Analysis of Structure-Borne and Radiated Sound using Component Modal Bases

J. E. Farstad, M.-R. Lee & R. Singh*

Acoustics and Dynamics Laboratory, Department of Mechanical Engineering, The Ohio State University, 206 West 18th Avenue, Columbus, Ohio 43210-1107, USA

ABSTRACT

Owing to the mathematical convenience and physical insight which they afford, analysis methods based on component modal functions are very useful for the study of structurally transmitted vibration in assemblies of machine elements, and acoustic radiation from vibrating structures. In this paper, a thorough discussion of prior work involving the use of component modal basis methods for estimating structure-borne and radiated sound is presented. In addition, formulations for two recently developed analysis frameworks for structurally transmitted vibration and acoustic radiation from vibrating planar radiators and, in particular, from a vibrating annular disk are given. To illustrate the use and relevance of these methods, an example of an elastically-supported rigid rotor joined to a flexible annular disk is considered.

1 INTRODUCTION

Noise produced by machines such as vehicles and home appliances is often due to forces and motions generated by internal components and transmitted structurally through joints to machine housings and other supporting structures, which, in turn, vibrate and radiate sound into the surrounding fluid. The usage of modal functions of the individual components joined to form machine assemblies as bases to analyze such problems is appealing not only because of its mathematical convenience, but also due to the unique understanding of sound transmission and radiation paths which it affords. In this paper, a detailed review of

* To whom correspondence should be addressed.
previous research on the application of component modal functions to problems of structure-borne noise and acoustical radiation is presented. In addition, two recently developed analysis methods based on the use of component modal bases are briefly presented. Methods for computing the structure-borne vibration transmitted among components in discretely joined assemblies and for computing the radiated sound power produced by planar acoustic radiators and, more specifically, by a vibrating annular disk are discussed. To illustrate the use of these methods, a simple example case of a joined rotor-disk system is considered.

2 LITERATURE REVIEW

The natural appeal of component modal bases has led to their frequent use for computing the modal properties of assembled structures. Such approaches, known as component mode synthesis methods, were first proposed in different forms by Hurty\textsuperscript{1} and Gladwell.\textsuperscript{2} Several improvements have been developed by subsequent investigators, most notably by Craig and Bampton,\textsuperscript{3} MacNeal,\textsuperscript{4} and Benfield and Hruda.\textsuperscript{5} These formulations differ primarily in the types of boundary conditions that are imposed on the individual components at joint locations to obtain the component natural frequencies and modes and in the way that the motion constraints at the joints are enforced. While these methods have been shown to be successful for predicting the natural frequencies and mode shapes of assemblies, no previous investigation has applied an approach based on component modal properties to problems of estimating the dynamic power transmitted among components when vibration of the assembly is forced. The structure-borne noise formulation presented in this paper is intended to fill this void. The component mode synthesis method used is an extension of that recently proposed by Min \textit{et al}.\textsuperscript{6} This method is similar to that of Hurty,\textsuperscript{1} but uses different interfacial boundary conditions on one of the components and enforces the motion constraints using Lagrange multipliers, as originally proposed by Dowell.\textsuperscript{7,8} Reviews of component mode synthesis formulations are given by Hale and Meirovitch\textsuperscript{9} and Craig.\textsuperscript{10}

Broad band analysis methods such as statistical energy analysis\textsuperscript{11} and asymptotic modal analysis\textsuperscript{12} have frequently been used to study dynamic power flow in component assemblies. These techniques were developed to predict high-frequency vibration for assemblies of components all having large numbers of participating modes, and yield only temporally, spectrally, and spatially averaged vibration level values for each component. Consequently, the usefulness of these methods is limited in cases
when one or more components has only a few modes participating in
the frequency range of interest, and they can offer no information for
determining which choice of spatial locations for joints connecting
components might reduce the structure-borne vibration transmitted
to one or more component, especially at low to moderate frequencies.
Furthermore, Woodhouse\textsuperscript{13} has shown that statistical energy analysis
is exactly correct only for the case of weakly coupled and lightly damped
oscillators.

Although less frequently applied than broad band methods, narrow
frequency band methods have also been developed to predict transmitted
vibration. Several previous investigations have focused on the narrow
band power flow associated with flexural and longitudinal vibrations
in one-dimensional assemblies of beams and rods.\textsuperscript{14-20} These studies
were based on wave propagation ideas, and several were limited to
assemblies of infinite or semi-infinite bodies.\textsuperscript{15-18} In particular, Gibbs and
Tattersall\textsuperscript{14} included the additional effect of power transmission due to
torsional waves, and Leung and Pinnington\textsuperscript{17,18} examined the power flow
through a single compliant and dissipative joint connecting two semi-
infinitume beams. Other investigators have applied similar methods to
assemblies of flat plates.\textsuperscript{21-27} Guyader \textit{et al.}\textsuperscript{21} and Boisson \textit{et al.}\textsuperscript{22,23}
computed transmitted power in systems of two plates by directly solving the
forced equations of motion for the assembled structure. Cuschieri\textsuperscript{24-26}
applied a method based on connection point mobilities of components to
L-shaped plate assemblies, and Lu, \textit{et al.}\textsuperscript{27} applied wave propagation
methods to joined systems of semi-infinite plates. Other investigations\textsuperscript{28-31}
have addressed problems associated with the power transmitted from
vibrating rigid bodies to flexible foundations through compliant mounts.
Chen and Soong\textsuperscript{32} examined the power transfer between two single-
degree-of-freedom oscillators joined through a compliant and dissipative
connection.

Because of their utility in computing the forced vibration of radiating
structures, modal bases have also been used to determine the acoustical
fields generated by vibrating planar radiators.\textsuperscript{33-41} Although considerable
effort has been expended in the past to compute the acoustic radiation
efficiencies associated with the rigid body motions of finite planar sur-
faces such as pistons or the flexural motion of infinite plates,\textsuperscript{42-49} little
work has focused on the radiation properties of elastic plates of finite
dimension. Among simple planar geometries, circular, annular, and rect-
angular plates are most frequently encountered in machinery acoustics
problems. In this paper, however, only radiation from an annular plate,
which has the most complicated modal functions among simple planar
radiators, is explicitly considered for the purpose of illustration.
The sound radiation characteristics of vibrating circular plates have been studied using analytical and numerical approaches. However, literature on the acoustic properties of annular disks, especially modal radiation properties, is rather sparse. Early studies of circular or annular planar radiator focused on plates vibrating in the piston or rocking-type rigid body modes. For instance, Gladwell analyzed a rigid circular piston by using the Hankel transform to map the velocity potential into the wave number space. Asymptotic results for the far-field velocity potential were found, and the radiation impedances were derived in terms of power series expansions of Bessel functions. Self- and mutual-radiation impedance methods have also been employed in studies of sound radiation due to rigid piston-type motion of circular and annular plates using different approaches. These include the surface pressure distribution method, a near-field approach, the Bouwkamp's integral method, in which the impedance is obtained by integration of the square of directivity function, and the Fourier transform of a generalized impulse response function. In general, these mathematical techniques are complex, and the final results are often expressed in terms of Bessel, Struve and hypergeometric functions. Greenspan extended the theory of piston radiation to radiators with special axisymmetric velocity distributions.

Investigation of the radiation characteristics of planar radiators vibrating in their elastic modes has been conducted both analytically and experimentally. Two experimental studies on clamped circular plates vibrating in their natural modes were conducted by Hansen and Bies and by Krishnappa and McDougal. Levine and Leppington analyzed modal radiated sound power using an exact integral representation for frequencies above the coincidence frequency. Asymptotic solutions for the modal radiation efficiencies of a clamped circular plate vibrating at high frequencies were derived. Similar analyses on beams and rectangular panels were performed by Levine in an earlier paper. Williams derived a power series representation of sound power in terms of wave number for planar radiators of arbitrary geometry. Others have performed related research on sound radiation dealing with the effects of distributed loads on rectangular plates, internal damping and viscous/thermal losses, and arbitrary boundary conditions.

Modal coupling effects are relatively important for accurate prediction of radiated sound power when more than one of the radiator's natural modes are excited, but the literature on this subject is very sparse and has considered only cases of one-dimensional beams and rectangular plates. Davies analyzed sound radiated from a simply supported rectangular plate into a half space filled with water. In vacuo
modes were used to derive a set of coupled linear equations involving plate modal velocities and pressures. Modal coupling coefficients were used to represent coupling effects between different elastic modes. Pope and Leibowitz\textsuperscript{51} developed a more complete formulation for the same physical system. Lomas and Hayek\textsuperscript{52} used a method similar to that of Davies\textsuperscript{50} to study radiation from an elastically supported rectangular plate. Keltie and Peng\textsuperscript{54} found that the coupling between two natural modes is as important as the effects of individual modes when both natural frequencies were much lower than the excitation frequency. Cunefare\textsuperscript{41,55} developed a quadratic expression for the radiation efficiency of a beam under multimodal excitation using a far-field intensity integration technique. It was shown that the minimization of the radiation efficiency implies an eigenvalue problem in a form identical to the Rayleigh’s quotient.

Statistical energy analysis has also been used for predicting radiated sound. As mentioned earlier, it was developed to predict power flow between coupled mechanical or acoustical oscillators at relatively high frequencies. Applications in this area are primarily limited to problems such as sound radiation from a vibrating structure with high modal density into a relatively large acoustic volume.\textsuperscript{56} As is the case for structure-borne sound, this technique yields only spatially averaged sound levels excited by a broad band random excitation. Consequently, no specific information about individual component modes can be obtained.

3 ANALYSIS OF STRUCTURE-BORNE SOUND

3.1 Modal synthesis

Consider a vibrating system built up from two components as indicated in Fig. 1. Let one of the components be arbitrarily denoted as the ‘main’ system, and let the other component be denoted as the ‘sub’ system. The assembled structure is assumed to be linear with proportional damping, so that its mode functions are real-valued. The sub system is connected to the main system at the \( j \) points \([x_1, x_2, \ldots, x_j]\). Let the vector \( \mathbf{d}_j \) be the displacement of joint \( j \) caused by its connection to the main system at \( x_j \). Although a joint may transmit both translational and rotational motions between components, in the following development only connections which transmit translational motions will be considered to clarify the analysis. Let \( H^m \) and \( H^s \) be the Hilbert spaces for motions of the main system and sub system, respectively.

First, consider separately the free vibration of the undamped com-
ponents. The displacements \( q^m \) of the main system and \( q^s \) of the subsystem with respect to the inertial reference frame \( o \) respectively satisfy

\[
M^m \ddot{q}^m + K^m q^m = 0 \quad (1a)
\]
\[
M^s \ddot{q}^s + K^s q^s = 0 \quad (1b)
\]

The self adjoint mass and stiffness operators \( M^m \) and \( K^m \) are those appropriate for the main system when all joint locations \( x_j \) are unconstrained, but other main system boundary conditions are enforced. The sub system mass and stiffness operators \( M^s \) and \( K^s \), however, are those corresponding to all motions constrained by joints set equal to zero, in addition to any other boundary conditions imposed upon the subsystem. Solution of the eigenvalue problems associated with eqns (1a and b) yield eigenvalues \( \xi_i^2 \) and \( \nu_i^2 \), and concomitant modes \( \phi_i \) and \( \psi_i \) for the main system and sub system, respectively. The modes may be scaled to be orthonormal with respect to the appropriate mass operators, so that

\[
\langle \phi_i, M^m \phi_j \rangle = \delta_{ij} \quad \text{and} \quad \langle \phi_i, K^m \phi_j \rangle = \xi_i^2 \delta_{ij} \quad (2a)
\]
\[
\langle \psi_i, M^s \psi_j \rangle = \delta_{ij} \quad \text{and} \quad \langle \psi_i, K^s \psi_j \rangle = \nu_i^2 \delta_{ij} \quad (2b)
\]

where the inner product is that associated with the appropriate Hilbert space and \( \delta_{ij} \) is the Kronecker delta. Observe that the sets \( \{ \phi_i \} \) and \( \{ \psi_i \} \), respectively, form bases for \( H^m \) and \( H^s \). Hence, it is reasonable to suppose that the motion of the assembled structure may be expressed as a linear combination of these component mode functions. The steady, harmonic motion of the main system after the components
are joined is

$$q^m = \sum_{i=1}^{N^m} a_i \phi_i e^{-j\Omega t}$$  \hspace{1cm} (3)

where $N^m$ is the number of main system modes included. If the main system has a large number of modes, it may not be practical to include all of them in the series expansion for the motion of the assembly. Consequently, the motion is projected onto a subspace of $H^m$ spanned by the $N^m$ mode functions included, and some error may be expected due to the incomplete nature of this set.

The motions of the subsystem after the components are joined requires more careful consideration. Since $\psi_k$ span the deformation of the subsystem relative to fixed conditions at the joint locations, it is reasonable to suppose that the motion of the subsystem may be obtained by superimposing upon these deformations due solely to displacements of the connection points. Let the static influence coefficient matrix $G$ be defined such that its columns are the subsystem internal displacement fields resulting from unit displacements imposed on the corresponding motion degrees of freedom constrained by joints. Hence, displacement of the subsystem due solely to the motions imposed by the joints is $q^s = Gd$, where $d = [d_1^T, d_2^T, \ldots, d_J^T]^T$ is the vector of all displacements constrained by joints to the main system. Superimposing this on the motion with respect to fixed joint displacements, the motion of the subsystem after assembly is

$$q^s = \left( \sum_{k=1}^{N^s} b_k \psi_k + Gd \right) e^{-j\Omega t}$$  \hspace{1cm} (4)

where $N^s$ is the number of modes of the substructure included in the expansion. For the remainder of this development, the ubiquitous term $e^{-j\Omega t}$ will be omitted with the reader's understanding that its presence is implied.

The kinetic energy of the vibrating assembled structure is given by the sum of the kinetic energies of its components, $T^m$ and $T^s$, respectively. Using the kinematic relations (3) and (4),

$$T^m = \frac{1}{2} \langle q^m, M^m q^m \rangle = \frac{1}{2} \Omega^2 \sum_{i=1}^{N^m} a_i^2$$

$$T^s = \langle q^s, M^s q^s \rangle = \frac{1}{2} \Omega^2 \left( \sum_{k=1}^{N^s} b_k^2 + 2 \sum_{k=1}^{N^s} b_k \langle \psi_k, M^s Gd \rangle + \langle Gd, M^s Gd \rangle \right)$$

(5a,b)
The potential energy for the assembly is computed in a similar fashion. For the main system,

\[ U^m = \frac{1}{2} \langle \mathbf{q}^m, \mathbf{K}^m \mathbf{q}^m \rangle = \frac{1}{2} \sum_{i=1}^{N^m} a_i^2 \xi_i^2 \] (6)

The potential energy of the subsystem may be considered as the sum of the work done to deform the subsystem into a particular configuration relative to fixed joint conditions and the additional work done on the subsystem by the forces acting at the joints. Defining \( \mathbf{S} \) to be the subsystem external static stiffness operator relating forces and displacements at the joint locations, the subsystem potential energy is

\[ U^s = \frac{1}{2} \sum_{k=1}^{N^s} b_k^2 v_k^2 + \frac{1}{2} \langle \mathbf{d}, \mathbf{S} \mathbf{d} \rangle \] (7)

The displacements of the joints connecting the main system and sub system occur in the expressions for both \( \mathbf{q}^m \) and \( \mathbf{q}^s \). Consequently, constraints on the joint motions \( \mathbf{d} \) are necessary. For ideal joints, the motions of the main system and subsystem at all joints must be identical. Defining an operator \( \mathbf{L} \) which evaluates the main system mode functions at the joint locations so that \( \mathbf{L} \phi = [\phi(x_1)^T, \phi(x_2)^T, \ldots, \phi(x_J)^T]^T \), the equations of constraint may be expressed as

\[ \mathbf{f} = \sum_{i=1}^{N^m} a_i \mathbf{L} \phi_i - \mathbf{d} = \mathbf{0} \] (8)

The equations of motion for free vibration of the assembled structure are obtained from Lagrange's equation, with Lagrange multipliers used to enforce the constraint conditions. The vector of generalized coordinates is

\[ \mathbf{y} = [a_1, \ldots, a_{N^m}, b_1 \ldots, b_{N^s}, \mathbf{d}^T]^T \] (9)

Since the coefficients in \( \mathbf{y} \) are time-invariant, the appropriate form of Lagrange's equation is

\[ \frac{\partial L}{\partial y_i} + \sum_{j=1}^{J} \lambda_j \frac{\partial f_j}{\partial y_i} = 0 \] (10)

where \( L = T - U \) is the Lagrangian function, \( \lambda_j \) is the Lagrange multiplier associated with the \( j \)th component of eqn (8), and \( J \) is the dimension of \( \mathbf{d} \).

Application of eqn (10) yields a total of \( N^m + N^s + J \) equations. If
these and the constraint eqns (8) are manipulated to eliminate all $d_j$ and $\lambda_j$, then the eigenvalue problem

$$Ax = \Omega^2 Bx$$  \hspace{1cm} (11)$$

is obtained, where $x = [a_1, \ldots, b_{N^s}]^T$ is the vector of unknown expansion coefficients and $A$ and $B$ are self-adjoint matrices with non-zero coefficients given by

$$A_{ij} = \xi_i^2 \delta_{ij} + (L\phi_j)^T SL\phi_i, \quad i,j \in N [1, N^m]$$  \hspace{1cm} (12a)$$

$$A_{ij} = \nu_i^2 \delta_{ij}, \quad i,j \in N [N^m + 1, N^m + N^s]$$  \hspace{1cm} (12b)$$

$$B_{ij} = \delta_{ij} + \langle GL\phi_i, M^s GL\phi_j \rangle, \quad i,j \in N [1, N^m]$$  \hspace{1cm} (12c)$$

$$B_{ij} = \delta_{ij}, \quad i,j \in N [N^m + 1, N^m + N^s]$$  \hspace{1cm} (12d)$$

$$B_{ij} = \langle \psi_k, M^s GL\phi_i \rangle, \quad i \in N [1, N^m],$$

$$j \in N [N^m + 1, N^m + N^s], \quad k = j - N^m$$  \hspace{1cm} (12e)$$

Equation (11) yields $N^m + N^s$ eigenvalues $\omega_i^2$ and eigenvectors $x_i$. The frequencies $\omega_i$ are the natural frequencies of the assembled structure, and the concomitant modes may be obtained from the eigenvectors $x_i$. Let the matrix $U$ be formed from the component modes such that

$$U = \text{diag} [\Phi \ \Psi]$$  \hspace{1cm} (13)$$

where $\Phi = [\phi_1, \ldots, \phi_{N^m}]$ and $\Psi = [\psi_1, \ldots, \psi_{N^s}]$.

The kinematic relations of eqns (3) and (4) are used to compute the assembled system mode functions. Let the coupled system displacement vector be $q = (q^m)^T(q^s)^T$. Then, the mode functions $\nu$ corresponding to $q$ are

$$\nu_i = (I + GL)UX_i$$  \hspace{1cm} (14)$$

where $I$ is an identity operator and

$$G = \text{diag} [0 \quad G]$$  \hspace{1cm} (15a)$$

$$L = \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}$$  \hspace{1cm} (15b)$$

Finally, the domains of the main system and subsystem intersect only at their boundaries. Hence, the main and subsystem share no common mass, and the form of $q$ ensures that the coupled system mass operator $M$ has the block diagonal form

$$M = \text{diag} [M^m \quad M^s]$$  \hspace{1cm} (16)$$

Consequently, if the coupled system stiffness operator is self-adjoint, then
the coupled system eigenvectors \( v_i \) will be orthogonal with respect to \( M \). Furthermore, they may be scaled to be orthonormal with respect to \( M \).

### 3.2 Intercomponent power transmission

The set \( \{ v_i \} \) forms an orthonormal basis for the motion of the assembled structure, and the normal mode method may be used to compute the forced response of the assembly. While the assembled stiffness matrix is not explicitly known, its self-adjoint property ensures that

\[
\langle v_i, K v_j \rangle = \omega_i^2 \delta_{ij}
\]

If \( \mathbf{f} e^{-j \Omega t} \) is the assembled system force vector, then the equation of motion for forced vibration is

\[
M \ddot{q} + K q = \mathbf{f} e^{-j \Omega t} \tag{18}
\]

Under the transformation

\[
q = V \eta e^{-j \Omega t} \tag{19}
\]

where \( V = [v_1, \ldots, v_{N^m+N^s}] \) eqn (18) is uncoupled. If modal damping \( \zeta_i \) is introduced, the modal participation factors \( \eta_i \) are given by

\[
\eta_i = \frac{v_i^H \mathbf{f}}{\omega_i^2 - \Omega^2 + j 2 \zeta_i \Omega \omega_i} \tag{20}
\]

The forced response of the structure may be used to compute the power transmitted among components through the joints. The present formulation is particularly well suited to this task, since the interfacial forces transmitted at the joints are precisely the Lagrange multipliers used to enforce the constraint conditions. In terms of \( q \),

\[
\lambda = \Omega^2 GM^s q^s - SL q^m \tag{21}
\]

where \( \lambda \) is the vector of Lagrange multipliers.

The power transmitted among the components is computed using the appropriate joint velocities. The total power injected into the structure is

\[
\Pi_{in} = \frac{1}{2} \text{Re} [j \Omega q^T \mathbf{f}^*] \tag{22}
\]

and the net power transmitted to the subsystem from the main system is

\[
\Pi_{ms} = \frac{1}{2} \text{Re} [j \Omega (Lq^m)^T \lambda^*] \tag{23}
\]

By considering cases when all components of \( \lambda \) are set to zero except those associated with a particular joint, one may examine the vibration transmitted through a particular structural path.
4 ANALYSIS OF RADIATED SOUND

Because structural modal functions are so useful for computing the forced vibration of machine elements, they also provide an efficient approach for computing the acoustic field produced by vibrating structures. The sound radiated from a planar radiator is typically calculated using one of two approaches: analysis of the far-field sound pressure and analysis of the acoustic intensity at the surface of the radiator.\(^3\) The first approach is based on integration of the far-field Green's function over a hemispherical surface in the far-field of the radiator. The second usually involves transforming the problem from the time domain into the wave number domain. Computing the sound power by using either approach then becomes a problem of accurately evaluating the complicated integrals which arise when numerical, analytical, or asymptotic expansion techniques are employed. In this paper, a case-specific formulation for the acoustic field produced by a vibrating annular disk using a far-field approach is developed. However, the conceptual formulation used is valid for planar radiators of arbitrary shape.

Frequently, an annular plate in a machine rotor is mounted on a shaft by collars; computer disks and circular saw blades are typical examples. Sound is radiated into the surrounding acoustic medium by vibration of the rotating disk. In this study, however, the relatively simple but practical case of a single stationary disk is considered as the vehicle for illustrating the methodology. It is assumed to be mounted flush with an infinite rigid baffle, and sound is radiated into a semi-infinite fluid-filled half space as indicated in Fig. 2. The medium is assumed to be air and the disk is assumed to be of sufficiently high impedance that the acoustical loading of the structure by surrounding medium over the frequency range of interest may be neglected.

4.1 Modal radiation efficiency formulations

To estimate the sound produced by a planar radiator under an arbitrary excitation, the surface velocity distribution must be expressed in a suitable form. A modal expansion approach is the most convenient way to achieve this, and also permits the radiated acoustic field to be expressed in terms of the structural modes of the radiator. The free-field sound pressure of a planar radiator vibrating at one of its flexural modes is given by Rayleigh's integral\(^4\)

\[
P_{mn} = \frac{j \omega_{mn} \rho_o}{2 \pi} \int \psi_{mn}(s) e^{-j k_{mn} d_{m}(s)} d_{m}(s) ds
\]

For a planar radiator of arbitrary geometry, the modal function is often
very complicated and may not be amenable to representation in terms of an explicit function. Consequently, integrations over the surface of the radiator must be evaluated numerically. Though the analytical modal functions for plates of simple geometries are available, explicit solutions can be obtained only for a few asymptotic cases.\textsuperscript{35,36} Once the sound pressure is calculated, either in the far-field or at the surface, the radiated sound power can easily be obtained. In the remainder of this section, these ideas are developed fully for the case on an annular disk radiator.

For an annular disk, the exact modal functions consist of Bessel functions and modified Bessel functions of the first and second kinds and of various orders which are cumbersome to work with and greatly complicate the formulation for radiated sound. Therefore, a more convenient polynomial approximation for the natural modes of the disk is employed, so that closed-form results may be obtained. For this analysis, the inner edge of the disk is assumed to be clamped and the outer edge is assumed to be free. The approximate disk velocity modal function which consists of trial functions that satisfy the geometric boundary conditions at \( r = a \) is

\[
\hat{\psi}_{mn} = \cos (n \varphi) \sum_{s=2}^{N(m)} c_{mn,s} (r - a)^s
\]  

(25)

The natural frequencies and the coefficients of the approximate modal functions are calculated by applying classical thin plate theory and the Rayleigh–Ritz method.\textsuperscript{57} The approximate modal functions have been found to accurately match the exact modal functions.\textsuperscript{53} Therefore, the approximate modal functions can be used to calculate the radiated sound without losing accuracy. Equation (25) is further expressed in terms of dimensionless radial coordinates as shown in equation (26).

\[
\psi_{mn} (r, \varphi) = \cos (n \varphi) \sum_{s=0}^{N} \tilde{c}_{mn,s} \left(\frac{r}{b}\right)^s
\]  

(26a)
where,

\[
\bar{c}_{mn,j} = \left( \frac{b}{a} \right)^j \sum_{i \geq j} \frac{(-1)^i + j}{i! (i-j)!} a^i c_{mn,j}^{(i)}
\]  

(26b)

Sound pressure for the system of Fig. 2 is then computed by using eqns (24) and (26) with the geometric far-field assumption that \( R \gg r \), so that

\[
P_{mn}(R, \theta, \beta) = \frac{j^{n+1} e^{-jkR}}{R} \rho_0 c k_{mn} \cos(n\varphi)
\]

\[
\times \sum_{s=0}^{N} \bar{c}_{mn,s} \int_{r_o}^{b} r^{s+1} J_n(k_{mn}r \sin \theta) \, dr
\]

(27)

where \( J_n \) is the Bessel function of the first kind of order \( n \). Since the far-field radiation impedance is the characteristic impedance \( \rho_0 c \) of the medium, the modal sound power \( \Pi_{mn} \) is obtained by integrating the far-field sound intensity over a hemispherical surface of large radius. By expressing the Bessel function as a power series, the formulation for modal sound power corresponding to the \((m,n)\) mode is expressed in terms of a convergent series of the modal wave number \( k_{mn} \).

\[
\Pi_{mn} = \frac{R^2}{2 \rho_0 c} \int_0^{2\pi} \int_0^{\pi/2} P_{mn}(R, \theta, \beta) P_{mn}(R, \theta, \beta)^* \sin \theta \, d\theta \, d\beta
\]

\[
= \frac{\rho_0 c \pi b^2}{\epsilon_n} (k_{mn}b)^{2(n+1)} \sum_{s=0}^{\infty} A_{mn,s} (k_{mn}b)^{2s}
\]

\[
\epsilon_n = \begin{cases} 
1 & n = 0 \\
2 & \text{otherwise}
\end{cases}
\]

(28)

Coefficients \( A_{mn,s} \) are functions of the geometry and material properties of the disk. The corresponding reference power obtained by using the corresponding approximate modal function is

\[
\Pi_{mn,\text{ref}} = \frac{\rho_0 c \pi b^2}{\epsilon_n} \sum_{t_1, t_2=0}^{N} \bar{c}_{mn,t_1} \bar{c}_{mn,t_2} \frac{1 - \alpha^{t_1 + t_2 + 2}}{t_1 + t_2 + 2}
\]

(29)

Consequently, the modal radiation efficiency \( \sigma_{mn} \) is

\[
\sigma_{mn} = \frac{\Pi_{mn}}{\Pi_{mn,\text{ref}}}
\]

(30)

Since the radiated sound is represented in terms of a convergent series of the wave number, it is convenient to relax the accuracy of the pre-
dictions and implement it numerically by including only a finite number of terms in the series. Predicted modal radiation efficiencies obtained for several selected vibration modes are compared with numerical results obtained from the boundary element program BEMAP$^{58}$ in Table 2 (see section 5). Excellent agreement between the two methods is apparent.

Two disk rigid body modes, the rigid piston and rocking modes, are also included. To distinguish them from the elastic modes, the radial modal index $m$ for these modes is assigned the value $-1$. The corresponding tangential modal index is $n = 0$ for the piston rigid body mode and $n = 1$ for the rocking rigid body mode. The radiation efficiencies for these two rigid body modes can be derived directly by using the formulation developed and expressing the normalized surface velocity functions of these two modes as

$$\hat{\psi}_{-1,0} = \frac{1}{\sqrt{(b^2 - a^2)\pi}}$$

$$\hat{\psi}_{-1,1} = \frac{2}{\sqrt{(b^4 - a^4)\pi}} r \cos (\varphi)$$

(31) (32)

4.2 Modal coupling effects

If several structural modes are excited simultaneously, coupling effects between the acoustic fields generated by different modes need to be included. From the formulation, it can be shown easily that non-zero sound power due to coupled modes exists only when the coupled modes have the same number of nodal diameters. Assume that the disk is excited by a harmonic force at frequency $\Omega$ and the surface velocity distribution can be expressed in terms of elastic and rigid body modes $\hat{\psi}$ normalized such that their norms are equal to one, such that

$$v = \hat{\eta}^T \hat{\psi}$$

(33)

Then, the far-field pressure is expressed by

$$P = \hat{\eta}^T P$$

(34a)

where,

$$P = \frac{jk \rho_c c}{2 \pi R} \int_S \hat{\psi} e^{-jkd} \, ds$$

(34b)

Consequently, the radiation efficiency is

$$\sigma = \frac{\hat{\eta}^H \Pi \hat{\eta}}{\frac{1}{2} \rho_c c \hat{\eta}^H G \hat{\eta}}$$

(35)
where,

\[
\Pi = \frac{R^2}{2 \rho_o c} \int_0^{2\pi} \int_0^{\pi/2} \mathbf{P}^H \sin \theta \, d\theta \, d\phi, \quad \mathbf{G} = \int_0^{2\pi} \int_a^b \hat{\psi}^T \hat{\psi} \, r \, dr \, d\phi = \begin{bmatrix} I_2 & \mathbf{M}^T \end{bmatrix} \begin{bmatrix} \mathbf{I}_m \ \mathbf{I}_{m+1} \end{bmatrix} \tag{36a,b}
\]

\[
\mathbf{M}^T = \begin{bmatrix}
\langle \hat{\psi}_{-1,0}, \hat{\psi}_{00} \rangle & 0 & 0 & \cdots & \langle \hat{\psi}_{-1,0}, \hat{\psi}_{m0} \rangle & 0 & 0 & \cdots & 0 \\
0 & \langle \hat{\psi}_{-1,1}, \hat{\psi}_{01} \rangle & 0 & \cdots & 0 & \langle \hat{\psi}_{-1,1}, \hat{\psi}_{m1} \rangle & 0 & \cdots & 0 \\
\end{bmatrix} \tag{36c}
\]

\[
\langle \hat{\psi}_{m1}, \hat{\psi}_{mn2} \rangle = \int_0^{2\pi} \int_a^b \hat{\psi}_{m1} \hat{\psi}_{mn2} \, r \, dr \, d\phi \tag{36d}
\]

and \( \mathbf{I}_n \) is an identity matrix of dimension \( n \).

For an arbitrary plate, the modal sound power matrix is a fully populated matrix, and each component needs to be evaluated numerically, as explained in the previous section. For the annular disk, the modal function vector is

\[
\hat{\psi} = [\hat{\psi}_{-1,0} \ \hat{\psi}_{-1,1} \ \hat{\psi}_{00} \ \hat{\psi}_{01} \ \cdots \ \hat{\psi}_{0n} \ \hat{\psi}_{11} \ \cdots \ \hat{\psi}_{mn}]^T \tag{37}
\]

Consequently, modal sound power matrix \( \Pi \) consists of \( (m+2) \) by \( (m+2) \) sub matrices so that

\[
\Pi = \begin{bmatrix}
\Pi^{1,-1} & \Pi^{1,0} & \cdots & \Pi^{1,m} \\
\Pi^{0,-1} & \Pi^{00} & \cdots & \Pi^{0m} \\
\vdots & \vdots & \ddots & \vdots \\
\Pi^{m,-1} & \Pi^{m0} & \cdots & \Pi^{mm} \\
\end{bmatrix} \tag{38a}
\]

where,

\[
\Pi^{ij} = \begin{bmatrix}
\Pi_{j0}^{i0} & 0 & \cdots & 0 \\
0 & \Pi_{j1}^{i1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Pi_{jn}^{in} \\
\end{bmatrix} \quad \text{for } i, j \neq -1 \tag{38b}
\]

\[
\Pi^{ij} = \begin{bmatrix}
\Pi_{j0}^{i0} & 0 & \cdots & 0 \\
0 & \Pi_{j1}^{i1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix} \quad \text{for } i, j \neq -1 \tag{38c}
\]

\[
\Pi_{jn1}^{mn2} = \frac{R^2}{2 \rho_o c} \int_0^{2\pi} \int_0^{\pi/2} P_{m1} P_{jn2}^* \sin \theta \, d\theta \, d\phi \tag{38d}
\]
Fig. 3. Selected self-radiated sound power for multi-modal excitation. (a) Axisymmetric modes, (b) asymmetric modes.

For $i = j$, the sub-matrices represent the sound power associated with the individual elastic or rigid body modes. Other sub-matrices are the contributions due to modal coupling effects between different modes. Every component of these sub-matrices is formulated in terms of a wave number power series as developed in the previous section. Figures 3–5 show selected self (single mode) and coupled (multiple mode) terms of radiated sound power of the example disk. From these results, it is observed that self-radiated sound power spectra associated with elastic modes which have the same number of modal circles (index $m$) essentially have similar patterns except that the humps shift to higher frequencies as the number of nodal diameters (index $n$) increases. For coupled modes,
cancellation phenomena are observed between some of the structural modes.

5 EXAMPLE

To illustrate the application of these new analysis methods for structure-borne and radiated sound, a simple example is included. Consider the system shown in Fig. 6(a), which consists of an elastically mounted rigid machine rotor with a flexible annular disk connected at one end by a rigid shaft of negligible mass. Properties of the disk and rotor for this
Fig. 5. Selected coupled radiated sound power between rigid body modes and elastic modes for multi-modal excitation. (a) Piston and axisymmetric elastic modes, (b) rocking and asymmetric elastic modes.

example are listed in Table 1 and Table 2, respectively. The rotor is directly excited by a harmonic force acting at its mass center and transmits vibration to the disk which, in turn, radiates sound. The structure-borne dynamic power transmitted from the rotor to the shaft and the acoustic power produced by the vibrating disk are sought.

Consider first the vibration transmitted to the disk. In accordance with the formulation of Section 3, the rotor will be considered the ‘main’ system, and the disk will be considered the ‘sub’ system. The main system is illustrated in Fig. 6(b); observe that free conditions exist at the joint location. The rotor is assumed to move only in the plane of Fig. 6(b), and
Fig. 6. Schematic of disk/rotor assembly studied as example case. (a) Assembled system. (b) three-degree-of-freedom rotor ‘main’ system, (c) annular disk ‘sub’ system.

<table>
<thead>
<tr>
<th>TABLE 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material Properties and Dimensions of the Annular Disk of the Example Case</td>
</tr>
<tr>
<td>Inner diameter $2a$                                               0.025 (m)</td>
</tr>
<tr>
<td>Outer diameter $2b$                                           0.15 (m)</td>
</tr>
<tr>
<td>Thickness $h$                                                        0.003 (m)</td>
</tr>
<tr>
<td>Young’s modulus                                                     19.5e10 (N/m²)</td>
</tr>
<tr>
<td>Density $\rho$                                                       7700 (kg/m³)</td>
</tr>
<tr>
<td>Possion’s ratio                                                      0.28</td>
</tr>
</tbody>
</table>
TABLE 2
Properties of the Rotor Main System Component for the Example Case

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass, ( m )</td>
<td>5.00 kg</td>
</tr>
<tr>
<td>Rotary inertia, ( J )</td>
<td>0.20 kg m²</td>
</tr>
<tr>
<td>Dimension ( l_x ) (Fig. 6(b))</td>
<td>0.10 m</td>
</tr>
<tr>
<td>Dimension ( l_y ) (Fig. 6(b))</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Dimension ( l_z ) (Fig. 6(b))</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Mount stiffnesses (Fig. 6(b))</td>
<td>( k_x = 3 \times 10^6 \text{ N/m}, k_y = 3 \times 10^5 \text{ N/m} )</td>
</tr>
<tr>
<td>Main system natural frequencies</td>
<td>( \xi_1 = 78 \text{ Hz}, \xi_2 = 246 \text{ Hz}, \xi_3 = 302 \text{ Hz} )</td>
</tr>
<tr>
<td>Main system mode vectors</td>
<td>( \phi_1 = [0 ~ 0.4472 ~ 0]^T )</td>
</tr>
<tr>
<td></td>
<td>( \phi_2 = [0.4472 ~ 0 ~ 0]^T )</td>
</tr>
<tr>
<td></td>
<td>( \phi_3 = [0 ~ 0 ~ 2.2361]^T )</td>
</tr>
</tbody>
</table>

consequently has only three degrees of freedom so that \( \mathbf{q}^m = [u_x, u_y, \gamma]^T \). The corresponding mass and stiffness matrices are

\[
\mathbf{M}^m = \begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & J
\end{bmatrix}, \quad \mathbf{K}^m = \begin{bmatrix}
4k_x & 0 & 0 \\
0 & 4k_y & 0 \\
0 & 0 & 4(k_y l_x + k_x l_y)
\end{bmatrix}
\]

(39a,b)

Numerical values for the main system natural frequencies and mode vectors are listed in Table 3, where the modes have been scaled to be orthonormal with respect to \( \mathbf{M}^m \).

The component modal properties of the disk sub system, shown in Fig.

TABLE 3
Selected Modal Radiation Efficiencies for the Disk of the Example Case

<table>
<thead>
<tr>
<th>Modal index ((m,n))</th>
<th>Natural frequency (\omega_{mn} ) (Hz)</th>
<th>Modal radiation efficiency (\sigma_{mn} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Proposed method</td>
<td>BEMAP (^b)</td>
</tr>
<tr>
<td>(0,0)</td>
<td>154</td>
<td>0.661e-1</td>
</tr>
<tr>
<td>(0,1)</td>
<td>139</td>
<td>0.850e-3</td>
</tr>
<tr>
<td>(0,2)</td>
<td>198</td>
<td>0.349e-4</td>
</tr>
<tr>
<td>(0,3)</td>
<td>406</td>
<td>0.532e-4</td>
</tr>
<tr>
<td>(0,4)</td>
<td>707</td>
<td>0.158e-3</td>
</tr>
<tr>
<td>(1,0)</td>
<td>951</td>
<td>0.411</td>
</tr>
<tr>
<td>(1,1)</td>
<td>1030</td>
<td>0.115</td>
</tr>
<tr>
<td>(1,2)</td>
<td>1280</td>
<td>0.278e-1</td>
</tr>
<tr>
<td>(1,3)</td>
<td>1750</td>
<td>0.303e-1</td>
</tr>
<tr>
<td>(2,0)</td>
<td>2780</td>
<td>0.333</td>
</tr>
<tr>
<td>(2,1)</td>
<td>2880</td>
<td>1.00</td>
</tr>
</tbody>
</table>

\(^{a}m: \) Number of nodal circles; \( n: \) number of nodal diameters.

\(^{b}Data taken from Ref. 58.\)
6(c), are computed approximately using the Rayleigh–Ritz method with shape functions given by eqn (25). Natural frequencies for the sub system are listed in Table 3. Because of its connection to the rotor, the disk may move in the \( y \)-direction as a rigid body in addition to the transverse motion normal to the disk considered previously. Consequently, each point on the disk surface has two degrees of freedom, so that the continuous sub system displacement is \( \mathbf{q}^s(r, \theta) = [w_x(r, \theta), w_y(r, \theta)]^T \). Corresponding subsystem mode functions are \( \psi_k(r, \theta) = [w_x^k(r, \theta) \ 0]^T \), where the transverse component is

\[
w_x^k(r, \theta) = \begin{cases} 0, & 0 \leq r \leq a \\ \sum_{i=2}^{8} C_i^k (r - a)^i f_k(\theta), & a \leq r \leq b \end{cases}
\]

(40a)

where

\[
f_k(\theta) = \begin{cases} 1, & n_k = 0 \\ \sin n_k \theta, & n_k > 0 \end{cases}
\]

(40b)

The zero values assigned to points \( r < a \) are needed so that the subsystem mode functions will be defined at the joint location, \( r = 0 \). Although the disk has an additional mode function for each natural frequency similar to that of eqn (40), but with \( \sin(n_k \theta) \) replaced by \( \cos(n_k \theta) \), the constraint that the rotor may move only in the plane of Fig. 6 ensures that the participation of any mode with an antinodal diameter at \( \theta = 0 \) will be zero. Consequently, the set of subsystem modes given by eqn (40) is sufficient for this example. The sub system static influence function matrix for this case corresponding to \( \mathbf{d} = [u_x \ u_y \ \gamma]^T \)_{\text{joint}} as indicated in Fig. 6(c) is

\[
G(r, \theta) = \begin{bmatrix} 1 & 0 & r \sin \theta \\ 0 & 1 & 0 \end{bmatrix}
\]

(41)

Because the disk is unconstrained except at the joint, static joint displacements result in no forces or moments at the joint. Consequently, the sub-system external static stiffness matrix \( \mathbf{S} \) is a null matrix for this example.

Although the continuous subsystem mode functions \( \psi_k(r, \theta) \) could be used in the modal synthesis formulation if the inner product corresponding to a planar polar coordinate space were used, computer implementation of the formulation is simplified significantly if the continuous disk mode functions are replaced by vectors of discrete values so that matrix methods may be used. Discrete mode vectors are computed by evaluating the continuous mode functions at discrete points on the disk. Since the disk is a two-dimensional component and one-dimensional discrete mode
vectors are desired, the following organizational scheme is used. Let the disk displacements be evaluated at the \((J_1 \times K_1)\) discrete points \((r_j, \theta_k)\), with \(j \in N[1,J_1]\) and \(k \in N[1,K_1]\). The coordinates of each point are \(r_j = \left(j - \frac{1}{2}\right)b/J_1\) and \(\theta_k = (k - 1)2\pi/K_1\). The vector \(\mathbf{q}_{sj}\) is defined to be the disk displacements at constant \(r_j\) and all \(\theta_k\) so that

\[
\mathbf{q}_{sj} = [\mathbf{q}_{s1}^T(r_j,\theta_1) \quad \mathbf{q}_{s1}^T(r_j,\theta_2) \ldots \quad \mathbf{q}_{s1}^T(r_j,\theta_{K_1})]^T
\]  

(42)

Corresponding to \(\mathbf{q}_{sj}\) is the partial subsystem mass matrix \(M_j = m_j \mathbf{I}\), where \(\mathbf{I}\) is the identity operator of the same dimension as \(\mathbf{q}_{sj}\) and \(m_j = \rho_h \pi b^2 \left(2j-1\right)/K_1 J_1^2\) is the mass of the annular sector at \((r_j,\theta_k)\). The discretized subsystem displacement vector \(\mathbf{q}_s\) is then

\[
\mathbf{q}_s = \begin{bmatrix} \mathbf{q}_{s1}^T & \mathbf{q}_{s2}^T & \ldots & \mathbf{q}_{K_1}^T \end{bmatrix}^T
\]

(43)

and the corresponding mass matrix for the entire subsystem is

\[
\mathbf{M}^s = \text{diag} \left[ M_1^s \quad M_2^s \ldots \quad M_{K_1}^s \right]
\]

(44)

The discretized subsystem mode vectors \(\psi_k\) and static influence function matrix \(\mathbf{G}\) are obtained by evaluating eqns (40) and (41), respectively, at the discrete point corresponding to \(\mathbf{q}_s\).

For this example, the main system selection operator \(\mathbf{L}\) relates the displacements of the connection point at the end of the rigid shaft to those of the rotor mass center. From Fig. 6(b), it is apparent that

\[
\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & (l_x + l_z) \\ 0 & 0 & 1 \end{bmatrix}
\]

(45)

The analysis procedure developed in Section 3 was applied to compute the modal properties of the assembly and the mechanical power transmitted between components. Although the subsystem has an infinite number of modes, an initial analysis included only the first six disk modes listed in Table 3 with natural frequencies less than 1000 Hz. Natural frequencies obtained for the assembly are listed in Table 4. Note that the assembly natural frequencies of 198 Hz, 406 Hz, and 708 Hz are identical to those for the disconnected disk subsystem. Inspection of the eigenvectors of component modal expansion coefficients \(x_i\) indicated that the corresponding assembly modes consisted solely of the subsystem modes with the same natural frequencies, and that these subsystem modes did not contribute to the other modes of the assembly. Hence, these disk modes are completely uncoupled from the other modes of the disk/rotor assembly. The forced response for the case of \(\mathbf{f} = [1 \ 1 \ 1 \ 0 \ \ldots \ 0]^T\) with uniform harmonic content was computed. A modal damping ratio \(\zeta = 0.02\) was applied to all assembly modes. The vibration at a typical point on the
disk is shown in Fig. 7(a) and (b). Observe that the participation of the uncoupled disk modes is zero, as no resonant peaks are seen at any of the uncoupled mode natural frequencies. Hence, one may conclude that these modes contribute nothing to the modal properties or the response of the assembly and may be neglected entirely. Since all of the uncoupled subsystem modes have more than one nodal diameter, one may conclude that only those disk modes with zero or one modal diameter are important for this example.

Because only those subsystem modes with \( n=0 \) or \( n=1 \) are needed, a second analysis was performed using all of the disk modes in Table 2 satisfying these criteria. Natural frequencies obtained for this case are also listed in Table 4. Note that the natural frequencies for those assembly modes predicted by both sets of subsystem modes are nearly identical, which indicates that the modal synthesis procedure has predicted these modes accurately. The vibration response at a typical point on the disk due to the same forcing function as for the previous case is also shown in Fig. 7(a) and (b) for frequencies up to 1.5 kHz. Note that resonant peaks are present at each of the assembly natural frequencies in this range, and that the response is identical to that obtained previously for frequencies less than approximately 750 Hz. Differences at the higher frequencies are due to participation of the mode with 1026 Hz natural frequency, which was not predicted when the lower frequency set of subsystem modes was used.

The mechanical power levels applied at the rotor and transferred to the disk are shown in Fig. 8. Observe that both input and transmitted
Fig. 7. Forced vibration of the disk at \((r, \theta) = (0.07 \text{ m}, 90^\circ)\) for different sets of disk modes used in the modal synthesis procedure. (a) Displacement in \(x\)-direction, (b) displacement in \(y\)-direction.

power levels are greatest in the neighborhoods of the assembly natural frequencies. While the input power level is particularly high near 68 Hz and 300 Hz, transmitted power levels are significantly lower at these frequencies. This indicates that at these frequencies most of the input power is consumed by internal damping in rotor, and relatively little is transmitted to the disk. Power levels for transmission by moment/rotation or force/displacement in the \(x\)-direction are comparable. The power transmitted by the force and displacement in the \(y\)-direction are not shown in Fig. 9 because it is approximately 200 dB lower than that due to the other two paths, and is not significant. This is interesting, since the mag-
Fig. 8. Input mechanical power to the rotor 'main' system and mechanical power transmitted to the disk 'sub' system through the joint.

Fig. 9. Acoustic power radiated from the vibrating disk for the example case.
The resulting sound power level is shown in Fig. 9. Note that the sound power spectrum exhibits peaks at the same frequencies where mechanical power transmitted to the disk is relatively great. Observe also that the sound power level is approximately 50 dB re 10^{-12} W at frequencies away from the assembly natural frequencies. The ratios of the total transmitted mechanical power and the acoustic power radiated from the disk to the mechanical power supplied are shown in Fig. 10. The ratio for the mechanical power transmitted to the disk is less than one at all frequencies, as it must be since power is directly applied only at the rotor. The power transmission to the disk is particularly efficient in the bands near 150 Hz and between 900 and 1100 Hz. The ratio of radiated sound power to
input power also indicates peaks in these bands, although the shape of
the sound power ratio curve is significantly different from that of the me-
chanical power ratio. Finally, note that the acoustic power radiated from
the disk is much less than either the input or transmitted mechanical
power over the entire frequency range.

6 CONCLUSION

This paper has focused on the application of component modal bases to
problems of structure-borne and radiated sound in machine assemblies.
These methods are likely to receive considerable attention in future be-
cause of their mathematical convenience and the unique physical insight
which they afford. The new formulations presented here and the example
case considered demonstrate their utility. Further research is needed to
apply these methods to the analysis of structure-borne sound in non-pro-
portionally damped component assemblies or in systems of components
connected by joints over continuous linear or area regions. More work is
also needed to extend modal basis methods to acoustical radiation prob-
lems involving radiation from generally damped structures with complex-
valued mode functions, since this would result in more general phase
relationships among the acoustic fields generated by different modes, and
would likely cause different modal coupling effects. Progress made in
some of these areas will be reported in forthcoming articles.59,60

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