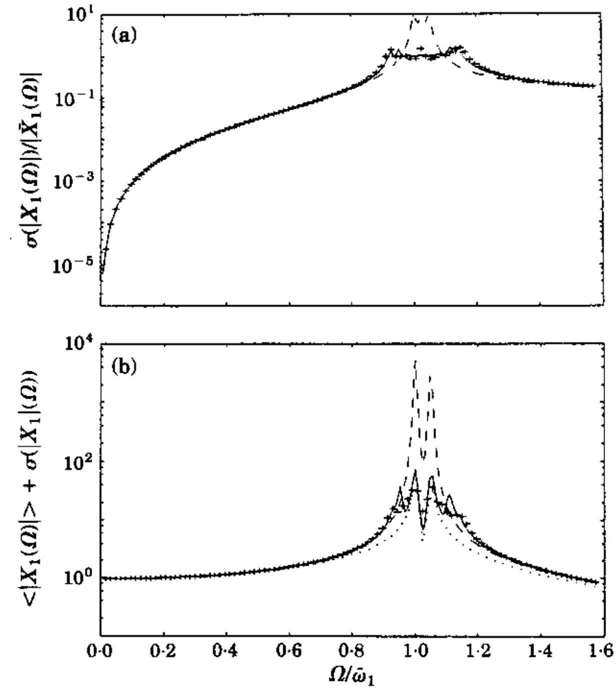

Figure 14. Comparison of $|X_1(\Omega)|$ spectra for example VII shown in Figure 1(c) with $\bar{\omega}_2/\omega_1$. Key and format as in Figure 10.

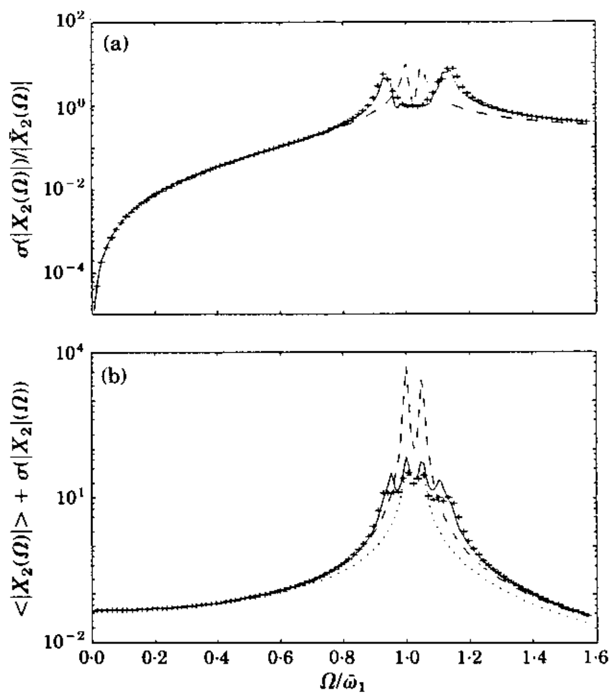

Figure 15. Comparison of $|X_2(\Omega)|$ spectra for example VII shown in Figure 1(c) with $\bar{\omega}_2/\bar{\omega}_1 = 1{\cdot}05$. Key and format as in Figure 10.

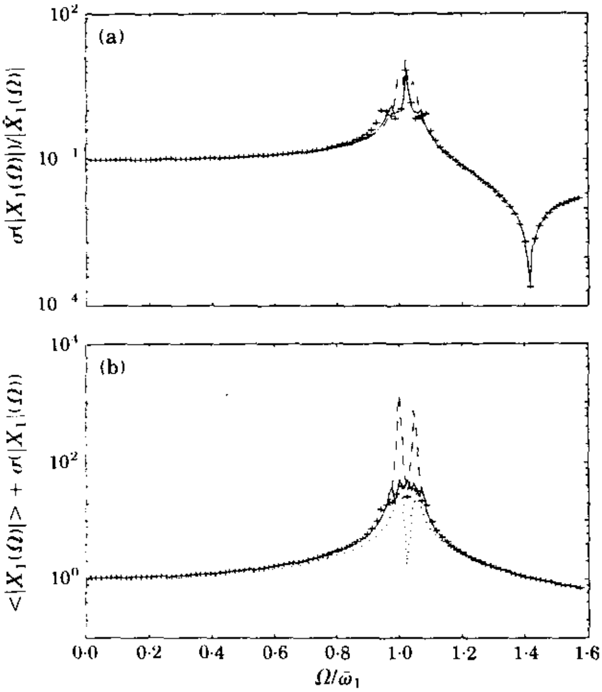

Figure 16. Comparison of $|X_1(\Omega)|$ spectra for example VIII shown in Figure 1(c) with correlated mass and stiffness. Key and format as in Figure 10.

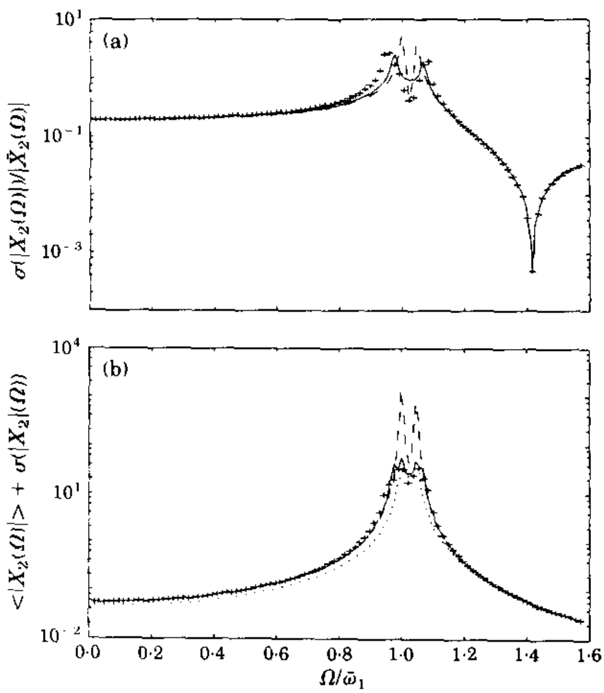

Figure 17. Comparison of $|X_2(\Omega)|$ spectra for example VIII shown in Figure 1(c) with correlated mass and stiffness. Key and format as in Figure 10.

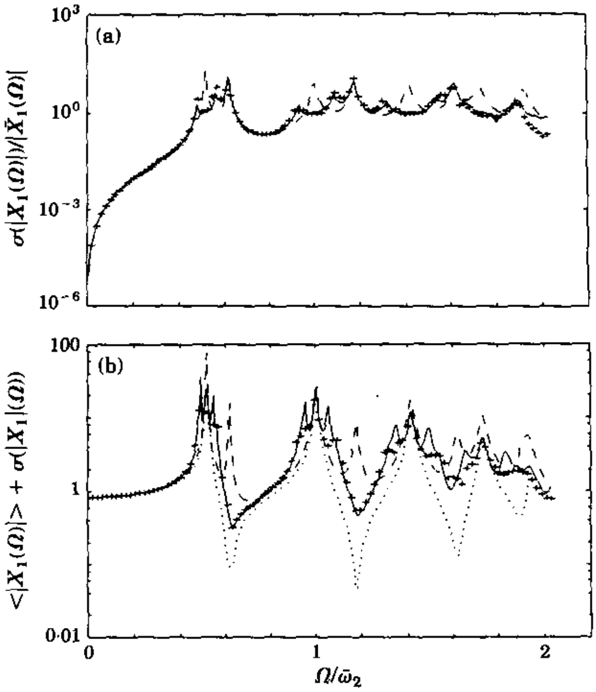

Figure 18. Comparison of $|X_1(\Omega)|$ spectra for example IX shown in Figure 1(d). Key and format as in Figure 10.

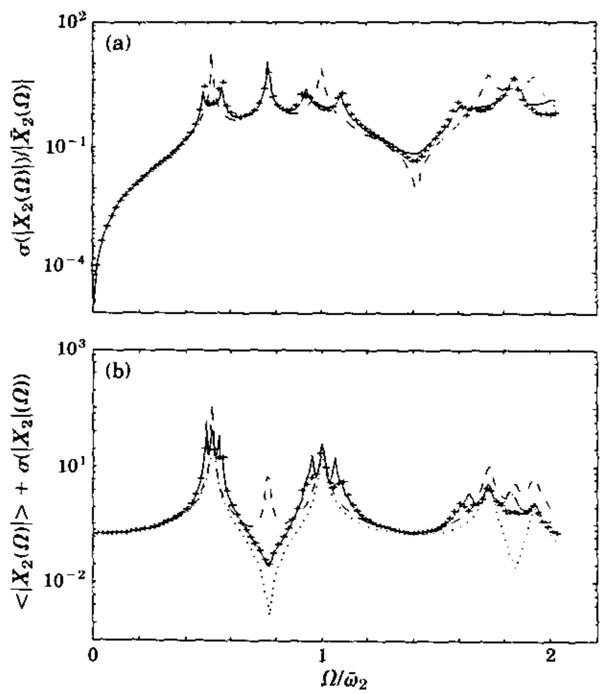

Figure 19. Comparison of $|X_2(\Omega)|$ spectra for example IX shown in Figure 1(d). Key and format as in Figure 10.

4.6. EXAMPLE IX: PERIODIC SYSTEM

The final case examines the periodic system of Figure 1(d) with $N = 5$. The deterministic modal database is as follows: $\bar{\omega}_1 = 0{\cdot}517$, $\bar{\omega}_2 = 1{\cdot}0$, $\bar{\omega}_3 = 1{\cdot}414$, $\bar{\omega}_4 = 1{\cdot}732$ and $\bar{\omega}_5 = 1{\cdot}932$; $\bar{\zeta}_1 = 0{\cdot}0026$, $\bar{\zeta}_2 = 0{\cdot}005$, $\bar{\zeta}_3 = 0{\cdot}007$, $\bar{\zeta}_4 = 0{\cdot}009$ and $\bar{\zeta}_5 = 0{\cdot}01$. Now we introduce randomness via $\mathbf{M}$ and choose $\mathbf{R}_{M,M} = 0{\cdot}01\mathbf{D}_{M,M}$. Spectral characteristics of X_1 and X_2 are shown in Figures 18 and 19. Again, we note that our proposed technique matches well with the numerical answers (sample size = 400), which are obviously valid within a certain confidence interval. Similar comparisons are found for X_3, X_4 and X_5, although not repeated here.

5. CONCLUDING REMARKS

A new analytical methodology has been developed to compute the statistical frequency response characteristics of a linear time invariant, discrete vibratory system with parameter uncertainties and/or random force amplitude. A direct product technique is proposed to estimate the mean and standard deviation of the displacement amplitude response at the deterministic excitation frequency. This theory is validated by comparing the results of several single- and multi-degree-of-freedom examples with the Monte-Carlo simulation. Predictions yielded by the perturbation methods are also given. It is seen clearly that the perturbation method deviates from the Monte-Carlo simulation within the resonance regime(s) because of the secular or near secular terms. However, our method overcomes this deficiency reasonably well, since predictions match resonably well with the numerical solutions. Also, our method is computationally faster when compared with the numerical simulation, especially for multi-degree-of-freedom systems. Although the proposed method is potentially powerful and overcomes a few deficiences of the existing methods, more efforts are definitely required to overcome some of the limitations identified in section 3.2. Furthermore, the proposed method, unlike the Monte-Carlo technique, cannot be applied to a non-linear problem.

ACKNOWLEDGMENTS

We thank the U.S. Army Research Office and The Ohio State University Gear Dynamics and Gear Noise Research Laboratory for supporting this research.

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APPENDIX A: MATRIX DIRECT PRODUCTS

A.1. REVIEW OF KEY CONCEPTS

Consider the direct product of two matrices $\mathbf{A}(p \times q)$ and $\mathbf{B}(s \times t)$, given by $\mathbf{A}\otimes\mathbf{B}$. It is a $ps \times qt$ dimensioned matrix defined as [14]

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1q}\mathbf{B} \\ a_{21}\mathbf{B} & & & \\ \vdots & & & \\ a_{p1}\mathbf{B} & & & a_{pq}\mathbf{B} \end{bmatrix}. \tag{A1}$$

A few well known identities are as follows; also see reference [14] for more details.

$$1\otimes\mathbf{A} = \mathbf{A} = \mathbf{A}\otimes 1,$$

$$(\mathbf{A}\otimes\mathbf{B})^{-1} = \mathbf{A}^{-1}\otimes\mathbf{B}^{-1} \qquad (\text{if } \mathbf{A}^{-1} \text{ and } \mathbf{B}^{-1} \text{ exist}),$$

$$(\mathbf{A}\otimes\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\otimes\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{AC}\otimes\mathbf{BD} = (\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}),$$

$$(\mathbf{A}+\mathbf{B})\otimes(\mathbf{C}+\mathbf{D}) = \mathbf{A}\otimes\mathbf{C} + \mathbf{A}\otimes\mathbf{D} + \mathbf{B}\otimes\mathbf{C} + \mathbf{B}\otimes\mathbf{D}, \tag{A2a–e}$$

Selected matrix operations are described below.

(i) cs(**A**): column transformation of a matrix $\mathbf{A}$ $(p \times q)$:

$$\mathrm{cs}(\mathbf{A}) = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_q \end{bmatrix}, \tag{A3}$$

where $\mathbf{A}_j$ is the jth column of $\mathbf{A}$.

(ii) Matrix derivative:

$$\mathscr{D}_{\mathrm{cs}(\mathbf{A})^{\mathrm{T}}}\mathbf{B} = \left[\frac{\partial\mathbf{B}}{\partial q_{11}}\left|\frac{\partial\mathbf{B}}{\partial a_{21}}\right|\frac{\partial\mathbf{B}}{\partial a_{31}}\right|\cdots\left|\frac{\partial\mathbf{B}}{\partial a_{pq}}\right], \tag{A4}$$

$$\mathscr{D}^2_{\mathrm{cs}(\mathbf{A})^{\mathrm{T}}}\mathbf{B} = \left[\frac{\partial^2\mathbf{B}}{\partial a_{11}^2}\left|\frac{\partial^2\mathbf{B}}{\partial a_{11}\,\partial a_{21}}\right|\frac{\partial^2\mathbf{B}}{\partial a_{11}\,\partial a_{31}}\right|\cdots\left|\frac{\partial^2\mathbf{B}}{\partial a_{pq}^2}\right]. \tag{A5}$$

(iii) Matrix Taylor expansion:

$$\mathbf{B}(\mathbf{A}) = \mathbf{B}(\bar{\mathbf{A}}) + \sum_{m=1}^{L}\frac{1}{m!}\mathscr{D}^m_{\mathrm{cs}(\tilde{\mathbf{A}})^{\otimes m\mathrm{T}}}\mathbf{B}|_{\bar{\mathbf{A}}}[\mathrm{cs}(\tilde{\mathbf{A}})^{\otimes m}\otimes\mathbf{I}_q] + \mathbf{O}(L+1), \tag{A6}$$

where $\mathbf{O}(L+1)$ is a remainder.

A.2. EXAMPLE

Consider a two-dimensional problem given by the random matrix $\mathbf{y}$ (2 × 2) and the deterministic vector $\bar{\mathbf{u}}$ (2 × 1) and define a random function $\mathbf{h}$ (2 × 1) such that $\mathbf{h} = \mathbf{y}^{-1}\bar{\mathbf{u}}$. See section 4.1 for its application. Using Taylor series, we obtain

$$\mathbf{h}(\mathbf{y}, \bar{\mathbf{u}}) = \mathbf{h}(\bar{\mathbf{y}}, \bar{\mathbf{u}}) + \mathscr{D}_{cs(y)^T}\mathbf{h}|_{\bar{\mathbf{h}}}[cs(\tilde{\mathbf{y}})] + \frac{1}{2!}\mathscr{D}^2_{cs(y)^{\otimes 2T}}\mathbf{h}|_{\bar{\mathbf{h}}}[cs(\tilde{\mathbf{y}})^{\otimes 2}]$$

$$+\frac{1}{3!}\mathscr{D}^3_{cs(y)^{\otimes 3T}}\mathbf{h}|_{\bar{\mathbf{h}}}[cs(\tilde{\mathbf{y}})^{\otimes 3}]\ldots, \tag{A7}$$

where

$$\mathscr{D}_{cs(y)^T}\mathbf{h}|_{\bar{\mathbf{h}}}[cs(\tilde{\mathbf{y}})] = \mathscr{D}_{cs(y)^T}\mathbf{y}^{-1}|_{\bar{\mathbf{y}}}[cs(\tilde{\mathbf{y}})\otimes\mathbf{I}_2]\bar{\mathbf{u}},$$

$$= \mathbf{y}^{-1}\left[\begin{bmatrix}1\,0 & 0\,0 & 0\,1 & 0\,0\\ 0\,0 & 1\,0 & 0\,0 & 0\,1\end{bmatrix}\right][cs(\tilde{\mathbf{y}})\otimes\mathbf{I}_2]\mathbf{y}^{-1}\bar{\mathbf{u}} = \mathbf{y}^{-1}\mathbf{y}_1\mathbf{y}^{-1}\bar{\mathbf{u}}, \tag{A8}$$

$$\frac{1}{2!}\mathscr{D}^2_{cs(y)^{\otimes 2T}}\mathbf{h}|_{\bar{\mathbf{h}}}[cs(\tilde{\mathbf{y}})^{\otimes 2}] = \frac{1}{2!}\mathscr{D}^2_{cs(y)^{\otimes 2T}}\mathbf{y}^{-1}|_{\bar{\mathbf{y}}}[cs(\tilde{\mathbf{y}})^{\otimes 2}\otimes\mathbf{I}_2]\bar{\mathbf{u}}$$

$$= \bar{\mathbf{y}}^{-1}\frac{1}{2!}\left[2\begin{bmatrix}1&0\\0&0\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}1&0\\0&0\end{bmatrix}\middle|\begin{bmatrix}1&0\\0&0\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}0&0\\1&0\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}0&0\\1&0\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}1&0\\0&0\end{bmatrix}\middle|\cdots\middle|2\begin{bmatrix}0&0\\0&1\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}0&0\\0&1\end{bmatrix}\right]$$

$$[cs(\tilde{\mathbf{y}})^{\otimes 2}\otimes\mathbf{I}_2]\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}} = \bar{\mathbf{y}}^{-1}\mathbf{y}_2\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}}, \tag{A9}$$

and $\mathbf{y}_3$ can be determined similarly. If $\mathbf{h}$ is approximated up to the third order term,

$$\mathbf{h}(\mathbf{y}, \bar{\mathbf{u}}) = \bar{\mathbf{h}} + \bar{\mathbf{y}}^{-1}\mathbf{y}_1\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}} + \bar{\mathbf{y}}^{-1}\mathbf{y}_2\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}} + \bar{\mathbf{y}}^{-1}\mathbf{y}_3\bar{\mathbf{y}}^{-1}\bar{\mathbf{u}}. \tag{A10}$$

APPENDIX B: LIST OF SYMBOLS

$C, \mathbf{C}$	damping
Cov	covariance
cs	column transformation
dy	incremental of y
$D, \mathbf{D}$	direct product of two deterministic matrices
e	exponential
f	force
$F, \mathbf{F}$	force amplitude
$G, \mathbf{G}$	dynamic stiffness
$\mathbf{G}_1, \mathbf{G}_2$	perturbed terms of $\mathbf{G}^{-1}$
$h, \mathbf{h}$	random function
$\mathbf{I}_N$	identity matrix of dimension N
j	$(-1)^{0.5}$
$K, \mathbf{K}$	stiffness
$M, \mathbf{M}$	mass
$P, \mathbf{P}$	variances by the proposed method
$Q, \mathbf{Q}$	variances by the first order perturbation
$R, \mathbf{R}$	covariance in direct product form
t	time
$u, \mathbf{u}$	a function
Var	variance
$\mathbf{x}$	displacement
$X, \mathbf{X}$	displacement
$y, \mathbf{y}$	random function
$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$	perturbed terms of $\mathbf{y}^{-1}$
δ	normalized parameter standard deviation
σ	standard deviation
ω	undamped natural frequency
Ω	excitation frequency
Φ	eigenvector
ζ	damping ratio

Superscripts

*	conjugate
-1	inverse
T	transpose
$\otimes 2$	second power of direct product
$\otimes 3$	third power of direct product
$\dot{}$	first time derivative
$\ddot{}$	second time derivative
$\tilde{}$	fluctuation or random component
$\bar{}$	deterministic component

Subscripts

cs	column transformation
C	damping
d	damped
F	force amplitude
G	dynamic stiffness
i	physical co-ordinate
K	stiffness
M	mass
P	variances by the proposed method
u	a function
X	displacement
y	random function
ω	undamped natural frequency

Operators

$\langle \, \rangle$	expectation operator
$\otimes$	direct or Kronecor product
!	factorial
$\| \, \|$	norm operator
$| \, |$	magnitude or absolute value
$|$	operating point
$\mathscr{D}$	derivative