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ANALYSIS OF DISCRETE VIBRATORY SYSTEMS WITH PARAMETER UNCERTAINTIES, PART I: EIGENSOLUTION

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A new analytical method is proposed for the estimation of eigensolutions for undamped and proportionally damped discrete vibratory systems when the system parameters are uncertain or random variables. Given several simplifying assumptions, a direct product technique is used to estimate the first and second moments of eigensolutions in terms of the moments of system parameter matrices. The proposed methodology is reasonably accurate and computationally fast, when compared with existing methods such as the first perturbation method and the Monte Carlo simulation. Application of the new method is demonstrated through several single- and multi-degree-of-freedom systems. It is seen that the standard deviation predictions match well with the results yielded by the Monte Carlo simulation, unlike the first order perturbation method, especially when moderately large random fluctuations are considered. The impulse response issue is addressed in the companion paper (Part II).

1. INTRODUCTION

The subject of parameter uncertainties has fascinated researchers for several decades, as is evident from extensive review articles by Ibrahim [1] and Benaroya and Rehak [2]. This problem is of interest in many disciplines, including vibrations and physical chemistry [1–7]. Random differential equations have also been studed extensively [8–11]. However, several research issues still remain unresolved [1, 2]. One such issue is the estimation of eigensolutions for viscously damped systems, which is addressed in this part. Yet another issue deals with impulse response characteristics, which forms the basis of the companion paper (Part II).

A few analytical techniques have been used to solve real or undamped eigenvalue problems. For instance, the transfer matrix method has been applied by Kener [12] and Soong and Bogdanoff [13] to study disordered periodic systems. Another method which has been used widely is the perturbation method [4, 9, 14, 17]. However, Pierre [6] has modified this perturbation method and applied it to disordered periodic systems. Typically, first order perturbations are included in the analysis since higher orders may make the analytical and numerical problems tedious. Finally, the Monte Carlo technique can be employed to find eigensolutions numerically [14, 15]. However, such a simulation is computationally intensive, since a large number of iterations is required to estimate the probability distributions. Based on this literature review and earlier assessments by prior investigators [1, 2, 6, 10], it is evident that new or improved analytical techniques are

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definitely required to overcome the deficiencies of existing methods, and still yield reasonably accurate results.

2. PROBLEM FORMULATION

Random differential equations for a linear time-invariant, proportionally damped vibratory system of dimension N can be given in the matrix form as follows:

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are system parameter matrices representing mass, viscous damping and stiffness, respectively. (Also refer to the Appendix of Part II for a complete list of symbols.) The following simplifying assumptions are made to develop new solution methodology: (i) random matrices and vectors of equation (1) can be given by the sum of deterministic or mean (identified by a bar) and random or fluctuating components (identified by a tilde), i.e., $\mathbf{M} = \bar{\mathbf{M}} + \tilde{\mathbf{M}}$, $\mathbf{C} = \bar{\mathbf{C}} + \tilde{\mathbf{C}}$, $\mathbf{K} = \bar{\mathbf{K}} + \tilde{\mathbf{K}}$ and $\mathbf{X}(t) = \bar{\mathbf{X}}(t) + \tilde{\mathbf{X}}(t)$; (ii) expected means of system matrices $\bar{\mathbf{M}} = \langle \mathbf{M} \rangle$, $\bar{\mathbf{C}} = \langle \mathbf{C} \rangle$ and $\bar{\mathbf{K}} = \langle \mathbf{K} \rangle$ are known, where $\langle\ \rangle$ is the expectation operator; (iii) the probability distributions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are of the same type and are known; (iv) the mean of a random parameter matrix is equal to zero, e.g., $\langle \tilde{\mathbf{M}} \rangle = \mathbf{0}$; (v) parameter fluctuations are much smaller compared to the deterministic values, i.e., $\|\tilde{\mathbf{M}}\| \ll \|\bar{\mathbf{M}}\|$, where $\|\ \|$ is the norm operator; and (vi) covariances of random fluctuations are known in the form of cross-correlation matrices such as $\mathbf{R}_{M,M} = \langle \tilde{\mathbf{M}} \otimes \tilde{\mathbf{M}} \rangle$ and $\mathbf{R}_{M,K} = \langle \tilde{\mathbf{M}} \otimes \tilde{\mathbf{K}} \rangle$, where $\otimes$ is the direct product of two matrices.

The scope of this paper is limited to the use of proportional damping. For the sake of convenience, consider the Rayleigh damping model given by $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$, where $\alpha = \bar{\alpha} + \tilde{\alpha}$ and $\beta = \bar{\beta} + \tilde{\beta}$ are random scalar variables. Depending upon α and β, one can obtain various cases in which $\mathbf{C}$ is uncorrelated, partially correlated and fully correlated with $\mathbf{M}$ and/or $\mathbf{K}$. The theory, based on a direct product technique presented in this paper, is developed first for the undamped case, i.e., $\alpha = \beta = 0$. Then, the proportionally damped system is examined. Finally, an undamped mass–spring system with randomly distributed masses [6, 13] is investigated, but periodic disordered systems [6, 16] are considered beyond the scope of this study. Of primary interest here is the prediction of the standard deviation of natural frequencies and damping ratios. In each case, the theory is illustrated through numerical examples and validated by comparing standard deviation predictions with the results yielded by either existing analytical techniques (whenever applicable) and/or the Monte Carlo simulation technique. The assumption of very small random fluctuations is relaxed in the numerical examples. For instance, in several cases, normalized standard deviation values of up to 0·5 are selected for analysis. The solutions yielded by all methods are compared.

3. UNDAMPED SYSTEM

3.1. THEORY

Consider the undamped system given by equation (1) with $\mathbf{C} = \mathbf{0}$. The eigensolution is given by $(\lambda_i\mathbf{I} - \mathbf{G})\mathbf{\Phi}_i = \mathbf{0}$, where $\mathbf{I}$ is an identity matrix, $\mathbf{G} = \mathbf{M}^{-1}\mathbf{K}$, $\lambda_i = \omega_i^2$ is the ith eigenvalue, ω_i is the natural frequency and $\mathbf{\Phi}_i$ is the corresponding eigenvector. Note that N eigenvalues λ_i of $\mathbf{G}$ are assumed to be distinct and $\mathbf{M}$ is considered to be non-singular. Deterministic eigenvalues $\bar{\lambda}_i$ and eigenvectors $\bar{\mathbf{\Phi}}^i$ are determined by $(\bar{\lambda}_i\mathbf{I} - \bar{\mathbf{G}})\bar{\mathbf{\Phi}}_i = \mathbf{0}$. Now express λ_i as $\lambda_i = \bar{\lambda}_i + \tilde{\lambda}_i = \mathbf{\Phi}_i^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_i / \mathbf{\Phi}_i^{\mathrm{T}}\mathbf{\Phi}_i$. If λ_i is estimated using deterministic $\bar{\mathbf{\Phi}}_i$, the

covariance between two eigenvalues is found to be as follows, where D is the direct product of two deterministic matrices, e.g., $\mathbf{D}_{\bar{\Phi},\bar{\Phi}} = \bar{\Phi} \otimes \bar{\Phi}$:

$$\begin{aligned}\mathrm{Cov}(\lambda_i, \lambda_j) &= \langle \lambda_i \otimes \lambda_j \rangle - \langle \lambda_i \rangle \otimes \langle \lambda_j \rangle \\ &= \frac{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_j}(\langle \mathbf{G} \otimes \mathbf{G} \rangle - \langle \mathbf{G} \rangle \otimes \langle \mathbf{G} \rangle)\mathbf{D}_{\Phi_i,\Phi_j}}{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_j}\mathbf{D}_{\Phi_i,\Phi_j}} \\ &= \frac{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_j}\mathbf{R}_{G,G}\mathbf{D}_{\Phi_i,\Phi_j}}{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_j}\mathbf{D}_{\Phi_i,\Phi_j}}, \qquad i,j = 1,2,\ldots,N.\end{aligned}$$

$$\langle \lambda_i \rangle = \bar{\Phi}_i^{\mathrm{T}} \langle \mathbf{G} \rangle \bar{\Phi}_i / \bar{\Phi}_i^{\mathrm{T}} \Phi_i, \qquad \mathbf{R}_{G,G} = \langle \mathbf{G} \otimes \mathbf{G} \rangle - \langle \mathbf{G} \rangle \otimes \langle \mathbf{G} \rangle$$

$$\langle \mathbf{G} \otimes \mathbf{G} \rangle = \langle (\mathbf{M} \otimes \mathbf{M})^{-1}(\mathbf{K} \otimes \mathbf{K}) \rangle, \qquad \langle \mathbf{G} \rangle = \langle \mathbf{M}^{-1}\mathbf{K} \rangle. \tag{2a–e}$$

It is possible that in many real mechanical systems M and K are uncorrelated, i.e., $\mathbf{R}_{M,K} = \mathbf{0}$, which should simplify equation (2). First, we identify the variance generated by the uncertainties of $\mathbf{M}^{-1} = (\bar{\mathbf{M}} + \tilde{\mathbf{M}})^{-1}$. The Taylor matrix expansion yields the following (refer to Appendix A for a brief review of matrix direct products):

$$\begin{aligned}\mathbf{M}^{-1} &= \bar{\mathbf{M}}^{-1} + \mathscr{D}_{\mathrm{cs}(M)^{\mathrm{T}}}\mathbf{M}^{-1}|_{\bar{M}}[\mathrm{cs}(\tilde{\mathbf{M}}) \otimes \mathbf{I}_N] + \frac{1}{2!}\mathscr{D}^2_{\mathrm{cs}(M)^{\otimes 2^{\mathrm{T}}}}\mathbf{M}^{-1}|_{\bar{M}}[\mathrm{cs}(\tilde{\mathbf{M}})^{\otimes 2} \otimes \mathbf{I}_N] + \cdots \\ &= \bar{\mathbf{M}}^{-1} + \bar{\mathbf{M}}^{-1}\mathbf{M}'\bar{\mathbf{M}}^{-1}[\mathrm{cs}(\tilde{\mathbf{M}}) \otimes \mathbf{I}_N] + \bar{\mathbf{M}}^{-1}\mathbf{M}''\bar{\mathbf{M}}^{-1}[\mathrm{cs}(\tilde{\mathbf{M}})^{\otimes 2} \otimes \mathbf{I}_N] + \cdots,\end{aligned} \tag{3}$$

where M′ and M″, etc. are the derivative terms. After some manipulation, equation (3) can be rewritten as

$$\begin{aligned}\mathbf{M}^{-1} &= \bar{\mathbf{M}}^{-1} + \bar{\mathbf{M}}^{-1}\mathbf{M}'[\mathrm{cs}(\tilde{\mathbf{M}}) \otimes \mathbf{I}_N]\bar{\mathbf{M}}^{-1} + \bar{\mathbf{M}}^{-1}\mathbf{M}''[\mathrm{cs}(\tilde{\mathbf{M}})^{\otimes 2} \otimes \mathbf{I}_N]\bar{\mathbf{M}}^{-1} + \cdots \\ &= \bar{\mathbf{M}}^{-1} + \bar{\mathbf{M}}^{-1}\mathbf{M}_1\bar{\mathbf{M}}^{-1} + \bar{\mathbf{M}}^{-1}\mathbf{M}_2\bar{\mathbf{M}}^{-1} + \cdots = \sum_{i=0}^{\infty} \bar{\mathbf{M}}^{-1}\mathbf{M}_i\bar{\mathbf{M}}^{-1}.\end{aligned} \tag{4}$$

Specifically, $\mathbf{M}_0 = \bar{\mathbf{M}}$ and $\mathbf{M}_1 = \tilde{\mathbf{M}}$. If the distribution of M is assumed to be symmetric, then odd moments of $\tilde{\mathbf{M}}$ are null matrices. The expected mean and variance of $\mathbf{M}^{-1}$ are given as

$$\langle \mathbf{M}^{-1} \rangle = \left\langle \sum_{i=0}^{\infty} \bar{\mathbf{M}}^{-1}\mathbf{M}_i\bar{\mathbf{M}}^{-1} \right\rangle,$$

$$\begin{aligned}\mathrm{Var}(\mathbf{M}^{-1}) &= \langle \mathbf{M}^{-1} \otimes \mathbf{M}^{-1} \rangle - \langle \mathbf{M}^{-1} \rangle \otimes \langle \mathbf{M}^{-1} \rangle \\ &= \left\langle \sum_{i=0}^{\infty} \bar{\mathbf{M}}^{-1}\mathbf{M}_i\bar{\mathbf{M}}^{-1} \otimes \sum_{i=0}^{\infty} \bar{\mathbf{M}}^{-1}\mathbf{M}_i\bar{\mathbf{M}}^{-1} \right\rangle \\ &\quad - \left\langle \sum_{i=0}^{\infty} \bar{\mathbf{M}}^{-1}\mathbf{M}_i\bar{\mathbf{M}}^{-1} \right\rangle \otimes \left\langle \sum_{i=0}^{\infty} \bar{\mathbf{M}}^{-1}\mathbf{M}_i\bar{\mathbf{M}}^{-1} \right\rangle \\ &= \left\langle \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbf{D}^{-1}_{M,M}\mathbf{M}_i \otimes \mathbf{M}_j\mathbf{D}^{-1}_{M,M} \right\rangle - \left\langle \sum_{i=0}^{\infty/2}\sum_{j=0}^{\infty/2} \mathbf{D}^{-1}_{M,M}\mathbf{M}_{2i} \otimes \mathbf{M}_{2j}\mathbf{D}^{-1}_{M,M} \right\rangle,\end{aligned} \tag{5a, b}$$

where $\mathbf{M}_i \otimes \mathbf{M}_j = \mathbf{0}$ when $i + j =$ odd. Two reasonable approximations of equation (5b) are presented as follows:

$$\mathrm{Var}(\mathbf{M}^{-1}) \approx (\mathbf{D}_{M,M} - \mathbf{R}_{M,M} - \langle \mathbf{M}_2 \otimes \mathbf{M}_2 \rangle)^{-1} - (\mathbf{D}_{M,M} - \langle \mathbf{M}_2 \rangle \otimes \langle \mathbf{M}_2 \rangle)^{-1}, \tag{6}$$

which can be simplified to

$$\mathrm{Var}(\mathbf{M}^{-1}) \approx (\mathbf{D}_{M,M} - \langle \mathbf{M}_1 \otimes \mathbf{M}_1 \rangle)^{-1} - \mathbf{D}^{-1}_{M,M} = (\mathbf{D}_{M,M} - \mathbf{R}_{M,M})^{-1} - \mathbf{D}^{-1}_{M,M}. \tag{7}$$

Consequently, $\mathbf{R}_{G,G}$ of equation (2) is now defined as follows by using equation (6):

$$\begin{aligned}\mathbf{R}_{G,G} &= \langle \mathbf{G}\otimes\mathbf{G}\rangle - \langle \mathbf{G}\rangle\otimes\langle \mathbf{G}\rangle = \langle (\mathbf{M}\otimes\mathbf{M})^{-1}(\mathbf{K}\otimes\mathbf{K})\rangle - \langle \mathbf{M}^{-1}\mathbf{K}\rangle\otimes\langle \mathbf{M}^{-1}\mathbf{K}\rangle \\ &\approx (\mathbf{D}_{M,M} - \mathbf{R}_{M,M} - \langle \mathbf{M}_2\otimes\mathbf{M}_2\rangle)^{-1}(\mathbf{D}_{K,K} + \mathbf{R}_{K,K}) \\ &\quad - (\mathbf{D}_{M,M} - \langle \mathbf{M}_2\rangle\otimes\langle \mathbf{M}_2\rangle)^{-1}\mathbf{D}_{K,K}. \end{aligned} \tag{8}$$

A simplified form of equation (8) results when equation (7) is used:

$$\mathbf{R}_{G,G} \approx (\mathbf{D}_{M,M} - \mathbf{R}_{M,M})^{-1}(\mathbf{D}_{K,K} + \mathbf{R}_{K,K}) - \mathbf{D}_{M,M}^{-1}\mathbf{D}_{K,K}. \tag{9}$$

Equation (9) can be simplified further when the parameter fluctuations are much smaller than the corresponding deterministic values: $\mathbf{R}_{G,G} \approx \mathbf{D}_{M,M}^{-1}\mathbf{R}_{M,M}\mathbf{D}_{M,M}^{-1}\mathbf{D}_{K,K} + \mathbf{D}_{M,M}^{-1}\mathbf{R}_{K,K}$. Note that this is compatible with the first order perturbation method. The covariance and higher order moments of $\mathbf{M}$ and $\mathbf{K}$ can be included in equations (8) and (9) by a similar manner, if they are statistically correlated.

Now let us go back to equation (2) and define the standard deviations of λ_i and ω_i as

$$\begin{aligned}\sigma(\lambda_i) &= (\langle \lambda_i\otimes\lambda_i\rangle - \langle \lambda_i\rangle^2)^{1/2} \\ &= \left(\frac{\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}(\langle \mathbf{G}\otimes\mathbf{G}\rangle - \langle \mathbf{G}\rangle\otimes\langle \mathbf{G}\rangle)\mathbf{D}_{\Phi_i,\Phi_i}}{\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{D}_{\Phi_i,\Phi_i}}\right)^{1/2} = \left(\frac{\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{R}_{G,G}\mathbf{D}_{\Phi_i,\Phi_i}}{\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{D}_{\Phi_i,\Phi_i}}\right)^{1/2} \\ \sigma(\omega_i) &\approx 0{\cdot}5\sigma(\lambda_i)/\bar{\omega}_i. \end{aligned} \tag{10a, b}$$

Similarly, the expected value of eigenvector formulation $\mathbf{\Phi}_i = \mathbf{G}\mathbf{\Phi}_i/\lambda_i$, estimated by using deterministic $\bar{\mathbf{\Phi}}_i$ and $\bar{\lambda}_i$, yields $\langle \mathbf{\Phi}_i\rangle = \langle \mathbf{G}\rangle\bar{\mathbf{\Phi}}_i/\bar{\lambda}_i$, which is then used to find the covariance between $\mathbf{\Phi}_i$ and $\mathbf{\Phi}_j$ in matrix form as

$$\begin{aligned}\mathrm{Cov}\,(\mathbf{\Phi}_i, \mathbf{\Phi}_j) &= \langle \mathbf{\Phi}_i\otimes\mathbf{\Phi}_j\rangle - \langle \mathbf{\Phi}_i\rangle\otimes\langle \mathbf{\Phi}_j\rangle \\ &= (\langle \mathbf{G}\otimes\mathbf{G}\rangle - \langle \mathbf{G}\rangle\otimes\langle \mathbf{G}\rangle)\mathbf{D}_{\Phi_i,\Phi_j}/\bar{\lambda}_i\bar{\lambda}_j = \mathbf{R}_{G,G}\mathbf{D}_{\Phi_i,\Phi_j}/\bar{\lambda}_i\bar{\lambda}_j. \end{aligned} \tag{11}$$

The covariance between λ_i and $\mathbf{\Phi}_j$ is found to be as follows by using the same technique:

$$\begin{aligned}\mathrm{Cov}\,(\lambda_i, \mathbf{\Phi}_j) &= \langle \lambda_i\otimes\mathbf{\Phi}_j\rangle - \langle \lambda_i\rangle\otimes\langle \mathbf{\Phi}_j\rangle \\ &= \mathbf{D}_{\Phi_i,\Phi_j}^{\mathrm{T}}(\langle \mathbf{G}\otimes\mathbf{G}\rangle - \langle \mathbf{G}\rangle\otimes\langle \mathbf{G}\rangle)\mathbf{D}_{\Phi_i,\Phi_j}/(\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\bar{\lambda}_j) \\ &= \mathbf{D}_{\Phi_i,\Phi_j}^{\mathrm{T}}\mathbf{R}_{G,G}\mathbf{D}_{\Phi_i,\Phi_j}/(\mathbf{D}_{\Phi_i,\Phi_j}^{\mathrm{T}}\bar{\lambda}_j). \end{aligned} \tag{12}$$

3.2. EXAMPLE I: SINGLE-DEGREE-OF-FREEDOM SYSTEM

Consider the undamped system shown in Figure 1(a) with $\bar{\lambda}_1 = \bar{\omega}_1^2 = 1$. Various random perturbations in mass (up to $\sigma_M = 0{\cdot}4$) are investigated. In Table 1 are listed the results predicted by the simplified, equations (9) and (10), and the exact, equations (8) and (10), as well as those yielded by the first order perturbation method [4]. All predictions are compared with those obtained by Monte Carlo simulation, which is considered as the benchmark. It is seen from Table 1 that the proposed methodology is in better agreement with the Monte Carlo simulation that the first order perturbation method. Observe the discrepancy between two analytical methods when σ_M is high. This occurs because the proposed methodology, unlike the perturbation method, maintains a few higher order terms of the mass randomness. It shows that the proposed method can indeed handle larger uncertainties of the system parameters reasonably well.

3.3. EXAMPLE II: TWO-DEGREE-OF-FREEDOM SYSTEM

The undamped system ($\mathbf{C} = \mathbf{0}$) of Figure 1(c) with $\bar{\omega}_1 = 0{\cdot}468$ and $\bar{\omega}_2 = 1{\cdot}510$ is investigated next. The randomness of the mass or stiffness matrix being fully correlated

is considered, and the corresponding results for $\sigma(\omega_1)$ and $\sigma(\omega_2)$ are presented in Table 2. Again, the proposed theory matches with the Monte Carlo simulation better than the first order perturbation method [4]. Like Example I, the error for the case $\mathbf{R}_{M,M} = 0{\cdot}04\mathbf{D}_{M,M}$ is associated with the higher order terms, which are obviously ignored by the first order perturbation method.

4. PROPORTIONALLY DAMPED SYSTEM

4.1. THEORY

Equation (1) of dimension N can be rewritten as follows, in accordance with the assumptions stated earlier:

$$\ddot{\mathbf{X}}(t) + \mathbf{B}\dot{\mathbf{X}}(t) + \mathbf{A}\mathbf{X}(t) = \mathbf{0}, \qquad \mathbf{A} = \mathbf{M}^{-1}\mathbf{K}, \qquad \mathbf{B} = \mathbf{M}^{-1}\mathbf{C}. \tag{13a–c}$$

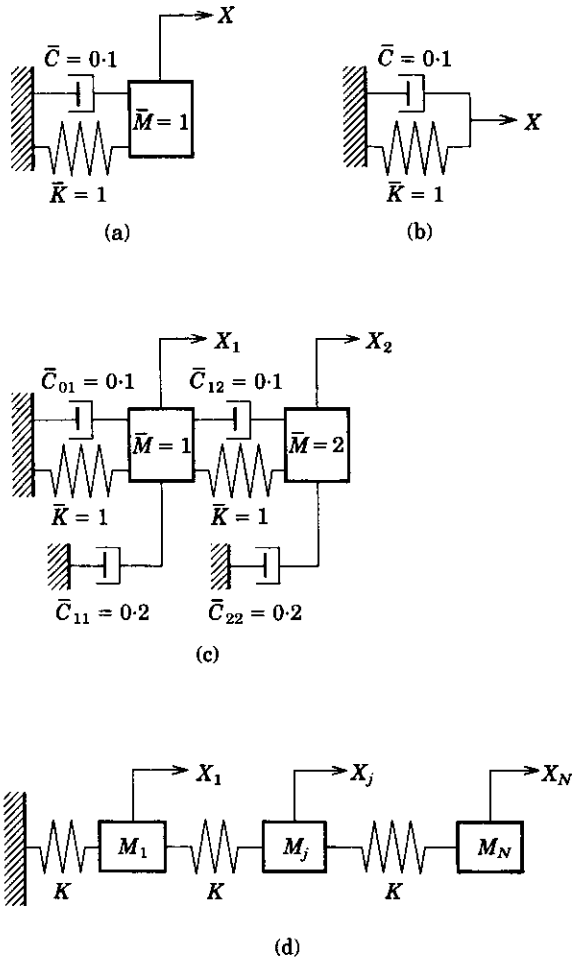

Figure 1. The physical systems used to illustrate the theory: (a) the single-degree-of-freedom system used for Examples I and III; (b) the viscoelastic model considered for Example IV; (c) the two-degree-of-freedom system used for Examples II and V; (d) the periodic system with randomly distributed masses used for Example VI.

TABLE 1

Standard deviation of ω_1 for Example I (Figure 1(a))

	$\sigma(\omega_1)$ and associated error ϵ†			
		Proposed method		
σ_M	Monte Carlo simulation	Simplified, equations (9) and (10)	Exact, equations (8) and (10)	First order perturbation method
0·1	0·0504	0·0503 ($\epsilon = 0{\cdot}2\%$)	0·0504 ($\epsilon = 0\%$)	0·050 ($\epsilon = 0{\cdot}8\%$)
0·2	0·106	0·102 ($\epsilon = 3{\cdot}8\%$)	0·104 ($\epsilon = 1{\cdot}9\%$)	0·100 ($\epsilon = 5{\cdot}7\%$)
0·3	0·176	0·157 ($\epsilon = 10{\cdot}8\%$)	0·163 ($\epsilon = 7{\cdot}4\%$)	0·150 ($\epsilon = 20{\cdot}5\%$)
0·4	0·271	0·218 ($\epsilon = 19{\cdot}6\%$)	0·235 ($\epsilon = 13{\cdot}3\%$)	0·200 ($\epsilon = 26{\cdot}2\%$)

† ϵ = (Monte Carlo simulation − analytical method)/Monte Carlo simulation.

Now let $\mathbf{X}(t) = \mathbf{\Phi}\mathbf{Y}(t)$, where $\mathbf{\Phi}$ is the modal matrix of the undamped system described earlier in section 3.1 and $\mathbf{Y}(t)$ is the principal co-ordinate vector. Furthermore, premultiply equation (13a) by $\mathbf{\Phi}^{\mathrm{T}}$ to yield the characteristic equation for the damped system:

$$\gamma_i^2 \mathbf{\Phi}_i^{\mathrm{T}} \mathbf{\Phi}_i + \gamma_i \mathbf{\Phi}_i^{\mathrm{T}} \mathbf{B} \mathbf{\Phi}_i + \mathbf{\Phi}_i^{\mathrm{T}} \mathbf{A} \mathbf{\Phi}_i = \mathbf{0}, \qquad i = 1, 2, \ldots, N. \tag{14}$$

TABLE 2

Standard deviation of ω_1 and ω_2 for Example II (Figure 1(c))

			$\sigma(\omega_i)$ and error ϵ†		
Mode i	$\mathbf{R}_{M,M}$	$\mathbf{R}_{K,K}$	Monte Carlo simulation	Proposed method, equations (9) and (10)	First order perturbation method
1	$0{\cdot}01\mathbf{D}_{M,M}$	0·0	0·0234	0·0235 ($\epsilon = -0{\cdot}4\%$)	0·0234 ($\epsilon = 0{\cdot}0\%$)
	$0{\cdot}01\mathbf{D}_{M,M}$	$0{\cdot}01\mathbf{D}_{K,K}$	0·0334	0·0333 ($\epsilon = 0{\cdot}3\%$)	0·0331 ($\epsilon = 0{\cdot}9\%$)
	$0{\cdot}04\mathbf{D}_{M,M}$	0·0	0·0496	0·0478 ($\epsilon = 3{\cdot}6\%$)	0·0468 ($\epsilon = 5{\cdot}6\%$)
2	$0{\cdot}01\mathbf{D}_{M,M}$	0·0	0·0755	0·0758 ($\epsilon = -0{\cdot}4\%$)	0·0755 ($\epsilon = 0\%$)
	$0{\cdot}01\mathbf{D}_{M,M}$	$0{\cdot}01\mathbf{D}_{K,K}$	0·1074	0·1070 ($\epsilon = 0{\cdot}4\%$)	0·1067 ($\epsilon = 0{\cdot}7\%$)
	$0{\cdot}04\mathbf{D}_{M,M}$	0·0	0·1599	0·1540 ($\epsilon = 3{\cdot}7\%$)	0·1510 ($\epsilon = 5{\cdot}5\%$)

† ϵ = (Monte Carlo simulation − analytical method)/Monte Carlo simulation.

Here, $\gamma_i = -\zeta_i\omega_i \pm j\omega_{di} = -\zeta_i\omega_i \pm j\omega_i\sqrt{1-\zeta_i^2}$ is the ith complex valued eigenvalue corresponding to the real eigenvector $\mathbf{\Phi}_i$, ζ_i is the damping ratio, ω_{di} is the damped natural frequency and $j = \sqrt{-1}$. Rewrite equation (14) as

$$\gamma_i = \frac{-\mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i \pm j\sqrt{4(\mathbf{\Phi}_i^T\mathbf{\Phi}_i\mathbf{\Phi}_i^T\mathbf{A}\mathbf{\Phi}_i) - (\mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i)^2}}{2\mathbf{\Phi}_i^T\mathbf{\Phi}_i} = \mathrm{Re}\,(\gamma_i) + j\,\mathrm{Im}\,(\gamma_i), \tag{15}$$

where

$$\mathrm{Re}\,(\gamma_i) = \frac{-\mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i}{2\mathbf{\Phi}_i^T\mathbf{\Phi}_i}, \qquad [\mathrm{Im}\,(\gamma_i)]^2 = \omega_{di}^2 = \frac{4\mathbf{\Phi}_i^T\mathbf{\Phi}_i\mathbf{\Phi}_i^T\mathbf{A}\mathbf{\Phi}_i - \mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i\mathbf{\Phi}_i^T\mathbf{B}^T\mathbf{\Phi}_i}{4(\mathbf{\Phi}_i^T\mathbf{\Phi}_i)^2}. \tag{16a, b}$$

If γ_i is estimated using $\bar{\mathbf{\Phi}}_i$, the expected value of γ_i yields

$$\langle \mathrm{Re}\,(\gamma_i)\rangle = -0{\cdot}5\bar{\mathbf{\Phi}}_i^T\langle\mathbf{B}\rangle\bar{\mathbf{\Phi}}_i/d_i,$$

$$\langle[\mathrm{Im}\,(\gamma_i)]^2\rangle = \bar{\omega}_{di}^2 = (4d_i\bar{\mathbf{\Phi}}_i^T\langle\mathbf{A}\rangle\bar{\mathbf{\Phi}}_i - \bar{\mathbf{\Phi}}_i^T\langle\mathbf{B}\bar{\mathbf{\Phi}}_i\bar{\mathbf{\Phi}}_i^T\mathbf{B}^T\rangle\bar{\mathbf{\Phi}}_i)/(4d_i^2),$$

$$d_i = \bar{\mathbf{\Phi}}_i^T\bar{\mathbf{\Phi}}_i. \tag{17a–c}$$

Since γ_i is complex valued, the standard deviation of γ_i is complex valued as well: $\sigma_{\gamma_i} = \sigma(\mathrm{Re}\,(\gamma_i)) + j\sigma(\mathrm{Im}\,(\gamma_i))$. It is determined by using the covariance matrices of $\mathbf{A}$ and $\mathbf{B}$. Furthermore, covariances of γ_i and ω_{di} can be given, as

$$\mathrm{Cov}\,(\mathrm{Re}\,(\gamma_i), \mathrm{Re}\,(\gamma_j)) = \langle \mathrm{Re}\,(\gamma_i)\otimes\mathrm{Re}\,(\gamma_j)\rangle - \langle \mathrm{Re}\,(\gamma_i)\rangle\otimes\langle \mathrm{Re}\,(\gamma_j)\rangle,$$

$$\mathrm{Cov}\,(\omega_{di}^2, \omega_{dj}^2) = \langle\omega_{di}^2\otimes\omega_{dj}^2\rangle - \langle\omega_{di}^2\rangle\otimes\langle\omega_{dj}^2\rangle. \tag{18a, b}$$

4.2. RAYLEIGH DAMPING MODEL

Two limiting cases of the Rayleigh damping model $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ are investigated next: (a) $\mathbf{C}$ is uncorrelated with $\mathbf{M}$ and $\mathbf{K}$, i.e., α and β are random variables; and (b) $\mathbf{C}$ is fully correlated with $\mathbf{M}$ and $\mathbf{K}$, i.e., $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ are deterministic.

4.2.1. *Case (a)*

If $\mathbf{C}$ is uncorrelated with $\mathbf{M}$ and $\mathbf{K}$, define the standard deviation of γ_i by using equations (16)–(18):

$$\begin{aligned}\sigma(\mathrm{Re}\,(\gamma_i)) &= (\langle \mathrm{Re}\,(\gamma_i)\otimes\mathrm{Re}\,(\gamma_i)\rangle - \langle \mathrm{Re}\,(\gamma_i)\rangle\otimes\langle \mathrm{Re}\,(\gamma_i)\rangle)^{1/2}\\ &= 0{\cdot}5(\mathbf{D}_{\Phi_i,\Phi_i}^T\mathbf{R}_{B,B}\mathbf{D}_{\Phi_i,\Phi_i}/(\mathbf{D}_{\Phi_i,\Phi_i}^T\mathbf{D}_{\Phi_i,\Phi_i}))^{1/2},\\ \mathbf{R}_{B,B} &\approx (\mathbf{D}_{M,M} - \mathbf{R}_{M,M})^{-1}(\mathbf{D}_{C,C} + \mathbf{R}_{C,C}) - \mathbf{D}_{M,M}^{-1}\mathbf{D}_{C,C}.\end{aligned} \tag{19a, b}$$

The standard deviation of ω_{di}^2 is

$$\begin{aligned}\sigma(\omega_{di}^2) &= (\langle\omega_{di}^2\otimes\omega_{di}^2\rangle - \langle\omega_{di}^2\rangle\otimes\langle\omega_{di}^2\rangle)^{1/2}\\ &\approx (\mathbf{D}_{\Phi_i,\Phi_i}^T(16d_i^2\mathbf{R}_{A,A} - 4d_i(\mathbf{R}_{A,B\bar{\Phi}_i\bar{\Phi}_i^TB^T} + \mathbf{R}_{B\bar{\Phi}_i\bar{\Phi}_i^TB^T,A})\\ &\quad + \mathbf{R}_{B\bar{\Phi}_i\bar{\Phi}_i^TB^T,B\bar{\Phi}_i\bar{\Phi}_i^TB^T})\mathbf{D}_{\Phi_i,\Phi_i}/(16d_i^4))^{1/2}.\end{aligned} \tag{20}$$

Note that the following terms appearing in equation (20) can be simplified as

$$\mathbf{R}_{A,A} \approx (\mathbf{D}_{M,M} - \mathbf{R}_{M,M})^{-1}(\mathbf{D}_{K,K} + \mathbf{R}_{K,K}) - \mathbf{D}_{M,M}^{-1}\mathbf{D}_{K,K},$$

$$\mathbf{R}_{A,B\bar{\Phi}_i\bar{\Phi}_i^TB^T} = \mathbf{R}_{M^{-1},B\bar{\Phi}_i\bar{\Phi}_i^TB^T}\mathbf{D}_{K,I}$$

$$\mathbf{D}_{\Phi_i,\Phi_i}^T\mathbf{R}_{B\bar{\Phi}_i\bar{\Phi}_i^TB^T,B\bar{\Phi}_i\bar{\Phi}_i^TB^T}\mathbf{D}_{\Phi_i,\Phi_i}/(16d_i^4) = 4[\mathrm{Re}\,(\bar{\gamma}_i)]^2\,\mathrm{Var}\,(\mathrm{Re}\,(\gamma_i)). \tag{21a–c}$$

The modal damping ratio ζ_i of the system is defined as $\zeta_i = 0{\cdot}5c_i/\sqrt{m_i k_i}$, where $c_i = \boldsymbol{\Phi}_i^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi}_i$, $m_i = \boldsymbol{\Phi}_i^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi}_i$ and $k_i = \boldsymbol{\Phi}_i^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi}_i$ are modal parameters. The corresponding standard deviations are

$$\sigma(\zeta_i) \approx \tfrac{1}{2}((R_{c_i,c_i}/\bar{m}_i\bar{k}_i) + (\bar{c}_i^2 R_{k_i,k_i}/4\bar{m}_i\bar{k}_i^3) + (\bar{c}_i^2 R_{m_i,m_i}/4\bar{m}_i^3\bar{k}_i))^{0{\cdot}5},$$

$$R_{c_i,c_i} = \mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{R}_{C,C}\mathbf{D}_{\Phi_i,\Phi_i}, \qquad R_{k_i,k_i} = \mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{R}_{K,K}\mathbf{D}_{\Phi_i,\Phi_i},$$

$$R_{m_i,m_i} = \mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{R}_{M,M}\mathbf{D}_{\Phi_i,\Phi_i}. \tag{22a–d}$$

If the system is lightly damped, the variance of ζ_i has to be much smaller than unity. Accordingly, $\mathrm{Im}(\gamma_i)$ is estimated by assuming $\sqrt{1-\zeta_i^2}$ to be deterministic. Thus

$$\langle[\mathrm{Im}(\gamma_i)]^2\rangle = \langle\omega_{di}^2\rangle = (1-\bar{\zeta}_i^2)\bar{\boldsymbol{\Phi}}_i^{\mathrm{T}}\langle\mathbf{A}\rangle\bar{\boldsymbol{\Phi}}_i/(\bar{\boldsymbol{\Phi}}_i^{\mathrm{T}}\bar{\boldsymbol{\Phi}}_i), \tag{23}$$

and the standard deviation of ω_{di} is calculated by using

$$\sigma(\omega_{di}) \approx 0{\cdot}5\sigma(\omega_{di}^2)/\bar{\omega}_{di},$$

$$\sigma(\omega_{di}^2) = (\langle\omega_{di}^2 \otimes \omega_{di}^2\rangle - \langle\omega_{di}^2\rangle \otimes \langle\omega_{di}^2\rangle)^{1/2}$$

$$\approx (1-\bar{\zeta}_i^2)(\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{R}_{A,A}\mathbf{D}_{\Phi_i,\Phi_i}/(\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}}\mathbf{D}_{\Phi_i,\Phi_i}))^{1/2}. \tag{24a, b}$$

4.2.2. *Case (b)*

If $\mathbf{C}$ is fully correlated with $\mathbf{M}$ and $\mathbf{K}$, the standard deviation of γ_i can be given in terms of $\sigma(\omega_i)$ since $\mathbf{C} = \bar{\alpha}\mathbf{M} + \bar{\beta}\mathbf{K}$. The eigenvalue has the form

$$\mathrm{Re}(\gamma_i) = -0{\cdot}5(\bar{\alpha} + \bar{\beta}\omega_i^2), \qquad [\mathrm{Im}(\gamma_i)]^2 = \omega_i^2 - [\mathrm{Re}(\gamma_i)]^2. \tag{25a, b}$$

The standard deviation of γ_i is defined as

$$\sigma(\mathrm{Re}(\gamma_i)) \approx 0{\cdot}5\bar{\beta}\sigma(\omega_i^2), \qquad \sigma([\mathrm{Im}(\gamma_i)]^2) \approx (1-\bar{\zeta}_i^2)\sigma(\omega_i^2). \tag{26a, b}$$

The modal damping ratio ζ_i is defined as follows, along with its standard deviation:

$$\zeta_i = 0{\cdot}5(\bar{\alpha}/\omega_i + \bar{\beta}\omega_i), \qquad \sigma(\zeta_i) \approx 0{\cdot}5\,\mathrm{abs}(\bar{\beta} - \bar{\alpha}/\bar{\omega}_i^2)\sigma(\omega_i). \tag{27a, b}$$

A few specific cases of interest are considered next: (i) if $\alpha = 0$ or $\mathbf{C} = \bar{\beta}\mathbf{K}$, $\sigma(\zeta_i) \cong 0{\cdot}5\bar{\beta}\sigma(\omega_i)$; (ii) if $\beta = 0$ or $\mathbf{C} = \bar{\alpha}\mathbf{M}$, $\sigma(\zeta_i) \approx 0{\cdot}5\bar{\alpha}\sigma(\omega_i)/\bar{\omega}_i^2$, and (iii) if $\bar{\alpha} = \bar{\beta}\bar{\omega}_i^2$ then $\sigma(\zeta_i) = 0$.

One interesting case arises when $\mathbf{C} = \bar{\beta}\mathbf{K}$ but only $\mathbf{M}$ is uncertain. The expression obtained for this fully correlated case should be the same as that found previously for the uncorrelated case, i.e.,

$$\sigma(\zeta_i) = \tfrac{1}{4}\left(\frac{\bar{c}_i^2 R_{m_i,m_i}}{\bar{m}_i^3\bar{k}_i}\right)^{0{\cdot}5}. \tag{28}$$

Similarly, each eigenvector is represented as

$$\boldsymbol{\Phi}_i = \mathbf{A}\boldsymbol{\Phi}_i/\omega_i^2. \tag{29}$$

Using the proposed direct product technique, the covariance matrix between eigenvectors is estimated by using deterministic $\bar{\omega}_i^2$ and $\bar{\boldsymbol{\Phi}}_i$ as

$$\mathrm{Cov}(\boldsymbol{\Phi}_i, \boldsymbol{\Phi}_j) = \langle\boldsymbol{\Phi}_i \otimes \boldsymbol{\Phi}_j\rangle - \langle\boldsymbol{\Phi}_i\rangle \otimes \langle\boldsymbol{\Phi}_j\rangle = \mathbf{R}_{A,A}\mathbf{D}_{\Phi_i,\Phi_j}/(\bar{\omega}_i^2\bar{\omega}_j^2). \tag{30}$$

It must be noted that all covariances between the real and imaginary parts of γ_i can also be calculated by using the proposed methodology.

TABLE 3

Standard deviation of $\mathrm{Re}(\gamma_i)$ *for Example III (Figure* 1(*a*))

Mode i	$R_{M,M}$	$R_{C,C}$	$R_{K,K}$	$\sigma(\mathrm{Re}(\gamma_i))$ and error ϵ†	
				Monte Carlo simulation	Proposed method
1	0·0	0·0	0·01	0·0	0·0 ($\epsilon = 0\%$)
	0·0	0·0004	0·01	0·0101	0·0100 ($\epsilon = 1\%$)
2	0·0	0·0	0·01	0·0500	0·0500 ($\epsilon = 0\%$)
	0·0	0·0004	0·01	0·0497	0·0500 ($\epsilon = 0{\cdot}6\%$)

† ϵ = (Monte Carlo simulation − analytical method)/Monte Carlo simulation.

4.3. EXAMPLE III: SINGLE-DEGREE-OF-FREEDOM SYSTEM

For the damped system of Figure 1(a), with $\bar{\gamma}_1 = -0{\cdot}05 \pm 0{\cdot}99875\mathrm{j}$, the standard deviation of γ_1 is calculated by using equations (19) and (24). Results are given in Table 3 for $R_{C,C} = 0$ and 0·0004; in the second case C is assumed to be uncorrelated with M and K. It can be seen from Table 3 that the proposed methodology is in excellent agreement with the Monte Carlo simulation (sample size = 2000).

4.4. EXAMPLE IV: VISCOELASTIC SYSTEM

A first order system $C\dot{X}(t) + KX(t) = 0$, as shown in Figure 1(b), with deterministic time constant $\bar{\tau} = \bar{C}/\bar{K} = 0{\cdot}1$, is investigated. When randomness is introduced in the spring, τ is a random variable. Given $R_{C,C} = 0{\cdot}01 D_{C,C}$, the proposed method finds $\sigma(\tau)$ to be 0·01. This compares very well with $\sigma(\tau) = 0{\cdot}0099$ yielded by the Monte Carlo simulation based on 1000 samples. The results of this case (with $\epsilon = 1\%$) verify the proposed theory.

4.5. EXAMPLE V: TWO-DEGREE-OF-FREEDOM DAMPED SYSTEM

Consider the damped system of Figure 1(c), with $\bar{\gamma}_1 = -0{\cdot}06096 \pm 0{\cdot}464\mathrm{j}$ and $\bar{\gamma}_2 = -0{\cdot}1640 \pm 1{\cdot}5013\mathrm{j}$. The results obtained by using equations (19) and (24) are compared with those yielded by the Monte Carlo simulation based on 2000 samples in Table 4. Again, the proposed methodology is found to be in excellent agreement with simulation.

Now we re-examine this physical system by using the Rayleigh damping model, with the assumption that $\mathbf{C}$ is fully correlated with $\mathbf{M}$ and/or $\mathbf{K}$; recall case (b) from section 4.2. In Table 5 are presented results for $\sigma(\zeta_1)$ and $\sigma(\zeta_2)$, given the following data set: $\mathbf{R}_{M,M} = 0{\cdot}01\mathbf{D}_{M,M}$, $\mathbf{R}_{K,K} = 0{\cdot}01\mathbf{D}_{K,K}$, $\bar{\omega}_1 = 0{\cdot}468$ and $\bar{\omega}_2 = 1{\cdot}510$. The standard deviations of ω_1 and ω_2 were found earlier in Table 2 by using the proposed methodology: $\sigma(\omega_1) = 0{\cdot}0333$ and $\sigma(\omega_2) = 0{\cdot}1070$. For this system, the Monte Carlo simulation is run twice, once with 1000 samples and the second time with 2000 samples. As expected, the proposed method matches better with the numerical simulation based on 2000 samples,

as is evident from $\epsilon \leqslant 3{\cdot}5\%$, and smaller sample size Monte Carlo simulation results deviate slightly more from the proposed method.

5. SYSTEM WITH ONLY RANDOMLY DISTRIBUTED MASSES

5.1. THEORY

Next, we examine the parameter case in which only one matrix is randomly distributed. For the sake of illustration, consider an undamped system with randomly distributed masses. If the expected mean mass matrix is described as $\bar{\mathbf{M}} = \bar{M}\mathbf{I}$ [13], the covariance mass matrix can be given as

$$\mathbf{R}_{M,M} = \sigma_M^2 \mathbf{S} = \sigma_M^2 \begin{bmatrix} r_{11}s_1^2 & & & & \\ & r_{12}s_1s_2 & & \mathbf{0} & \\ & & \cdots & & \\ & & & r_{1N}s_1s_N & \\ & & & & \cdots \\ & \mathbf{0} & & & \cdots \\ & & & & r_{NN}s_N^2 \end{bmatrix}, \tag{31}$$

where $r_{jk} = \langle \tilde{M}_j \tilde{M}_k \rangle / \sigma_{M_j} \sigma_{M_k}$ is the correlation coefficient between the $\tilde{M}_j$ and $\tilde{M}_k$ terms, $s_j = \sigma_{M_j}/\sigma_M$ and $\sigma_M = \max(\sigma_{M_j})$; note that $-1 \leqslant r_{jk} \leqslant 1$ and $0 \leqslant s_j \leqslant 1$. The standard

TABLE 4

Standard deviation of γ_1 and γ_2 for Example V (Figure 1(c))

				$\sigma(\gamma_i)$ and error ϵ†	
$\mathbf{R}_{M,M}$	$\mathbf{R}_{C,C}$	$\mathbf{R}_{K,K}$	γ_i	Monte Carlo simulation	Proposed method, equations (9) and (10)
0	**0**	$0{\cdot}01\mathbf{D}_{K,K}$	Re (γ_1)	0·0	0·0 ($\epsilon = 0{\cdot}0\%$)
			Im (γ_1)	0·0235	0·0232 ($\epsilon = 1{\cdot}3\%$)
			Re (γ_2)	0·0	0·0 ($\epsilon = 0{\cdot}0\%$)
			Im (γ_2)	0·0756	0·0751 ($\epsilon = 0{\cdot}7\%$)
0	$0{\cdot}01\mathbf{D}_{C,C}$	$0{\cdot}01\mathbf{D}_{K,K}$	Re (γ_1)	0·00606	0·0061 ($\epsilon = -0{\cdot}6\%$)
			Im (γ_1)	0·0235	0·0232 ($\epsilon = 1{\cdot}3\%$)
			Re (γ_2)	0·0163	0·0164 ($\epsilon = -0{\cdot}6\%$)
			Im (γ_2)	0·0753	0·0751 ($\epsilon = 0{\cdot}7\%$)

† ϵ = (Monte Carlo simulation − analytical method)/Monte Carlo simulation.

TABLE 5

Standard deviation of ζ_1 for Example V (Figure 1(c))

Mode i	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\zeta}_i$	$\sigma(\zeta_i)$ and error ϵ† Monte Carlo simulation 1000 samples	Monte Carlo simulation 2000 samples	Proposed method, equation (28)
1	0·1	0·0	0·0107	0·00784	0·00774	0·00762 ($\epsilon = 1{\cdot}6\%$)
	0·0	0·1	0·0234	0·00176	0·00170	0·00167 ($\epsilon = 1{\cdot}8\%$)
	0·1	0·1	0·1302	0·00615	0·00605	0·00595 ($\epsilon = 1{\cdot}7\%$)
2	0·1	0·0	0·0331	0·00243	0·00240	0·00235 ($\epsilon = 2{\cdot}1\%$)
	0·0	0·1	0·0755	0·00550	0·00550	0·00534 ($\epsilon = 2{\cdot}9\%$)
	0·1	0·1	0·1086	0·00307	0·00310	0·00299 ($\epsilon = 3{\cdot}5\%$)

† A Monte Carlo simulation with 2000 samples is taken as the benchmark for error; ϵ = (Monte Carlo simulation − analytical method)/Monte Carlo simulation.

deviation of random eigensolutions can be simplified by using equations (7)–(12). Substituting equation (31) into equations (10) and (11), we obtain

$$\sigma(\lambda_i) \approx \left(\frac{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}[(\mathbf{D}_{I,I} - \bar{\sigma}_M^2 \mathbf{S})^{-1}\mathbf{D}_{I,I}]\mathbf{D}_{\Phi_i,\Phi_i}}{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}\mathbf{D}_{\Phi_i,\Phi_i}} \right)^{1/2} \bar{\lambda}_i, \qquad \sigma(\boldsymbol{\Phi}_{ij}) \approx \left(\frac{1}{1 - \bar{\sigma}_M^2 S_j^2} - 1 \right)^{1/2} \bar{\boldsymbol{\Phi}}_{ij}, \tag{32a, b}$$

where $\bar{\sigma}_M = \sigma_M / \bar{M}$. The standard deviation of the ith normalized eigenvalue $\lambda_{oi} = \lambda_i / \bar{\lambda}_o$ can be expressed as follows, given that $\bar{\lambda}_o = \bar{K}/\bar{M}$, $\bar{\lambda}_{oi} = \bar{\lambda}_i / \bar{\lambda}_o$ and $\lambda_{oi} = \bar{\lambda}_{oi} + \tilde{\lambda}_{oi}$:

$$\sigma(\lambda_{oi}) \approx \left(\frac{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}[(\mathbf{D}_{I,I} - \bar{\sigma}_M^2 \mathbf{S})^{-1} - \mathbf{D}_{I,I}]\mathbf{D}_{\Phi_i,\Phi_i}}{\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}\mathbf{D}_{\Phi_i,\Phi_i}} \right)^{1/2} \bar{\lambda}_{oi}. \tag{33}$$

The effect of the randomness of the stiffness matrix alone can be estimated in a similar manner. Also, all covariances of eigenvalues and eigenvectors can also be calculated by using the methodology described earlier in section 3.

5.2. LIMITING CASES OF RANDOM MASS MATRIX

5.2.1. *Case (a)*

If the randomness of each diagonal mass element is identical and they are fully correlated with each other, i.e., $\mathbf{S} = \mathbf{I}$, equations (33) and (32b) reduce to

$$\sigma(\lambda_{oi}) \approx \left(\frac{1}{1 - \bar{\sigma}_M^2} - 1 \right)^{1/2} \bar{\lambda}_{oi}, \qquad \sigma(\boldsymbol{\Phi}_{ij}) \approx \left(\frac{1}{1 - \bar{\sigma}_M^2} - 1 \right)^{1/2} \bar{\boldsymbol{\Phi}}_{ij}. \tag{34a, b}$$

5.2.2. *Case (b)*

If the randomness of each diagonal mass element is identical but they are uncorrelated with each other, i.e., $r_{jk} = \delta_{jk}$, where δ_{jk} is the Kronecker delta and $s_j = s_k = 1$, then equations (33) and (32b) reduce to

$$\sigma(\lambda_{oi}) \approx \left(\frac{\sum_{j=1}^{N} \Phi_{ij}^4}{\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}} \mathbf{D}_{\Phi_i,\Phi_i}} \right)^{1/2} \left(\frac{1}{1-\bar{\sigma}_M^2} - 1 \right)^{1/2} \bar{\lambda}_{oi}, \qquad \sigma(\Phi_{ij}) = \left(\frac{1}{1-\bar{\sigma}_M^2} - 1 \right)^{1/2} \Phi_{ij}. \quad (35a, b)$$

When the random fluctuations are very small, i.e., $\bar{\sigma}_M \ll 1$, these solutions can be approximated by using the first order perturbation technique. For instance, for case (b), equation (35) is simplified to

$$\sigma(\lambda_{oi}) \approx \left(\frac{\sum_{j=1}^{N} \Phi_{ij}^4}{\mathbf{D}_{\Phi_i,\Phi_i}^{\mathrm{T}} \mathbf{D}_{\Phi_i,\Phi_i}} \right)^{1/2} \bar{\sigma}_M \bar{\lambda}_{oi}, \qquad \sigma(\Phi_{ij}) \approx \bar{\sigma}_M \Phi_{ij}. \quad (36a, b)$$

5.3. EXAMPLE VI: MULTI-DEGREE-OF-FREEDOM SYSTEM

The undamped system of Figure 1(d), with random masses, is studied to illustrate the proposed theory. The randomness of the masses being uncorrelated, case (b) as described in section 5.2 is investigated first for $N = 5$. Equation (35) is used to obtain results for σ_{λ_1}, given various values of σ_M. Results are compared in Table 6 with predictions yielded by the Monte Carlo simulation (sample size = 1000) and the first order perturbation method [6]. When $\sigma_M \ll 1$, all four predictions are virtually identical. However as σ_M increases, the first order perturbation method begins to deviate from the Monte Carlo simulation. The proposed method, as described by equation (35), gives excellent results up to $\sigma_M = 0{\cdot}3$, using the Monte Carlo simulation as a benchmark. When σ_M is as high as 0·5, only a discrepancy of $\epsilon = 6{\cdot}6\%$ is seen, as opposed to $\epsilon = 19{\cdot}2\%$ associated with the first order perturbation method. This example demonstrates clearly that the proposed method is superior to the existing analytical techniques based on the first order perturbations.

TABLE 6

Standard deviation of λ_1 for Example VI (Figure 1(d)) with $N = 5$

	σ_{λ_1} and error ϵ†		
σ_M	Monte Carlo simulation	Proposed method, equation (17)	First order perturbation method [5]
0·05	0·00217	0·00212 ($\epsilon = 2{\cdot}3\%$)	0·00211 ($\epsilon = 2{\cdot}8\%$)
0·15	0·00645	0·00642 ($\epsilon = 0{\cdot}5\%$)	0·00634 ($\epsilon = 1{\cdot}7\%$)
0·3	0·01358	0·01331 ($\epsilon = 2\%$)	0·01269 ($\epsilon = 6{\cdot}6\%$)
0·5	0·02616	0·02443 ($\epsilon = 6{\cdot}6\%$)	0·02115 ($\epsilon = 19\%$)

† ϵ = (Monte Carlo simulation − analytical method)/Monte Carlo simulation.

TABLE 7

Standard deviation of λ_{oi} for Example VI (Figure 1(d)) with $N = 10$

		$\sigma(\lambda_{oi})$	
i	$\bar{\lambda}_{oi}$	Proposed method, equation (11)	First perturbation method [6]
1	0·0223	0·00849	0·00844
2	0·1981	0·07523	0·07486
3	0·5339	0·20203	0·20179
4	1·000	0·3799	0·37796
5	1·555	0·5906	0·5877
6	2·1495	0·8164	0·8124
7	2·7307	1·0373	1·0321
8	3·247	1·233	1·227
9	3·652	1·387	1·381
10	3·911	1·486	1·478

The standard deviation of normalized eigenvalues associated with the random masses of Figure 1(d), with $N = 10$, is shown in Table 7. The results obtained by using equation (30) are slightly higher than those obtained earlier by Pierre [6] and Soong and Bogdanoff [13]. This is expected, since the proposed method includes the higher order terms.

6. CONCLUDING REMARKS

A new analytical technique for random eigensolutions of undamped and proportionally damped discrete systems has been presented and validated by using two existing techniques, namely the first order perturbation method and the Monte Carlo simulation. It is potentially a powerful technique because of its reasonable accuracy coupled with computational ease. Yet another key advantage of the proposed method is its application to the determination of impulse and frequency response characteristics, which will be the subject of the companion paper (Part II). The proposed method has been found to be more accurate than the commonly used first order perturbation methods, especially when random fluctuations are no longer very small. Also, it has been observed to be computationally faster than the Monte Carlo simulation, at least for several problems. For instance, the computational speed ratio between the proposed method and Monte Carlo simulation for Example V was of the order of 10^2 to 10^3 when it was solved on the same computer. Such a ratio may be even higher for a complex vibratory system of higher N; this is due to the fact that a large number of iterations is required to estimate the probability distribution in a stochastic simulation [19]. However, it should be noted that there are uncertainties for various numerical examples associated with Monte Carlo predictions. Therefore, comparisons given by ϵ values should be considered valid within the confidence interval associated with a sample size.

The major weaknesses of the proposed analytical method come from various simplifying assumptions made during development, some of which are as follows: (i) this method loses accuracy when random fluctuations are very high (say, when the normalized standard deviation is higher than 0·5); (ii) it cannot address localization effects in a periodic disordered system [16]; and (iii) non-proportionally damped vibratory systems cannot be analyzed. Research is underway to overcome these shortcomings, as well as extend the proposed method to continuous systems. Also, an effort will be made to investigate the possibility of using a faster Monte Carlo algorithm [19].

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APPENDIX A: MATRIX DIRECT PRODUCTS

A.1. REVIEW OF KEY CONCEPTS

Consider the direct product of two matrices $\mathbf{A}(p \times q)$ and $\mathbf{B}(s \times t)$, given by $\mathbf{A}\otimes\mathbf{B}$. It is a $ps \times qt$ dimensioned matrix, defined as [18]

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ a_{21}\mathbf{B} & & & \\ \cdots & & & \\ a_{p1}\mathbf{B} & & & a_{pq}\mathbf{B} \end{bmatrix}. \tag{A1}$$

A few well known identities are as follows; also see reference [14] for more details:

$$1\otimes\mathbf{A}=\mathbf{A}=\mathbf{A}\otimes 1, \qquad (\mathbf{A}\otimes\mathbf{B})^{-1}=\mathbf{A}^{-1}\otimes\mathbf{B}^{-1} \text{ (if } \mathbf{A}^{-1} \text{ and } \mathbf{B}^{-1} \text{ exist)}, \quad \text{(A2a, b)}$$

$$(\mathbf{A}\otimes\mathbf{B})^{\mathrm{T}}=\mathbf{A}^{\mathrm{T}}\otimes\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{AC}\otimes\mathbf{BD}=(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}), \quad \text{(A2c, d)}$$

$$(\mathbf{A}+\mathbf{B})\otimes(\mathbf{C}+\mathbf{D})=\mathbf{A}\otimes\mathbf{C}+\mathbf{A}\otimes\mathbf{D}+\mathbf{B}\otimes\mathbf{C}+\mathbf{B}\otimes\mathbf{D}. \quad \text{(A2e)}$$

Selected matrix operations are described below.

(i) cs(**A**), column transformation of a matrix $\mathbf{A}(p\times q)$:

$$\mathrm{cs}(\mathbf{A})=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\\ \cdots\\ \mathbf{A}_q\end{bmatrix}, \quad \text{(A3)}$$

where $\mathbf{A}_j$ is the jth column of **A**.

(ii) Matrix derivative:

$$\mathscr{D}_{\mathrm{cs}(A)^{\mathrm{T}}}\mathbf{B}=\left[\frac{\partial\mathbf{B}}{\partial a_{11}}\middle|\frac{\partial\mathbf{B}}{\partial a_{21}}\middle|\frac{\partial\mathbf{B}}{\partial a_{31}}\middle|\cdots\middle|\frac{\partial\mathbf{B}}{\partial a_{pq}}\right], \quad \text{(A4)}$$

$$\mathscr{D}^2_{\mathrm{cs}(A)^{\mathrm{T}}}\mathbf{B}=\left[\frac{\partial^2\mathbf{B}}{\partial a_{11}^2}\middle|\frac{\partial^2\mathbf{B}}{\partial a_{11}\,\partial a_{21}}\middle|\frac{\partial^2\mathbf{B}}{\partial a_{11}\,\partial a_{31}}\middle|\cdots\middle|\frac{\partial^2\mathbf{B}}{\partial a_{pq}^2}\right]. \quad \text{(A5)}$$

(iii) Matrix Taylor expansion:

$$\mathbf{B}(\mathbf{A})=\mathbf{B}(\bar{\mathbf{A}})+\sum_{m=1}^{L}\frac{1}{m!}\mathscr{D}^m_{\mathrm{cs}(\tilde{A})^{\otimes m^{\mathrm{T}}}}\mathbf{B}\,|_{\bar{A}}[\mathrm{cs}(\tilde{A})^{\otimes m}\otimes\mathbf{I}_q]+\mathbf{O}(L+1), \quad \text{(A6)}$$

where $\mathbf{O}(L+1)$ is a remainder.

A.2. EXAMPLE

Consider a two-dimensional problem given by random matrix $\mathbf{y}(2\times 2)$ and deterministic vector $\bar{\mathbf{u}}(2\times 1)$ and define a random function $\mathbf{h}(2\times 1)$ such as $\mathbf{h}=\mathbf{y}^{-1}\bar{\mathbf{u}}$. See section 4.1 for its application. Using a Taylor series, we obtain

$$\mathbf{h}(\mathbf{y},\bar{\mathbf{u}})=\mathbf{h}(\bar{\mathbf{y}},\bar{\mathbf{u}})+\mathscr{D}_{\mathrm{cs}(y)^{\mathrm{T}}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})]+(1/2!)\mathscr{D}^2_{\mathrm{cs}(y)^{\otimes 2^{\mathrm{T}}}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})^{\otimes 2}]$$

$$+(1/3!)\mathscr{D}^3_{\mathrm{cs}(y)^{\otimes 3^{\mathrm{T}}}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})^{\otimes 3}]+\cdots, \quad \text{(A7)}$$

where

$$\mathscr{D}_{\mathrm{cs}(y)^{\mathrm{T}}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})]=\mathscr{D}_{\mathrm{cs}(y)^{\mathrm{T}}}\mathbf{y}^{-1}|_{\bar{Y}}[\mathrm{cs}(\tilde{\mathbf{y}})\otimes\mathbf{I}_2]\bar{\mathbf{u}}$$

$$=\mathbf{y}^{-1}\left[\begin{array}{cc|cc|cc|cc}1&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&1\end{array}\right][\mathrm{cs}(\tilde{\mathbf{y}})\otimes\mathbf{I}_2]\mathbf{y}^{-1}\bar{\mathbf{u}}=\mathbf{y}^{-1}\mathbf{y}_1\mathbf{y}^{-1}\bar{\mathbf{u}}, \quad \text{(A8)}$$

$$
\begin{aligned}
&(1/2!)\mathscr{D}^2_{\mathrm{cs}(y)^{\otimes 2}\mathrm{T}}\mathbf{h}|_{\bar{h}}[\mathrm{cs}(\tilde{\mathbf{y}})^{\otimes 2}] \\
&\quad = (1/2!)\mathscr{D}^2_{\mathrm{cs}(y)^{\otimes 2}\mathrm{T}}\mathbf{y}^{-1}|_{\bar{y}}[\mathrm{cs}(\tilde{\mathbf{y}})^{\otimes 2}\otimes \mathbf{I}_2]\tilde{\mathbf{u}} \\
&\quad = \bar{\mathbf{y}}^{-1}\frac{1}{2!}\left[2\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\middle|\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right. \\
&\qquad \left. + \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\middle|\cdots\middle|2\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\bar{\mathbf{y}}^{-1}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right][\mathrm{cs}(\tilde{\mathbf{y}})^{\otimes 2}\otimes \mathbf{I}_2]\bar{\mathbf{y}}^{-1}\tilde{\mathbf{u}} \\
&\quad = \bar{\mathbf{y}}^{-1}\mathbf{y}_2\bar{\mathbf{y}}^{-1}\tilde{\mathbf{u}}, \qquad \text{(A9)}
\end{aligned}
$$

and $\mathbf{y}_3$ can be determined similarly. If $\mathbf{h}$ is approximated up to the third order term,

$$\mathbf{h}(\mathbf{y}, \tilde{\mathbf{u}}) = \bar{\mathbf{h}} + \bar{\mathbf{y}}^{-1}\mathbf{y}_1\bar{\mathbf{y}}^{-1}\tilde{\mathbf{u}} + \mathbf{y}^{-1}\mathbf{y}_2\bar{\mathbf{y}}^{-1}\tilde{\mathbf{u}} + \bar{\mathbf{y}}^{-1}\mathbf{y}_3\bar{\mathbf{y}}^{-1}\tilde{\mathbf{u}}. \qquad \text{(A10)}$$