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ANALYSIS OF DISCRETE VIBRATORY SYSTEMS WITH PARAMETER UNCERTAINTIES, PART II: IMPULSE RESPONSE

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This paper is a continuation of its companion (Part I) in which a new analytical method was proposed to estimate the first and second moments of eigenvalues for a linear time-invariant, proportionally damped, discrete vibratory system with uncertain parameters. In this part, the force amplitude is also considered as a random variable, but its time history is assumed to be deterministic. Based on certain simplifying assumptions and given probabilistic eigenvalue solutions, the first two moments of response in both modal and physical domains are estimated. First, an impulse excitation is considered and single- and two-degree-of-freedom system examples are used to illustrate the proposed method. Unlike the commonly used first order perturbation technique, our method does not include any secular terms. It is verified by comparing predictions with the benchmark Monte Carlo simulation. Second, a convolution integral formulation is developed for harmonic and other excitations. One example case is considered to illustrate and validate the proposed approach. Overall, our method overcomes the deficiencies of first order perturbation technique and is reasonably accurate and computationally inexpensive.

1. INTRODUCTION

The impulse response of a single- or multi-degree-of-freedom system with uncertain mass, stiffness and damping parameters has been the subject of several investigations [1–5]. Primarily, the first order perturbation method has been used, but with limited success [1–3, 5]. It has been seen that such perturbation methods are valid only for a short time duration, since the standard deviation of the transient response becomes unbounded with time, especially for a virtually undamped system. Stochastic finite element methods have the same limitations, since they are also based on the theory of first order perturbations [6, 7]. Such problems including mathematical singularities arise due to the presence of secular terms in analytical solutions. Accordingly, the first main objective of this paper is to propose a new analytical method which does not include any secular terms in the impulse response expression. It should be noted that direct Monte Carlo simulation can always be employed for these problems [8]. However, this simulation is computationally intensive, since a large number of iterations is required to estimate the probabilistic distributions. Hence, there is interest in developing new or improved analytical or semi-analytical methods which overcome the disadvantages of existing methods and still yield reasonably accurate solutions. Review articles by Ibrahim [9] and Benaroya and Rehak [10] discuss these and other related research issues.

In a parallel development, we have proposed a new direct product technique which estimates the statistical frequency response of a damped vibratory system [11]. This method has been found to be efficient and reasonably accurate at and near resonance(s), unlike

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the first order perturbation method [13]. However, the system matrix is inverted numerically, which poses computational problems, especially when the damping ratio is very low. Also, this method does not use or predict random eigensolutions. Therefore, in this paper an alternative technique, based on the eigensolutions formulated in Part I [12] and the impulse response characteristics given in this paper, will be proposed to estimate the frequency response. Even though this paper is a continuation of Part I, it is written such that it is self-sufficient. However, for the sake of brevity, the reader is asked to refer to Part I for a few items.

2. PROBLEM FORMULATION

The major objective of this paper is to develop a new analytical method for the dynamic response of a linear time-invariant, proportionally damped vibratory system of dimension N with uncertain parameter matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$; each matrix is assumed to be symmetric and positive definite. The amplitude of the excitation, $\mathbf{MF}$, is also randomly distributed, but the time history $\xi(t)$ is deterministic and arbitrary; impulse and sinusoidal functions are chosen here to illustrate the method. The random differential equation can be given in the matrix form as follows (see the Appendix of this part for a list of notation)—since only the forced response is of interest, the initial displacement $\mathbf{X}(0)$ and velocity $\dot{\mathbf{X}}(0)$ are assumed to be null vectors:

$$\mathbf{M}\ddot{\mathbf{X}}(t)+\mathbf{C}\dot{\mathbf{X}}(t)+\mathbf{K}\mathbf{X}(t)=\mathbf{MF}\xi(t), \qquad \mathbf{X}(0)=\dot{\mathbf{X}}(0)=\mathbf{0}. \tag{1}$$

Specifically, the first two moments of the ensuing response are estimated, by making a few simplifying assumptions, from the knowledge of statistical eigenvalues of Part I [13]. Both single- and two-degree-of-freedom system examples are considered. The proposed method is validated by comparing it with the Monte Carlo numerical simulation [10], which is considered as the benchmark. Predictions yielded by the first order perturbation technique are also given when appropriate. Like Part I, the following assumptions are made to develop the new solution methodology: (i) random matrices and vectors of equation (1)

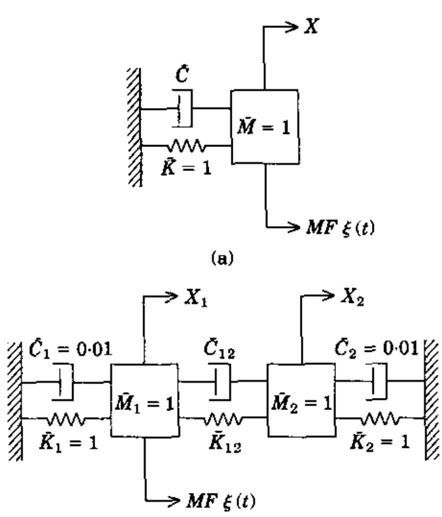

Figure 1. The physical systems used to illustrate and validate the proposed method: (a) the single-degree-of-freedom system (Examples I, II and IV); (b) the two-degree-of-freedom system (Example III). Deterministic system parameters are given here. The force locations are also shown.

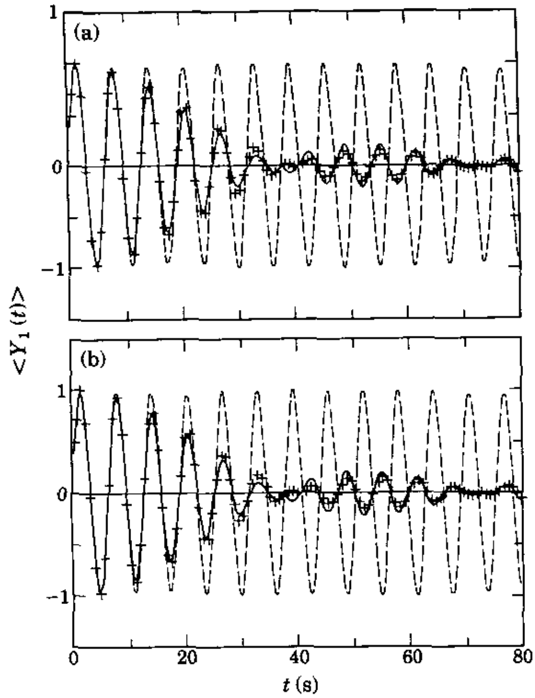

Figure 2. The expected mean of the impulse response for Example I, shown in Figure 1(a): (a) $\langle Y_1(t)\rangle$ by using equation (24); (b) $\langle Y_1(t)\rangle$ by using equation (27). ——, Proposed method; ----, first order perturbation technique [1]; + + +, Monte Carlo simulation.

can be given by the sum of deterministic or mean (identified by a bar), and random or fluctuating components (identified by a tilde), i.e., $\mathbf{M}=\bar{\mathbf{M}}+\tilde{\mathbf{M}}$, $\mathbf{C}=\bar{\mathbf{C}}+\tilde{\mathbf{C}}$, $\mathbf{K}=\bar{\mathbf{K}}+\tilde{\mathbf{K}}$, $\mathbf{F}=\bar{\mathbf{F}}+\tilde{\mathbf{F}}$ and $\mathbf{X}=\bar{\mathbf{X}}+\tilde{\mathbf{X}}$; (ii) the expected means of the system matrices $\bar{\mathbf{M}}=\langle\mathbf{M}\rangle$, $\bar{\mathbf{C}}=\langle\mathbf{C}\rangle$, $\bar{\mathbf{K}}=\langle\mathbf{K}\rangle$ and of the excitation amplitude $\langle\mathbf{F}\rangle=\bar{\mathbf{F}}$ are known; (iii) the probability distributions of $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{F}$ are of the same type and are known; (iv) the means of a random parameter matrix and $\tilde{\mathbf{F}}$ are equal to null, e.g., $\langle\tilde{\mathbf{M}}\rangle=\mathbf{0}$, $\langle\tilde{\mathbf{F}}\rangle=\mathbf{0}$; (v) parameter fluctuations are much smaller compared to the deterministic values, i.e., $\|\tilde{\mathbf{M}}\|\ll\|\bar{\mathbf{M}}\|$; (vi) the covariances of the parameter fluctuations are known in the form of cross-correlation matrices such as $\mathbf{R}_{M,M}=\langle\tilde{\mathbf{M}}\otimes\tilde{\mathbf{M}}\rangle=\mathrm{Var}\,(\mathbf{M})$ and $\mathbf{R}_{M,K}=\langle\tilde{\mathbf{M}}\otimes\tilde{\mathbf{K}}\rangle=\mathrm{Cov}\,(\mathbf{M},\mathbf{K})$; and (vii) $\mathbf{F}$ is uncorrelated with the system parameter matrices, i.e., $\mathbf{R}_{F,M}=\mathbf{R}_{F,C}=\mathbf{R}_{F,K}=\mathbf{0}$. Additional assumptions will be specified as part of the analytical development.

3. PROPOSED ANALYTICAL METHOD

3.1. EIGENVALUES

Even though the theory presented in Part I is summarized here, it is written in a manner which facilitates further analytical development. Assuming $\mathbf{F}=\mathbf{0}$, equation (1) is expressed as follows, in accordance with the assumptions stated earlier:

$$\ddot{\mathbf{X}}(t)+\mathbf{B}\dot{\mathbf{X}}(t)+\mathbf{A}\mathbf{X}(t)=\mathbf{0},\qquad \mathbf{A}=\mathbf{M}^{-1}\mathbf{K},\qquad \mathbf{B}=\mathbf{M}^{-1}\mathbf{C}.\qquad (2a\text{–}c)$$

Now let $\mathbf{X}(t)=\mathbf{\Phi}\mathbf{Y}(t)$, where $\mathbf{\Phi}$ is the modal matrix of the undamped system and $\mathbf{Y}(t)$ is the normal co-ordinate vector. Furthermore, premultiply equation (2a) by $\mathbf{\Phi}^T$ to yield the following characteristic equation for the proportionally damped system

$$\gamma_i^2\mathbf{\Phi}_i^T\mathbf{\Phi}_i + \gamma_i\mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i + \mathbf{\Phi}_i^T\mathbf{A}\mathbf{\Phi}_i = \mathbf{0}, \qquad i = 1, 2, \ldots, N. \tag{3}$$

Here $\gamma_i = -\zeta_i\omega_i \pm \mathrm{j}\omega_{di} = -\zeta_i\omega_i \pm \mathrm{j}\omega_i\sqrt{1-\zeta_i^2}$ is the ith complex valued eigenvalue corresponding to the real eigenvector $\mathbf{\Phi}_i$, ζ_i is the damping ratio, ω_i is the undamped natural frequency, ω_{di} is the damped natural frequency and $\mathrm{j} = \sqrt{-1}$. The eigenvalues are assumed to be distinct. Rewrite equation (3) as

$$\gamma_i = \frac{-\mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i \pm \mathrm{j}\sqrt{4(\mathbf{\Phi}_i^T\mathbf{\Phi}_i\mathbf{\Phi}_i^T\mathbf{A}\mathbf{\Phi}_i) - (\mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i)^2}}{2\mathbf{\Phi}_i^T\mathbf{\Phi}_i} = \mathrm{Re}\,(\gamma_i) + \mathrm{j}\,\mathrm{Im}\,(\gamma_i), \tag{4}$$

where

$$\mathrm{Re}\,(\gamma_i) = -\mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i/2\mathbf{\Phi}_i^T\mathbf{\Phi}_i,$$

$$[\mathrm{Im}\,(\gamma_i)]^2 = \omega_{di}^2 = (4\mathbf{\Phi}_i^T\mathbf{\Phi}_i\mathbf{\Phi}_i^T\mathbf{A}\mathbf{\Phi}_i - \mathbf{\Phi}_i^T\mathbf{B}\mathbf{\Phi}_i\mathbf{\Phi}_i^T\mathbf{B}^T\mathbf{\Phi}_i)/4(\mathbf{\Phi}_i^T\mathbf{\Phi}_i)^2. \tag{5a, b}$$

If the system is lightly damped, the variance of ζ_i has to be much smaller than unity. Accordingly, Im (γ_i) is estimated by assuming $\sqrt{1-\zeta_i^2}$ to be deterministic. Thus,

$$\langle[\mathrm{Im}\,(\gamma_i)]^2\rangle = \langle\omega_{di}^2\rangle \approx (1-\bar{\zeta}_i^2)\langle\omega_i^2\rangle \approx (1-\bar{\zeta}_i^2)\bar{\mathbf{\Phi}}_i^T\langle\mathbf{A}\rangle\bar{\mathbf{\Phi}}_i/(\bar{\mathbf{\Phi}}_i^T\bar{\mathbf{\Phi}}_i). \tag{6}$$

If γ_i is estimated using the deterministic eigenvector $\bar{\mathbf{\Phi}}_i$, the expected value of γ_i yields

$$\langle\mathrm{Re}\,(\gamma_i)\rangle = -0{\cdot}5\bar{\mathbf{\Phi}}_i^T\langle\mathbf{B}\rangle\bar{\mathbf{\Phi}}_i/(\bar{\mathbf{\Phi}}_i^T\bar{\mathbf{\Phi}}_i),$$

$$\langle[\mathrm{Im}\,(\gamma_i)]^2\rangle = \langle\omega_{di}^2\rangle \approx (1-\bar{\zeta}_i^2)\langle\omega_i^2\rangle \approx (1-\bar{\zeta}_i^2)\bar{\mathbf{\Phi}}_i^T\langle\mathbf{A}\rangle\bar{\mathbf{\Phi}}_i/(\bar{\mathbf{\Phi}}_i^T\bar{\mathbf{\Phi}}_i). \tag{7a, b}$$

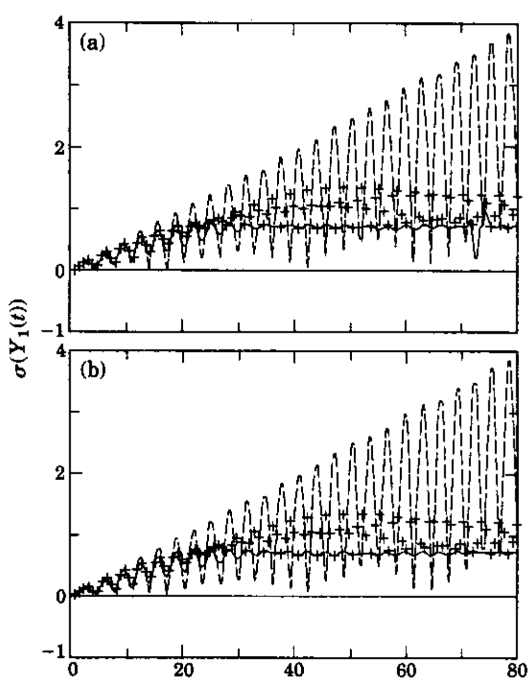

Figure 3. The standard deviation of the impulse response $\sigma(Y_1(t))$ for Example I, shown in Figure 1(a): (a) the exact form yielded by equations (24) and (25); (b) the approximation solution yielded by equations (27) and (28). Key as in Figure 2.

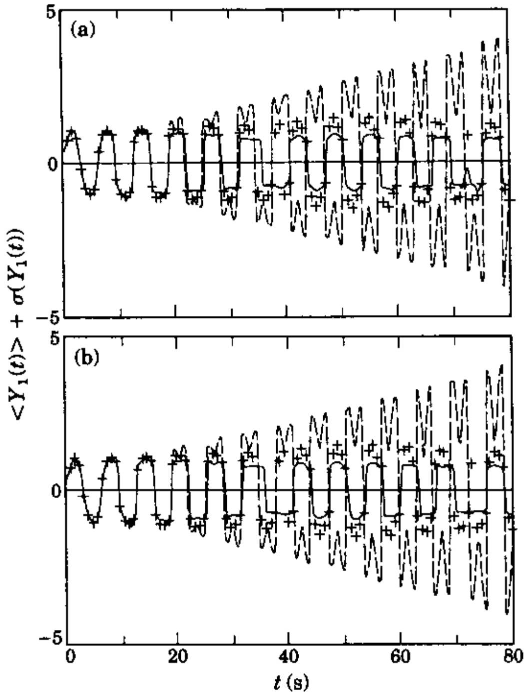

Figure 4. The upper bound of the impulse response $\langle Y_1(t)\rangle + \sigma(Y_1(t))$ for Example I: (a) the exact form yielded by equations (24) and (25); (b) the approximate solution yielded by equations (27) and (28). Key as in Figure 2.

Since γ_i is complex valued, its standard deviation is complex valued as well: $\sigma_{\gamma_i} = \sigma(\mathrm{Re}\,(\gamma_i)) + \mathrm{j}\sigma(\mathrm{Im}\,(\gamma_i))$. It is determined by using the covariance matrices of **A** and **B**. Furthermore, covariances of $\mathrm{Re}\,(\gamma_i)$ and ω_{di}^2 can be given as

$$\mathrm{Cov}\,(\mathrm{Re}\,(\gamma_i), \mathrm{Re}\,(\gamma_j)) = \langle \mathrm{Re}\,(\gamma_i) \otimes \mathrm{Re}\,(\gamma_j)\rangle - \langle \mathrm{Re}\,(\gamma_i)\rangle \otimes \langle \mathrm{Re}\,(\gamma_j)\rangle,$$
$$\mathrm{Cov}\,(\omega_{di}^2, \omega_{dj}^2) = \langle \omega_{di}^2 \otimes \omega_{dj}^2\rangle - \langle \omega_{di}^2\rangle \otimes \langle \omega_{dj}^2\rangle. \qquad (8a, b)$$

3.2. MOMENTS OF MODAL RESPONSE

Equation (1) can be given in normal co-ordinates as follows, where $\mathbf{\Phi Y}(t) = \mathbf{X}(t)$:

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}\ddot{\mathbf{Y}}(t) + \mathbf{\Phi}^{\mathrm{T}}\mathbf{B}\mathbf{\Phi}\dot{\mathbf{Y}}(t) + \mathbf{\Phi}^{\mathrm{T}}\mathbf{A}\mathbf{\Phi}\mathbf{Y}(t) = \mathbf{\Phi}^{\mathrm{T}}\mathbf{F}\xi(t). \qquad (9)$$

Since the system is assumed to be proportionally damped, equation (9) becomes uncoupled and takes the following form

$$\ddot{\mathbf{Y}}(t) + \begin{bmatrix} \ddots & & \\ & 2\zeta_i\omega_i & \\ & & \ddots \end{bmatrix}\dot{\mathbf{Y}}(t) + \begin{bmatrix} \ddots & & \\ & \omega_i^2 & \\ & & \ddots \end{bmatrix}\mathbf{Y}(t) = \frac{\mathbf{\Phi}^{\mathrm{T}}\mathbf{F}}{\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}}\xi(t). \qquad (10)$$

Like the deterministic system, equation (10) yields N uncoupled equations, each describing a single-degree-of-freedom in the modal domain. The response $Y_i(t)$ for the ith mode is determined by

$$Y_i(t) = \int \frac{\mathbf{\Phi}_i^{\mathrm{T}}\mathbf{F}}{\mathbf{\Phi}_i^{\mathrm{T}}\mathbf{\Phi}_i} \mathrm{e}^{-a_i\tau} \frac{\sin b_i\tau}{b_i} \xi(t-\tau)\,\mathrm{d}\tau, \qquad t \geqslant 0, \qquad (11)$$

where $a_i = \mathrm{Re}(\lambda_i)$ and $b_i = \mathrm{Im}(\lambda_i)$ are random variables. The corresponding response of the deterministic system is

$$\bar{Y}_i(t) = \int \frac{\bar{\Phi}_i^T \bar{F}}{\bar{\Phi}_i^T \bar{\Phi}_i} e^{-\bar{a}_i \tau} \frac{\sin \bar{b}_i \tau}{\bar{b}_i} \xi(t-\tau)\, d\tau, \qquad t \geqslant 0. \tag{12}$$

The statistical behavior of the eigenvalues is found by using the direct product technique based on the covariance matrices of system parameters in section 3.1; also refer to Part I [12]. Without losing any accuracy, the first two moments are estimated solely on the basis of knowledge of the deterministic modal vector $\bar{\mathbf{\Phi}}_i$. Therefore, the expected mean and mean-square values of $Y_i(t)$ are found to be

$$\langle Y_i(t)\rangle = \int \rho_i \iint e^{-a_i \tau} \frac{\sin b_i \tau}{b_i} p(a_i, b_i)\, da_i\, db_i\, \xi(t-\tau)\, d\tau,$$

$$\langle Y_i^2(t)\rangle = \iint \mu_i \iint e^{-a_i(\tau+\upsilon)} \frac{\sin b_i \tau \sin b_i \upsilon}{b_i^2} p(a_i, b_i)\, da_i\, db_i\, \xi(t-\tau)\xi(t-\upsilon)\, d\tau\, d\upsilon, \tag{13a, b}$$

where $p(a_i, b_i)$ is the joint density function, and ρ_i and μ_i are the expected modal participation factors, defined as

$$\rho_i = \frac{\langle \bar{\mathbf{\Phi}}_i^T \mathbf{F}\rangle}{\bar{\mathbf{\Phi}}_i^T \bar{\mathbf{\Phi}}_i} = \frac{\bar{\mathbf{\Phi}}_i^T \bar{\mathbf{F}}}{\bar{\mathbf{\Phi}}_i^T \bar{\mathbf{\Phi}}_i}, \qquad \mu_i = \frac{\mathbf{D}_{\Phi_i,\Phi_i}^T \langle \mathbf{F} \otimes \mathbf{F}\rangle}{(\bar{\mathbf{\Phi}}_i^T \bar{\mathbf{\Phi}}_i)^2} = \frac{\mathbf{D}_{\Phi_i,\Phi_i}^T (\mathbf{D}_{F,F} + \mathbf{R}_{F,F})}{(\bar{\mathbf{\Phi}}_i^T \bar{\mathbf{\Phi}}_i)^2}. \tag{14a, b}$$

3.3. UNCORRELATED DAMPING

The first limiting case of the Rayleigh damping model $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ is investigated next when **C**, **M** and **K** are statistically independent, i.e., α and β are random variables; recall

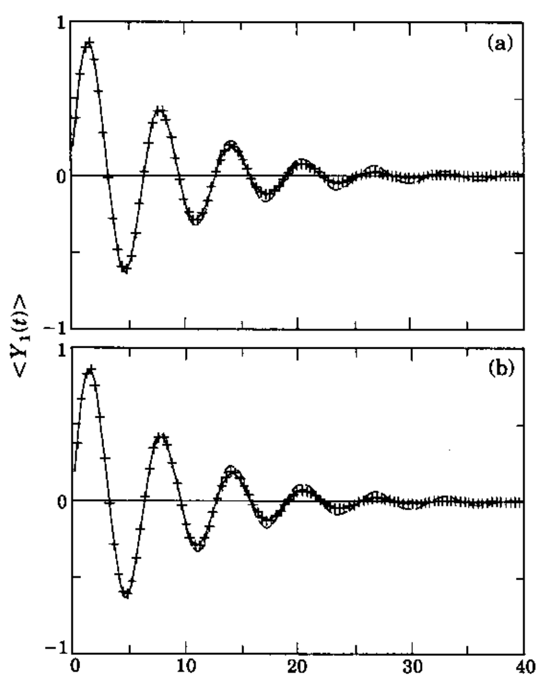

Figure 5. The expected mean of the impulse response for a lightly damped system with $\bar{C} = 0{\cdot}2$, shown in Figure 1(a) (Example II): (a) $\langle Y_1(t)\rangle$ by using equation (24); (b) $\langle Y_1(t)\rangle$ by using equation (27). Key as in Figure 2.

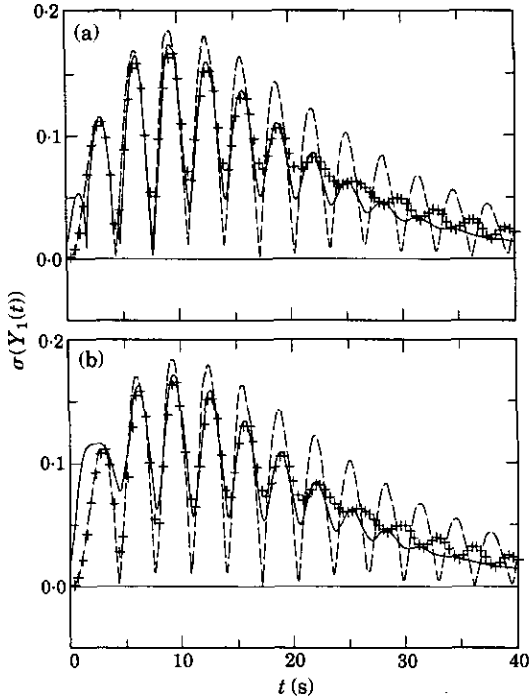

Figure 6. The standard deviation of the impulse response $\sigma(Y_1(t))$ for Example II: (a) the exact form yielded by equations (24) and (25); (b) the approximate solution yielded by equations (27) and (28). Key as in Figure 2.

Part I [12]. Define the standard deviation of $\mathrm{Re}\,(\gamma_i)$ by using equation (8a):

$$\sigma(\mathrm{Re}\,(\gamma_i)) = (\langle \mathrm{Re}\,(\gamma_i) \otimes \mathrm{Re}\,(\gamma_i)\rangle - \langle \mathrm{Re}\,(\gamma_i)\rangle \otimes \langle \mathrm{Re}\,(\gamma_i)\rangle)^{1/2}$$

$$= 0{\cdot}5(\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}\mathbf{R}_{B,B}\mathbf{D}_{\Phi_i,\Phi_i}/(\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}\mathbf{D}_{\Phi_i,\Phi_i}))^{1/2}, \qquad (15)$$

$$\mathbf{R}_{B,B} \approx (\mathbf{D}_{M,M} - \mathbf{R}_{M,M})^{-1}(\mathbf{D}_{C,C} + \mathbf{R}_{C,C}) - \mathbf{D}^{-1}_{M,M}\mathbf{D}_{C,C}, \qquad (16)$$

where $\mathbf{D}$ is the direct product of two deterministic matrices, e.g., $\mathbf{D}_{M,M} = \bar{\mathbf{M}} \otimes \bar{\mathbf{M}}$. The standard deviation of ω_{di} is calculating by using

$$\sigma(\omega_{di}) \approx 0{\cdot}5\sigma(\omega^2_{di})/\bar{\omega}_{di}, \qquad \sigma(\omega_i) \approx 0{\cdot}5\sigma(\omega^2_i)/\bar{\omega}_i,$$

$$\sigma(\omega^2_{di}) = (\langle \omega^2_{di} \otimes \omega^2_{di}\rangle - \langle \omega^2_{di}\rangle \otimes \langle \omega^2_{di}\rangle)^{1/2}$$

$$\approx (1 - \bar{\zeta}^2_i)\sigma(\omega^2_i) \approx (1 - \bar{\zeta}^2_i)(\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}\mathbf{R}_{A,A}\mathbf{D}_{\Phi_i,\Phi_i}/(\mathbf{D}^{\mathrm{T}}_{\Phi_i,\Phi_i}\mathbf{D}_{\Phi_i,\Phi_i}))^{1/2},$$

$$\mathbf{R}_{A,A} = (\mathbf{D}_{M,M} - \mathbf{R}_{M,M})^{-1}(\mathbf{D}_{K,K} + \mathbf{R}_{K,K}) - \mathbf{D}^{-1}_{M,M}\mathbf{D}_{K,K}. \qquad (17\text{a–d})$$

If $\mathbf{M}$ is deterministic but $\mathbf{K}$ is still random, a_i and b_i must be uncorrelated. Therefore, mean and mean-square values of $Y_i(t)$ are

$$\langle Y_i(t)\rangle = \int \rho_i \int \mathrm{e}^{-a_i\tau} p(a_i)\,\mathrm{d}a_i \int \frac{\sin b_i\tau}{b_i} p(b_i)\,\mathrm{d}b_i\,\xi(t-\tau)\,\mathrm{d}\tau,$$

$$\langle Y^2_i(t)\rangle = \iint \mu_i \iint \mathrm{e}^{-a_i(\tau+\upsilon)} \frac{\sin b_i\tau \, \sin b_i\upsilon}{b^2_i} p(a_i)\,\mathrm{d}a_i\, p(b_i)\,\mathrm{d}b_i\,\xi(t-\tau)\xi(t-\upsilon)\,\mathrm{d}\tau\,\mathrm{d}\upsilon,$$

(18a, b)

where $p(a_i)$ and $p(b_i)$ are marginal density functions.

3.4. FULLY CORRELATED DAMPING

The second limiting case is examined by assuming that **C** is fully correlated with **M** and **K**. Recall from Part I [12] that the standard deviation of γ_i can be given in terms of $\sigma(\omega_i)$ since $\mathbf{C} = \bar{\alpha}\mathbf{M} + \bar{\beta}\mathbf{K}$. The eigenvalue has the form

$$\mathrm{Re}\,(\gamma_i) = -0{\cdot}5(\bar{\alpha} + \bar{\beta}\omega_i^2), \qquad [\mathrm{Im}\,(\gamma_i)]^2 = \omega_i^2 - [\mathrm{Re}\,(\gamma_i)]^2 = \omega_{di}^2 = (1 - \bar{\zeta}_i^2)\omega_i^2. \quad (19, 20)$$

The standard deviation of γ_i is defined as

$$\sigma(\mathrm{Re}\,(\gamma_i)) = 0{\cdot}5\bar{\beta}\sigma(\omega_i^2), \quad (21)$$

however, $\sigma(\omega_{di}^2)$ is still given by equation (17). Since a_i and b_i are fully correlated, they can be given in terms of a single random variable, say ω_i. Expected first and second moments of $Y_i(t)$ are consequently given as follows, where $p(\omega_i)$ is the density function

$$\langle Y_i(t)\rangle = \int \rho_i \int \mathrm{e}^{-0{\cdot}5(\alpha + \beta\omega_i^2)\tau} \frac{\sin\sqrt{1-\bar{\zeta}_i^2}\,\omega_i\tau}{\sqrt{1-\bar{\zeta}_i^2}\,\omega_i} p(\omega_i)\,\mathrm{d}\omega_i\,\xi(t-\tau)\,\mathrm{d}\tau, \quad (22)$$

$$\langle Y_i^2(t)\rangle = \int\int \mu_i \int \mathrm{e}^{-0{\cdot}5(\alpha + \beta\omega_i^2)(\tau + \upsilon)} \frac{\sin\sqrt{1-\bar{\zeta}_i^2}\,\omega_i\tau \,\sin\sqrt{1-\bar{\zeta}_i^2}\,\omega_i\upsilon}{(1-\bar{\zeta}_i^2)\omega_i^2}$$
$$\times p(\omega_i)\,\mathrm{d}\omega_i\,\xi(t-\tau)\xi(t-\upsilon)\,\mathrm{d}\tau\,\mathrm{d}\upsilon. \quad (23)$$

4. IMPULSE RESPONSE

4.1. MOMENTS OF GREEN'S FUNCTION: ANALYTICAL DEVELOPMENT

Equations (22) and (23) illustrate that $\langle Y_i(t)\rangle$ and $\langle Y_i^2(t)\rangle$ must be evaluated numerically because of the double integrals and probabilistic density functions involved. In order

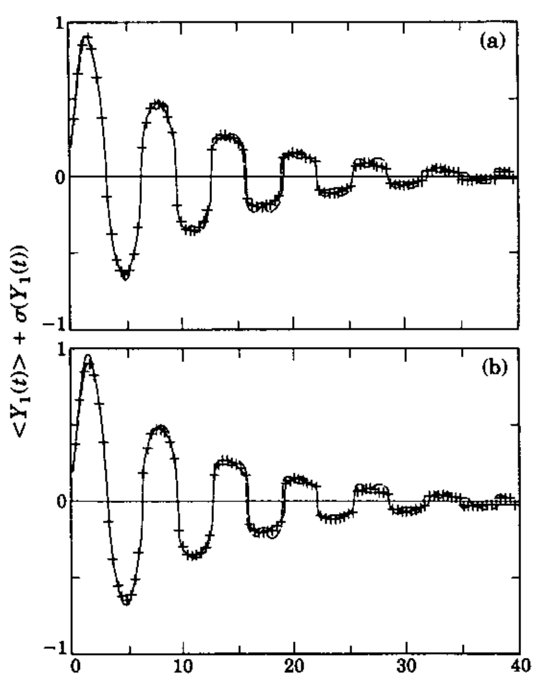

Figure 7. The upper bound of the impulse response $\langle Y_1(t)\rangle + \sigma(Y_1(t))$ for Example II: (a) the exact form yielded by equations (24) and (25); (b) the approximate solution yielded by equations (27) and (28). Key as in Figure 2.

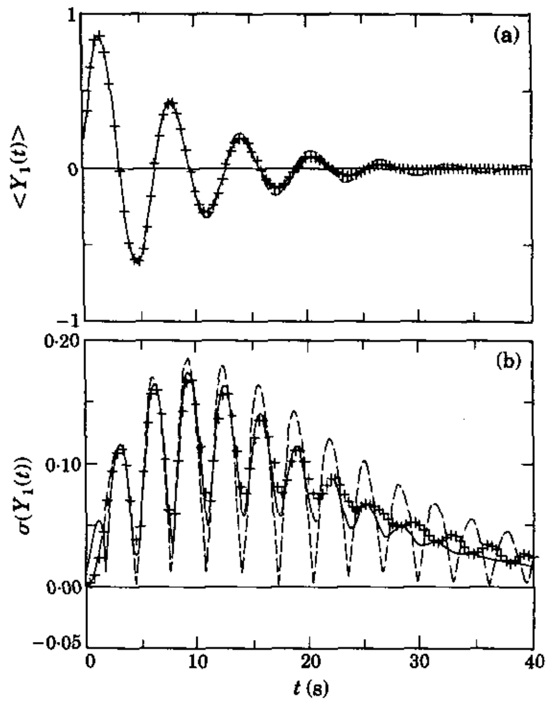

Figure 8. The effect of damping fluctuation for a lightly damping system with $\bar{C} = 0{\cdot}2$ and $R_{C,C} = 4 \times 10^{-4}$, shown in Figure 1(a) (Example II): (a) $\langle Y_1(t)\rangle$ by using equation (24); (b) $\sigma(Y_1(t))$ by using equations (24) and (25). Key as in Figure 2.

to illustrate the proposed method, a simplified case is presented analytically. Consider $\xi(t)$ to be an unit impulse function. Assume that the random variables a_i and b_i are uniformly distributed and uncorrelated with each other. The expected mean of impulse response or Green's function is obtained from equations (18a) as

$$\langle Y_i(t)\rangle = \rho_i \hbar(t) = \rho_i \left\langle e^{-a_i t} \frac{\sin b_i t}{b_i} \right\rangle = \rho_i \frac{p(a_i)p(b_i)}{-t}\, e^{-a_i t}\Big|_{a_i^-}^{a_i^+}\, \mathrm{Si}\,(b_i t)\Big|_{b_i^-}^{b_i^+},$$

$$p(a_i) = 1/(2\sqrt{3}\sigma_{a_i}), \qquad p(b_i) = 1/(2\sqrt{3}\sigma_{b_i}), \tag{24a–c}$$

where $\mathrm{Si}\,(b_i t) = \int_{b_i^-}^{b_i^+} ((\sin b_i t)/b_i)\,\mathrm{d}b_i$ is the sine integral, $a_i^+ = \bar{a}_i + \sqrt{3}\sigma_{a_i}$, $a_i^- = \bar{a}_i - \sqrt{3}\sigma_{a_i}$, $b_i^+ = \bar{b}_i + \sqrt{3}\sigma_{b_i}$ and $b_i^- = \bar{b}_i - \sqrt{3}\sigma_{b_i}$. The expected mean-square value of the impulse response or Green's function is expressed as

$$\langle Y_i^2(t)\rangle = \mu_i g(t) = \mu_i \left\langle e^{-2a_i t} \frac{\sin^2 b_i t}{b_i^2} \right\rangle = \mu_i p(a_i)p(b_i) \frac{e^{-2a_i t}}{-2t}\Big|_{a_i^-}^{a_i^+} \int_{b_i^-}^{b_i^+} \frac{\sin^2 b_i t}{b_i^2}\,\mathrm{d}b_i$$

$$= \mu_i p(a_i)p(b_i) \frac{e^{-2a_i t}}{-2t}\Big|_{a_i^-}^{a_i^+} ((-1 + \cos 2b_i t)/b_i + 2t\,\mathrm{Si}\,(2b_i t))\Big|_{b_i^-}^{b_i^+},$$

$$\mathrm{Si}\,(2b_i t) = \int_{b_i^-}^{b_i^+} \frac{\sin 2b_i t}{b_i}\,\mathrm{d}b_i, \tag{25a, b}$$

and the standard deviation of the impulse response $Y_i(t)$ is expressed as

$$\sigma_{Y_i}(t) = (\langle Y_i^2(t)\rangle - \langle Y_i(t)\rangle^2)^{1/2}, \tag{26}$$

which can be determined from equations (24) and (25). The moments of impulse response in terms of the generalized co-ordinate can now be estimated by the deterministic modal matrix $\bar{\mathbf{\Phi}}$, i.e., $\mathbf{X}(t) \approx \bar{\mathbf{\Phi}}\mathbf{Y}(t)$.

4.2. APPROXIMATE SOLUTION

Observe that the sine integrals of equations (24) and (25) can be evaluated analytically or numerically. However, to demonstrate the procedure again, we find an approximate analytical solution by assuming first order perturbations of only b, not a. Therefore the expected first and second moments are as follows, from equations (24) and (25), while the rest of the procedure remains the same:

$$\langle Y_i(t)\rangle = \rho_i h(t) \approx \rho_i \left\langle \frac{2\bar{b}_i - b_i}{\bar{b}_i^2} e^{-a_i t} \sin b_i t \right\rangle$$

$$= \rho_i p(a_i) p(b_i) \frac{e^{-a_i t}}{-t}\bigg|_{a_i^-}^{a_i^+} \frac{-1}{\bar{b}_i^2}\left(\frac{(2\bar{b}_i - b_i)\cos b_i t}{t} + \frac{\sin b_i t}{t^2}\right)\bigg|_{b_i^-}^{b_i^+}, \tag{27}$$

$$\langle Y_i^2(t)\rangle = \mu_i g(t) \approx \mu_i \left\langle \frac{2\bar{b}_i^2 - b_i^2}{\bar{b}_i^4} e^{-2a_i t} \sin^2 b_i t \right\rangle = \mu_i p(a_i) p(b_i) \frac{e^{-2a_i t}}{-2t}\bigg|_{a_i^-}^{a_i^+} *$$

$$* \frac{1}{\bar{b}_i^4}\left(\bar{b}_i^2\left(b_i - \frac{\sin 2b_i t}{2t}\right) - \left(\frac{b_i^3}{6} - \frac{b_i^2 \sin 2b_i t}{4t} - \frac{b_i \cos 2b_i t}{4t^2} + \frac{\sin 2b_i t}{8t^3}\right)\right)\bigg|_{b_i^-}^{b_i^+}, \tag{28}$$

since

$$\frac{1}{b_i} = \frac{1}{\bar{b}_i + \tilde{b}_i} \approx \frac{\bar{b}_i - \tilde{b}_i}{\bar{b}_i^2} = \frac{2\bar{b}_i - b_i}{\bar{b}_i^2}, \qquad \frac{1}{b_i^2} = \frac{1}{(\bar{b}_i + \tilde{b}_i)^2} \approx \frac{\bar{b}_i^2 - 2\bar{b}_i\tilde{b}_i + \tilde{b}_i^2}{\bar{b}_i^4} = \frac{2\bar{b}_i^2 - b_i^2}{\bar{b}_i^4}. \tag{29a, b}$$

4.3. EXAMPLE I: UNDAMPED SINGLE-DEGREE-OF-FREEDOM SYSTEM

The parameters of the system shown in Figure 1(a) are chosen to be $\bar{M} = 1$, $\bar{C} = 0$ and $\bar{K} = 1$, and hence $\bar{\gamma}_i = -0 \pm 1\mathrm{j}$. The excitation is an unit impulse with $\bar{F} = 1$ and $R_{F,F} = 0$. Random variation is considered only in the stiffness and it is assumed to be uniformly

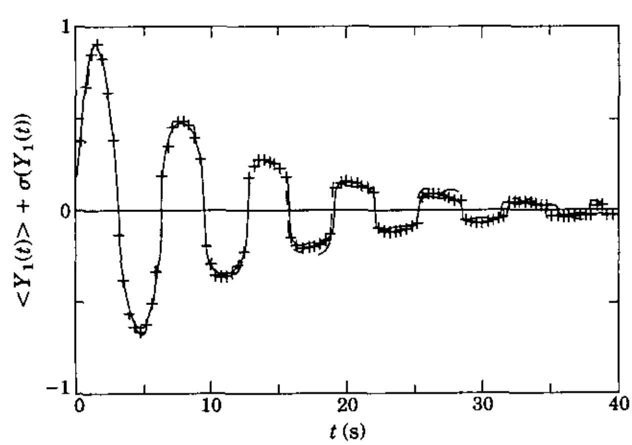

Figure 9. The effect of damping fluctuation on upper-bound impulse response $\langle Y_1(t)\rangle + \sigma(Y_1(t))$ for a lightly damped system, shown in Figure 1(a) (Example II) with $\bar{C} = 0.2$ and $R_{C,C} = 4 \times 10^{-4}$. Key as in Figure 2.

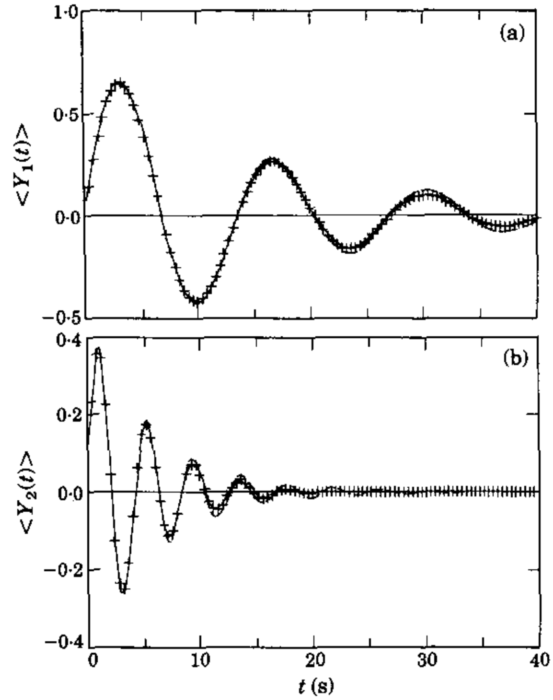

Figure 10. The expected mean of the impulse response in the normal mode form for Example III, shown in Figure 1(b): (a) $\langle Y_1(t)\rangle$; (b) $\langle Y_2(t)\rangle$. ——, Proposed method; + + +, Monte Carlo simulation.

distributed, i.e., $R_{K,K}=0{\cdot}01$, $R_{M,M}=R_{C,C}=0$ and $\sigma(\omega_{d1})=0{\cdot}05$. The corresponding expected mean $\langle Y_i(t)\rangle$, standard deviation $\sigma_{Y_i}(t)$ and upper-bound $\sigma_{Y_i}(t)+\langle Y_i(t)\rangle$ time histories, as yielded by the proposed method, both in the exact form by using equations (24) and (25) and in the simplified form given by equations (27) and (28), are given in Figures 2–4. Results are also compared with predictions yielded by Monte Carlo simulation (sample size = 400) and the first order perturbation method [1]. From Figures 2(a) and 2(b), it can be seen that the results of $\langle Y_1(t)\rangle$, as predicted by the proposed method, unlike the first order perturbation [1], match well with the Monte Carlo simulation. Observe that both exact and simplified forms yield virtually the same answers, except in the vicinity of $t=0$. Our method, unlike the first order perturbation technique [1], shows a decay in amplitudes in an otherwise undamped system. It must be noted that the Monte Carlo simulation has its uncertainties as well, since the sample size is finite. We observe from Figures 3(a) and 3(b) that standard deviations predicted by the proposed method (equations (24) and (25) or (27) and (28)) are in reasonably good agreement with the Monte Carlo simulation, except that the oscillations in the numerical simulation results die out slowly. It is seen that the standard deviation amplitude, as yielded by the first order perturbation method [1], grows boundlessly as time increases because of the secular terms which are inherently present. Note that the simplified form of our method, as given by equations (27) and (28), does not show the presence of any secular terms, even though first order perturbations of one random variable were assumed in section 4.2. The results of the upper-bound response, $\sigma_{Y_1}(t)+\langle Y_1(t)\rangle$, are shown in Figure 4; again, the proposed method matches well with the numerical simulation.

4.4. EXAMPLE II: DAMPED SINGLE-DEGREE-OF-FREEDOM SYSTEM

Reconsider the physical system of Example I with $\bar{C} = 0{\cdot}2$ and $\bar{\gamma}_1 = -0{\cdot}1 \pm 0{\cdot}995\mathrm{j}$. Two cases of randomness are considered, as follows.

(1) $R_{K,K} = 0{\cdot}01$ and $\sigma_{\omega_{d1}} = 0{\cdot}04975$: we observe from Figures 5–7 that predictions obtained by the proposed method (equations (24) and (25) or (27) and (28)) are in virtual agreement with the results of the Monte Carlo simulation. It is seen again that the first order perturbation method [1] deviates from the Monte Carlo simulation.

(2) $R_{K,K} = 0{\cdot}01$, $R_{C,C} = 4 \times 10^{-4}$, $\sigma_{\zeta_1} = 0{\cdot}02$ and $\sigma_{\omega_{d1}} = 0{\cdot}04975$: in Figures 8 and 9 are shown the results predicted by equations (24) and (25) and those obtained by the existing methods. The proposed method is more accurate than the first order perturbation method [1] when compared with the Monte Carlo simulation. A comparison of both damping cases demonstrates that a small fluctuation in damping ratio does not affect the impulse response significantly.

4.5. EXAMPLE III: TWO-DEGREE-OF-FREEDOM SYSTEM

Consider the physical system shown in Figure 1(b) with $\bar{\gamma}_1 = -0{\cdot}061 \pm 0{\cdot}464\mathrm{j}$ and $\bar{\gamma}_2 = -0.164 \pm 1{\cdot}501\mathrm{j}$. We consider randomness in the stiffness matrix only and choose it as $\mathbf{R}_{K,K} = 0{\cdot}01\mathbf{D}_{K,K}$, i.e., $\sigma_{\omega_{d1}} = 0{\cdot}0232$ and $\sigma_{\omega_{d2}} = 0{\cdot}07506$. Statistical moments of impulse response in terms of the generalized co-ordinate vector $\mathbf{X}(t)$ (Figures 13–15) are estimated from the knowledge of the statistical response in the normal mode vector $\mathbf{Y}(t)$ form (Figures 10–12) and the deterministic modal matrix $\bar{\mathbf{\Phi}}$. It is seen from Figures 10 and 13 that the expected mean values as predicted by the proposed method (equations (24) and (25)) are in very good agreement with the Monte Carlo simulation. The predicted standard

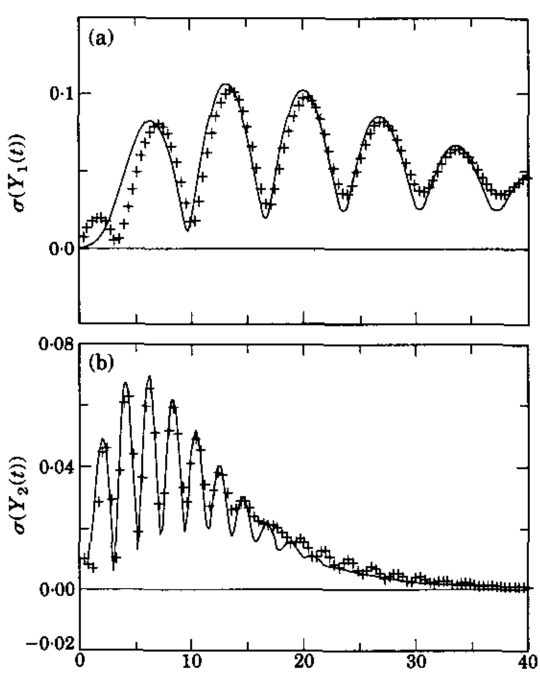

Figure 11. The standard deviation of the impulse response in the normal mode form for Example III, shown in Figure 1(b): (a) $\sigma(Y_1(t))$; (b) $\sigma(Y_2(t))$. Key as in Figure 10.

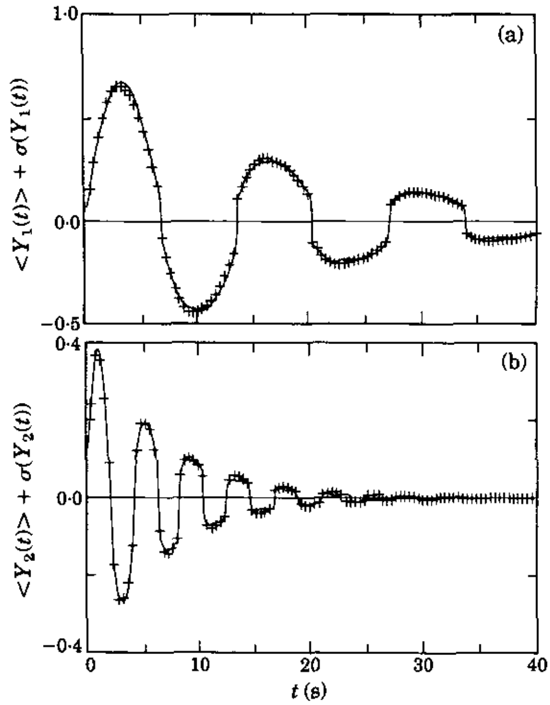

Figure 12. The standard deviation of the impulse response in the normal mode for Example III, shown in Figure 1(b): (a) $\langle Y_1(t)\rangle + \sigma(Y_1(t))$: (b) $\langle Y_2(t)\rangle + \sigma(Y_2(t)$. Key as in Figure 10.

deviations and upper-bound responses are also reasonably accurate, for both $\mathbf{Y}(t)$ and $\mathbf{X}(t)$, as is evident from Figures 11 and 12 and 14 and 15, respectively.

5. FREQUENCY RESPONSE

5.1. ARBITRARY EXCITATION

The first two moments of the dynamic response $Y_i(t)$ due to an arbitrary but deterministic excitation $\xi(t)$ can be obtained from equation (18) as

$$\langle Y_i(t)\rangle = \int \rho_i \mathfrak{h}_i(\tau)\xi(t-\tau)\,\mathrm{d}\tau, \qquad \langle Y_i^2(t)\rangle = \iint \mu_i \mathfrak{g}_i(\tau,\upsilon)\xi(t-\tau)\xi(t-\upsilon)\,\mathrm{d}\tau\,\mathrm{d}\upsilon, \quad (30\text{a, b})$$

where $\mathfrak{h}_i(\tau)$ and $\mathfrak{g}_i(\tau,\upsilon)$ are the first and second moments of Green's function, defined as follows by using equations (24) and (25):

$$\mathfrak{h}_i(\tau)=\left\langle \frac{2\bar{b}_i-b_i}{\bar{b}_i^2}\mathrm{e}^{-a_i t}\sin b_i\tau \right\rangle = p(a_i)p(b_i)\left.\frac{\mathrm{e}^{-a_i\tau}}{-\tau}\right|_{a_i^-}^{a_i^+}\frac{-1}{\bar{b}_i^2}\left.\left(\frac{(2\bar{b}_i-b_i)\cos b_i\tau}{\tau}+\frac{\sin b_i\tau}{\tau^2}\right)\right|_{b_i^-}^{b_i^+},$$

$$\mathfrak{g}_i(\tau,\upsilon)=\left\langle \frac{2\bar{b}_i^2-b_i^2}{\bar{b}_i^4}\mathrm{e}^{-a_i(\tau+\upsilon)}\sin b_i\tau\,\sin b_i\upsilon \right\rangle = p(a_i)p(b_i)\left.\frac{\mathrm{e}^{-a_i(\tau+\upsilon)}}{-(\tau+\upsilon)}\right|_{a_i^-}^{a_i^+}\frac{1}{2\bar{b}_i^4}*$$

$$*\left.\left[\begin{array}{l} 2\bar{b}_i^2\left(\dfrac{\sin b_i(\tau-\upsilon)}{(\tau-\upsilon)}\right)-\dfrac{2b_i(\tau-\upsilon)\cos b_i(\tau-\upsilon)+((\tau-\upsilon)^2b_i^2-2)\sin b_i(\tau-\upsilon)}{(\tau-\upsilon)^3} \\ -2\bar{b}_i^2\left(\dfrac{\sin b_i(\tau+\upsilon)}{(\tau+\upsilon)}\right)+\dfrac{2b_i(\tau+\upsilon)\cos b_i(\tau+\upsilon)+((\tau+\upsilon)^2b_i^2-2)\sin b_i(\tau+\upsilon)}{(\tau+\upsilon)^3}\end{array}\right]\right|_{b_i^-}^{b_i^+}. \quad (31\text{a, b})$$

5.2. HARMONIC EXCITATION

Consider harmonic excitation as $\xi(t) = \sin \Omega t$. The frequency response of a discrete system is, in general, given by $Y_i(\Omega) = \mathscr{A}(\Omega) + j\mathscr{B}(\Omega)$, where $\mathscr{A}$ and $\mathscr{B}$ are coincident and quadrature components. The first and second moments of $Y_i(\Omega)$ can be estimated by using the Fourier series expansion of $h_i(\tau)$ and $g_i(\tau, \upsilon)$ as follows, irrespective of the analytical method used:

$$\langle Y_i(\Omega_1)\rangle = \rho_i \mathscr{H}_i(\Omega), \qquad \langle Y_i(\Omega_1) Y_i(\Omega_2)\rangle = \mu_i \mathscr{G}_i(\Omega_1, \Omega_2), \tag{32a, b}$$

where $\mathscr{H}_i(\Omega_1)$ and $\mathscr{G}_i(\Omega_1, \Omega_2)$ are the Fourier transforms of $h_i(\tau)$ and $g_i(\tau, v)$. Similarly, the auto-power spectrum $\langle Y_i(\Omega_1) Y_i^*(\Omega_2)\rangle$ can be obtained. The one-dimensional and two-dimensional Fourier transforms of the sampled time data are computed as

$$\mathscr{H}_i(m\Delta\Omega_1) = \Delta t_1 \sum_{k=0}^{M-1} h_i(k\Delta t_1)[\cos(w_1) - j\sin(w_1)],$$

$$\mathscr{G}_i(m\Delta\Omega_1, n\Delta\Omega_2) = \Delta t_1 \Delta t_2 \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g_i(l\Delta t_1, k\Delta t_2)$$

$$\times [\cos(w_1) - j\sin(w_1)][\cos(w_2) - j\sin(w_2)], \tag{33a, b}$$

where $m, n = 1, 2, 3, \ldots,$ $\Delta t_1 = T_1/M,$ $\Delta t_2 = T_2/N,$ $\Delta\Omega_1 = 2\pi/T_1,$ $\Delta\Omega_2 = 2\pi/T_2,$ $w_1 = m\Delta\Omega_1 k\Delta t_1$, $w_2 = n\Delta\Omega_2 l\Delta t_2$, M and N are the total number of points sampled in the time domain, and T_1 and T_2 are time windows. Furthermore, the variance and covariance

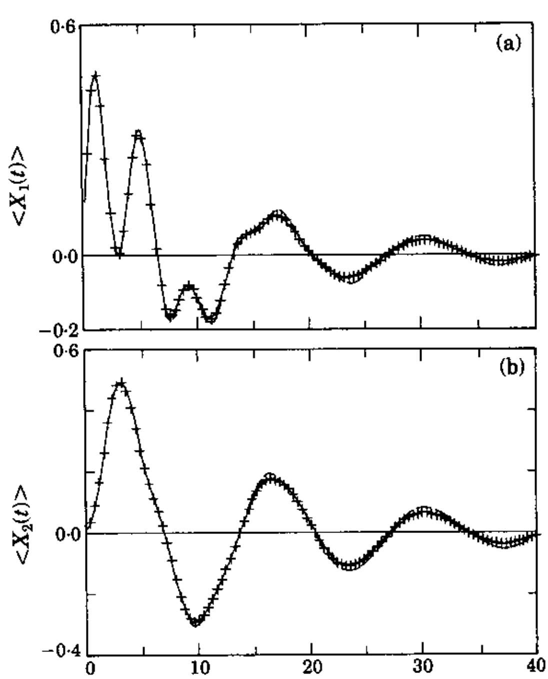

Figure 13. The expected mean of the impulse response in the generalized vector form for Example III, shown in Figure 1(b): (a) $\langle X_1(t)\rangle$; (b) $\langle X_2(t)\rangle$. Key as in Figure 10.

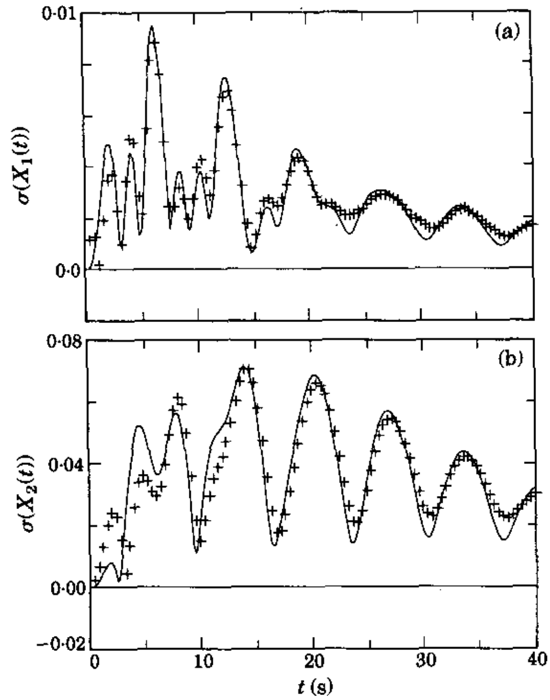

Figure 14. The standard deviation of the impulse response in the generalized vector form for Example III shown in Figure 1(b): (a) $\sigma(X_1(t))$; (b) $\sigma(X_2(t))$. Key as in Figure 10.

of the real and imaginary parts of $Y_i(\Omega)$ can be obtained from its second moment and auto-power spectrum definitions. For instance [14],

$$2\langle \mathscr{A}_i^2(\Omega)\rangle = \mathrm{Re}\,\langle Y_i(\Omega)Y_i(\Omega)\rangle + \mathrm{Re}\,\langle Y_i(\Omega)Y_i^*(\Omega)\rangle, \qquad \langle \mathscr{A}_i(\Omega)\rangle = \mathrm{Re}\,\langle Y_i(\Omega)\rangle. \tag{34a, b}$$

Therefore, the standard deviation of amplitude at Ω, as effected by the parameter uncertainties, e.g., the variance and covariance of the mass and stiffness can be found as

$$\sigma(A_i(\Omega)) = (\langle A_i^2(\Omega)\rangle - \langle A_i(\Omega)\rangle^2)^{0\cdot5}. \tag{35}$$

5.3. EXAMPLE IV: SINGLE-DEGREE-OF-FREEDOM SYSTEM

Consider the single-degree-of-freedom system of Figure 1(a) again, with $\bar{M} = 1$, $\bar{C} = 0{\cdot}4$, $\bar{K} = 1$, $\bar{F} = 1$ and $\xi(t) = \sin \Omega t$; hence $\bar{\gamma}_1 = -0{\cdot}2 \pm 0{\cdot}98\mathrm{j}$. We consider the uncertainty in stiffness only and assume it to be $R_{K,K} = 0{\cdot}01$, i.e., $\sigma(\omega_{d1}) = 0{\cdot}049$. The one- and two-dimensional FFT algorithms of CTRL-C software [15] are used to calculate the mean and mean-square spectra through equations (30) and (31). For this study the following sampling parameters were chosen: $M = N = 64$ and $T_1 = T_2 = 20\pi$. It is seen from Figure 16 that the expected mean and standard deviation of the $Y_i(\Omega)$ spectra, as predicted by the Green's function approach presented here and the direct product method discussed elsewhere in parallel development [11], are in very good agreement. It must be noted that the selections of sampling parameters influence the accuracy of FFT algorithm. Therefore, the prediction should improve for higher values of M and N. However, given the computationally intensive nature of the proposed approach, the alternative method of reference [11], which can also handle non-proportionally damped systems, is preferred.

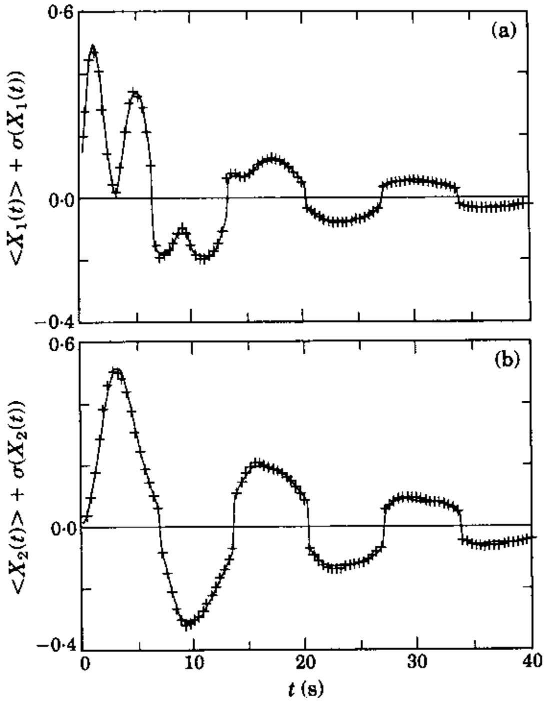

Figure 15. The standard deviation of the impulse response in the generalized vector form for Example III, shown in Figure 1(b): (a) $\langle X_1(t)\rangle + \sigma(X_1(t))$; (b) $\langle X_2(t)\rangle + \sigma(X_2(t))$. Key as in Figure 10.

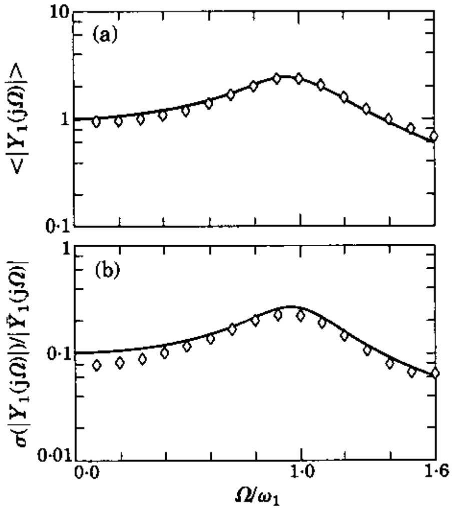

Figure 16. The frequency response spectra of Example IV, shown in Figure 1(a), with $\bar{C} = 0{\cdot}4$: (a) $\langle |Y_1(j\Omega)|\rangle$; (b) $\sigma(|Y_1(j\Omega)|)/|\bar{Y}_1(j\Omega)|$. Here, $|\bar{Y}_1(j\Omega)|$ is the amplitude of the corresponding deterministic system. ——, Direct product method [11]; ◇ ◇ ◇, proposed method of section 5.2.

6. CONCLUDING REMARKS

A new analytical method for the solutions of the impulse and frequency responses of a linear time-invariant, proportionally damped, discrete vibratory system with uncertain system parameters and force amplitude has been developed and validated through single- and two-degree-of-freedom system examples. Even though only uncorrelated damping and uniform distribution cases are presented analytically for the sake of illustration, the proposed method is also applicable to other cases. The proposed method overcomes some of the deficiencies of the existing methods. It was found to be clearly superior to the commonly used first order perturbation technique [1–3], since it removes the effect of secular terms generated by the perturbation method. Also, the proposed method is computationally faster than the Monte Carlo simulation, at least for several problems, since a large number of iterations is not required. The limitations of the proposed approach are already documented in Part I. Future efforts will be focused on overcoming some of the shortcomings of the method formulated here.

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APPENDIX: LIST OF SYMBOLS

a, b	random variables
$\mathbf{A}, \mathbf{B}$	random parameter matrices
c_i	modal damping
$C, \mathbf{C}$	damping
Cov	covariance
cs	column transformation
$\mathrm{d}y$	incremental of y
$\mathbf{D}$	direct product of two deterministic parameter matrices
$F, \mathbf{F}$	force amplitude
g	jointly density function
$h, \mathbf{h}$	random function
$\mathbf{I}$	identity matrix
Im	imaginary part
j	$=\sqrt{-1}$
k_i	modal stiffness
$K, \mathbf{K}$	stiffness
m_i	modal mass
$M, \mathbf{M}$	mass
r	correlation coefficient
$R, \mathbf{R}$	covariance in direct product form
Re	real part
s_j	normalized standard deviation of mass
$\mathbf{S}$	normalized covariance matrix
Si	sine integral
t	time
T	time window
$u, \mathbf{u}$	a function
$\xi(t)$	time history of force
Var	variance
$X, \mathbf{X}$	generalized co-ordinate
$y, \mathbf{y}$	random function
$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$	perturbed terms of $\mathbf{y}^{-1}$
$Y, \mathbf{Y}$	normal co-ordinate
$\mathscr{g}$	second moment of Green's function
$\mathcal{G}$	second moment of Fourier transformation
$\mathscr{h}$	first moment of Green's function
$\mathcal{H}$	first moment of Fourier transformation
α, β	random variable
μ	modal participation factor
ρ	modal participation factor
δ	normalized standard deviation
λ	eigenvalue
γ	complex eigenvalue
σ	standard deviation
τ, υ	time
ω	undamped natural frequency (rad/s)
Ω	excitation frequency (rad/s)
$\mathbf{\Phi}$	eigenvector
ζ	damping ratio

Superscripts

$*$	conjugate
-1	inverse
T	transpose
	second power of direct product
$\otimes 3$	third power of direct product
$\dot{}$	first time derivative
$\ddot{}$	second time derivative
$'$	derivative terms
$+$	upper bound
$-$	lower bound
$\tilde{}$	fluctuation or random component
$\bar{}$	deterministic component

Subscripts

A, B	random parameter matrices
c_i	modal damping
cs	column transformation
C	damping
d	damped
F	force amplitude
i, j	modal indices
I	identity matrix
k_i	modal stiffness
K	stiffness
m_i	modal mass
M	mass
oi	ith normalized eigenvalue
P	variance by the proposed method
u	a function
X	generalized co-ordinate
y	random function
Y	normal co-ordinate
γ, λ	eigenvalue
$\mathbf{\Phi}$	eigenvector
ω	undamped natural frequency (rad/s)

Operators

max	maximum value
abs	absolute value
Δ	incremental
$\langle\ \rangle$	expectation operator
$\otimes$	direct or Kronecker product
!	factorial
$\Vert\ \Vert$	norm operator
$\vert\ \vert$	magnitude or absolute value
$\vert$	operating point