LINEAR DYNAMIC ANALYSIS OF MULTI-MESH TRANSMISSIONS CONTAINING EXTERNAL, RIGID GEAR

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This paper extends the multi-body dynamics modeling strategy for a gear pair [6] to multi-mesh transmissions with external, fixed center, helical or spur gears. Each gear is modeled as a rigid body with six degrees of freedom. A multi-dimensional, position-dependent formulation is used to describe the gear mesh stiffness which is assumed to be distributed along the line of action. A simplified model of the shaft-bearing subsystems is included since the focus of this study is on the gear dynamics. Excitation to the system is considered in the form of either external torque pulsation or internal static transmission error. The governing equations are linearized to yield a formulation with position or time-varying coefficients (LTV). Subsequently, three examples of linearized time-invariant (LTI) transmission systems are solved, and eigensolution predictions of the multi-body dynamics model compare very well with finite element calculations. Then the periodic response of a non-unity gear pair system is studied in depth. New results including a comparison between LTI and LTV models are presented. It has been demonstrated that both time and frequency domain solutions can be efficiently and accurately constructed by using the multi-term harmonic balance method, provided that several shaft and gear mesh harmonics are included.
The force coupling phenomenon between the torsional, flexural and rocking modes of vibration is important in a geared-shaft system, especially for helical gears operating at high speeds. Rigid gears with six degrees of freedom have been modelled by some researchers, but the prior efforts to model the multi-dimensional mesh interface dynamics have been limited. For example, Kahraman [7] has used such a vectorial, lumped mesh stiffness model to study the effect of gear orientation on the multi-mesh system, but without including the distribution of mesh stiffness along the line of action. Kosuba and August [8] proposed a model for an epicyclic gear train with each gear having torsional and flexural degrees of freedom. More recently, Donley and Steyer [9] used a finite element model to analyze a planetary gear system consisting of a ring gear, a sun gear and four planetary gears. Saada and Velex [10] have retained all of the six degrees of freedom for each rigid gear in their model of a planetary drive while using a lumped mesh stiffness description. Recently, Blankenship and Singh [6] proposed a true six-dimensional mesh model for a gear pair that includes the results of teeth contact distribution. This model is the starting point for our multi-mesh formulation.

A few researchers have included a spatially distributed mesh stiffness expression in their multi-mesh models. For instance, Amirouche et al. [11, 12] used a combination of finite elements and multi-body dynamic formulation based on Kane’s equations [13, 14] to develop a composite model of gear and teeth. This method, though sufficiently general, needs a priori determination of some partial generalized velocities, which may be very difficult to obtain for a complicated gear train. Also, the usage of hybrid finite elements makes the overall problem computationally intensive. Wang and Huston [15, 16] have derived a modified form of Kane’s equations such that the generalized velocities need not be determined prior to the analysis. This modified method combined with Amirouche’s hybrid finite element formulation [13, 14] could speed up the analysis of multi-mesh geared systems but it would still be computationally intensive. Conversely, the generalized Newton–Euler equation method developed by Shabana [17–19] and others seems to be more suitable for the multi-body dynamics analysis of a geared system since it does not assume any prior knowledge of the physical system and could be applied to complex systems with moderate computational efforts. This approach is followed here conceptually, even though our multi-body dynamics methodology is developed from the basic principles while keeping the gearing problem in context, as evident from the single gear pair formulation [2, 6].

Given the comprehensive nature of the problem, the scope of this paper has been limited to the examination of only external, involute gears and each spur or helical gear is assumed to be rigid with a fixed center of rotation. Figure 1 illustrates three generic configurations which will be used to illustrate our methodology. Specific objectives of this paper are as follows: (i) to modify the mesh stiffness matrix which couples all six degrees of freedom between gear teeth while describing position-varying teeth contacts; (ii) to extend the prior multi-body dynamics strategy [6] to multi-mesh geared systems; (iii) to develop tractable linearized equations with time or position-varying coefficient (LTV) and the linear time-invariant (LTI) formulation; (iv) to validate the proposed methodology by comparing the resulting eigensolutions for the configurations shown in Figure 1 with those predicted by the finite element method; (v) to develop an efficient procedure for calculating the steady state response of the LTV model by using the multi-term harmonic balance method; and (vi) to compare the LTV and LTI models for a non-unity gear pair problem.
2. SINGLE MESH FORMULATION

2.1. GOVERNING EQUATIONS

The single gear pair mesh dynamics is reviewed briefly since our multi-mesh formulation is a direct extension of the theory proposed by Blankenship and Singh [6]. The equations of motion for the gear \(i\) in a pair \(i-j\) are given in the dual domain \((t, \theta^*)\) form as follows, where \(t\) is time, \(\theta^*=\int_0^t \Omega^* \, dt\) is mean rotational component and \(\Omega^*\) is the mean rotational velocity of gear \(i\):

\[
M_i(\theta^*)q_{i.m}(t) + C_{i.m}^{-1}(\theta^*)q_{i.m}(t) - C_{i.m}^{-1}(\theta^*)q_{i.m}^*(t) + K_{i.m}^{-1}(\theta^*)q_{i.m}^* - K_{i.m}^{-1}(\theta^*)q_{i,m}^*(t) + K_{i.s.b}(\theta^*)q_{i,m}(t) = Q_{i.m}(t) + Q_i(t),
\]

where \(M_i\) is the inertia matrix, \(C_{i.m}^{-1}\) and \(C_{i.m}^{-1}\) are the generalized mesh damping matrices, \(K_{i.m}^{-1}\) and \(K_{i.m}^{-1}\) are the generalized mesh stiffness matrices, \(Q_{i.m}\) is the parametric force due to transmission error, \(Q_i\) is the external generalized force on gear \(i\) and \(q_i\) is the generalized co-ordinate associated with the gear \(i\). Also refer to Appendix A for the identification of symbols.

Figure 2 shows a few co-ordinate systems for a typical external gear body where \(X-Y-Z\) is an inertial reference frame and \(X_{i.G} - Y_{i.G} - Z_{i.G}\) and \(X_{i.m.G} - Y_{i.m.G} - Z_{i.m.G}\) are non-inertial frames necessary to define the motion of the gear body completely. Body co-ordinate system \(X_{i.G} - Y_{i.G} - Z_{i.G}\) is fixed to the gear blank \(i\) and hence it represents the true motion of the gear. The generalized co-ordinates of each gear are given as \(q = [R_{i.G}^T \, \theta]^T\), where \(R_{i.G}\) are the translational and \(\theta\) are the rotational co-ordinates. The decomposition of these co-ordinates into a mean (subscript \(o\)) and a dynamic (subscript \(m\)) component is carried out as outlined by Blankenship and Singh [6] and it is assumed that the dynamic components are small compared to the corresponding mean components.
The origin of the geometric co-ordinate system $X_iGm–Y_iGm–Z_iGm$ is coincident with that of the body co-ordinate and is fixed to the gear blank. This system co-ordinate, however, is a non-rotating type and its orientation is represented only by the dynamic component. The translational motion of this and the body co-ordinate system consists of the mean and vibratory components. The mean motion is significant for non-fixed centered gears, i.e., planet gears in an epicyclic transmission system. A mesh co-ordinate system $x_{ij}–g_{ij}–f_{ij}$ is fixed at the pitch point. Here, $g_{ij}–f_{ij}$ lie in the plane of action while $x_{ij}$ is normal to it, and $f_{ij}$ is parallel to $Z$-axis in the initial state. Yet another co-ordinate system $x_{ij}–s_{ij}–h_{ij}$ is necessary for the helical gears where the line of action is inclined at an helix angle of $\psi_{ij}$ to $\phi_{ij}$. For a spur gear, $s_{ij}–h_{ij}$ and $g_{ij}–f_{ij}$ are equivalent.

2.2. **Dynamic Mesh Force: Modified Formulation**

A six-dimensional mesh force was introduced by Blankenship and Singh [6] as $Q(t) = Q_m(t) + Q_v = [F(t)^T T(t)^T]^T$, where $Q_m(t)$ and $Q_v(t)$ are mean and vibratory components, respectively. The dynamic component $Q_v(t) = Q_{vm}(t) + Q_{vd}(t)$ consists of an elastic force $Q_{vm}(t)$ and a dissipative force $Q_{vd}(t)$. The elastic mesh force $Q_{vm}(t) = Q_m(t) + Q_{mv}(\theta^*)$ consists of $Q_m(t) = K_m(t)[\delta_i - \delta_j]$ where $\delta_i - \delta_j$ is the gross motion of the blanks and a parametric excitation force $Q_{mv}(t) = K_{mv}(t)[\delta_i - \delta_j]$ due to the static transmission error $STE = \delta_i - \delta_j$. Here $K_m$ is the generalized mesh stiffness matrix.
The analytical model of reference [6] is modified to make it more suitable for the analysis of multi-mesh, multi-gear systems. For instance, our model is formulated entirely in the generalized co-ordinate system, unlike the previous model. Consider an external helical gear of Figure 2 as described in the mesh co-ordinates $\chi^i - \eta^j$. The mesh is modeled by a linear array of springs distributed over the length of contact $l^j$, as proposed in reference [6], which depends on the tooth surface modifications, gear shaft misalignments and other mounting errors. The net contact zone may be off-center on the tooth facewidth by a length $h^j$. The elastic mesh force $F^j_{v'}$ at a point $P_{v'}$ in the direction $\hat{p}^j$ is

$$\|F^j_{v'}(t)\| = K^j(t)[\|\delta r^j_{p,v} - \delta r^j_{p,v'}\|] \delta \sigma,$$

where $K^j(t)$ is a scalar value for mesh stiffness per unit length of contact. Here, $r^j_{p,v}$ and $r^j_{p,v'}$ give the position of $P_{v'}$ in the geometric co-ordinates attached to the gears $i$ and $j$, respectively, as

$$r^j_{p,v} = R^j_{j} + A^j u^j_{v}, \quad r^j_{p,v'} = R^j_{j} + A^j u^j_{v'}.$$

The position vectors $u^j_{v}$ and $u^j_{v'}$ are in the geometric co-ordinates $(X_{im} - Y_{im} - Z_{im})$ and $(X_{im} - Y_{im} - Z_{im})$ of gears $i$ and $j$, respectively. Further, $A^j$ and $A^j$ are the rotational transformation matrices formed for these co-ordinate systems, given as

$$A(t) = \begin{bmatrix} 1 & -\theta^j_{im} & \theta^j_{im} \\ \theta^j_{im} & 1 & -\theta^j_{im} \\ -\theta^j_{im} & \theta^j_{im} & 1 \end{bmatrix}, \quad A'(t) = \begin{bmatrix} 1 & -\theta^j_{im} & \theta^j_{im} \\ -\theta^j_{im} & 1 & -\theta^j_{im} \\ -\theta^j_{im} & \theta^j_{im} & 1 \end{bmatrix}.$$

where angles $\theta^j_{im}$, $\theta^j_{im}$, $\theta^j_{im}$ are assumed to be very small such that $\cos \theta^j_{im} \approx 1$ and $\sin \theta^j_{im} \approx \theta^j_{im}$, etc. Now, $\delta r^j_{p,v}$ and $\delta r^j_{p,v'}$ can be derived from equation (3) as

$$\delta r^j_{p,v} = \delta R^j_{j} + \delta(A^j u^j_{v}) \quad \text{and} \quad \delta r^j_{p,v'} = \delta R^j_{j} + \delta(A^j u^j_{v'}).$$

Since $\langle \theta^j_{im} \rangle = 0$ and $\langle R^j_{j} \rangle = 0$, we obtain

$$\delta r^j_{p,v} = A^j u^j_{v} G^j \delta \theta^j_{im} + A \delta u^j_{v},$$

$$\delta r^j_{p,v'} = A^j u^j_{v} G^j \delta \theta^j_{im} + A \delta u^j_{v'}. \quad (5a, b)$$

Here, $\omega^j = G^j \theta^j_{im}$ is the angular velocity and $\theta^j_{im} = [\theta^j_{im}, \theta^j_{im}, \theta^j_{im}]^T$, where the superscript T implies the transpose of the matrix. Since $\theta^j_{im}$'s are infinitesimally small, $\omega^j \approx \theta^j_{im}$ and $G^j \approx I$, where $I$ is an identity matrix. Further, $\ddot{u}^j_{v}$ is an asymmetrical matrix formed from $u^j_{v} = [\ddot{u}^j_{v} \ddot{u}^j_{v} \ddot{u}^j_{v}]^T$ and is given by

$$E \ddot{u}^j_{v} = \begin{bmatrix} 0 & -\ddot{u}^j_{v} & \ddot{u}^j_{v} \\ \ddot{u}^j_{v} & 0 & -\ddot{u}^j_{v} \\ -\ddot{u}^j_{v} & \ddot{u}^j_{v} & 0 \end{bmatrix}.$$

With reference to Figure 2, $\ddot{u}^j_{v}$ can be given by the sum of $\ddot{u}^j_{v}$, the position vector of the pitch point in geometric co-ordinates and the unit mesh vector $\hat{q}^j$ as

$$\ddot{u}^j_{v} = \ddot{u}^j_{v} + \sigma^j \hat{q}^j. \quad \text{The pitch position vectors are} \quad \ddot{u} = A^j \xi[R^j_{j} - R^j_{v}], \quad \text{and} \quad \ddot{u}^j_{v} = A^j \xi[R^j_{j} - R^j_{v}],$$

where $\xi = \phi^j / (\phi^j + \phi^j)$. These can be now used to obtain expressions for $\ddot{u}^j_{v}$ and $\ddot{u}^j_{v}$ as

$$\ddot{u}^j_{v}(t) = \ddot{u}^j_{v}(t) + \sigma^j \ddot{q}^j, \quad \ddot{u}^j_{v}(t) = \ddot{u}^j_{v}(t) + \sigma^j \ddot{q}^j. \quad (7a, b)$$
Expressions for the mesh unit vectors \( \hat{u}^i \), \( \hat{v}^i \), \( \hat{q}^i \) and \( \hat{p}^i \) have already been derived by Blankenship and Singh [6]. These can be used to obtain

\[
\delta \hat{q}^i = \delta \left[ L^i \begin{bmatrix} \hat{u}^i \\ \hat{v}^i \end{bmatrix} \right],
\]

where

\[
L^i = \begin{bmatrix} \cos \psi^i_b & -\sin \psi^i_b \\ \sin \psi^i_b & \cos \psi^i_b \end{bmatrix},
\]

which can be substituted in equation (7) to yield

\[
\delta \mathbf{r}_{p,i}(t, \theta^*) = \mathbf{R}_{cm}(t) - \mathbf{A}(t)\mathbf{U}^i(t, \theta^*)\theta^i(t) + \xi \mathbf{A}(t)\delta[\mathbf{A}^T(t)[\mathbf{R}_d(t, \theta^*) - \mathbf{R}_d(t, \theta^*)]]
\]

\[
+ \sigma \delta \left[ L^i \begin{bmatrix} \mathbf{A}(t)\hat{u}^i \\ \mathbf{A}(t)\hat{v}^i \end{bmatrix} \right] \mathbf{R}_d(t, \theta^*) - \mathbf{R}_d(t, \theta^*)],
\]

\[
\delta \mathbf{r}_{p,j}(t, \theta^*) = \mathbf{R}_{cm}(t) - \mathbf{A}(t)\mathbf{U}^j(t, \theta^*)\theta^j(t) + \xi \mathbf{A}(t)\delta[\mathbf{A}^T(t)[\mathbf{R}_d(t, \theta^*) - \mathbf{R}_d(t, \theta^*)]]
\]

\[
+ \sigma \delta \left[ L^j \begin{bmatrix} \mathbf{A}(t)\hat{u}^j \\ \mathbf{A}(t)\hat{v}^j \end{bmatrix} \right] \mathbf{R}_d(t, \theta^*) - \mathbf{R}_d(t, \theta^*)].
\]

(8a, b)

This can be used to determine \( \delta \mathbf{r}_{p,i} \) and \( \delta \mathbf{r}_{p,j} \) at any instant of time \( t \) and nominal angular positions \( \theta^* \) and \( \theta^* \). Subsequently, instantaneous mesh force \( \mathbf{F}_{\theta^*} \) at the point \( P_{\theta^*} \) can be obtained. The generalized force \( Q_{\theta^*} \) due to the instantaneous mesh point force \( F_{\theta^*} \) is given by \( Q_{\theta^*} \approx [I \mathbf{A}^T \mathbf{A}]^{-1} \mathbf{F}_{\theta^*} = [I \mathbf{A}^T \mathbf{A}]^{-1} \mathbf{p}^i \mathbf{p}^j \). Substitution in equation (2) yields

\[
Q_{\theta^*} = \left[ \begin{bmatrix} 1 \\ \mathbf{u}_i^T \mathbf{A}^T \end{bmatrix} \right] \mathbf{p}^i \mathbf{p}^j \left\{ \delta \mathbf{r}_{p,i}(t, \theta^*) - \delta \mathbf{r}_{p,j}(t, \theta^*) \right\} \delta \sigma.
\]

(9)

Assuming that the contact occurs over the entire zone of contact along the line of action between gears \( i \) and \( j \), the total generalized mesh force on gear \( i \) is

\[
Q^i_{\theta^*} = \int_{0}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[ \begin{bmatrix} 1 \\ \mathbf{u}_i^T \mathbf{A}^T \end{bmatrix} \right] \mathbf{p}^i \mathbf{p}^j \left\{ \delta \mathbf{r}_{p,i}(t, \theta^*) - \delta \mathbf{r}_{p,j}(t, \theta^*) \right\} \delta \sigma. \]

(10)

Similarly, the parametric excitation force \( Q_{\theta^*} \) due to kinematic errors between teeth and elastic deflections of teeth is

\[
Q_{\theta^*} = \int_{0}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[ \begin{bmatrix} 1 \\ \mathbf{u}_i^T \mathbf{A}^T \end{bmatrix} \right] \mathbf{p}^i \mathbf{p}^j \delta \mathbf{e}_{cm} \delta \mathbf{e}_{cm}, \]

(11)

where \( \delta_i^j \) and \( \delta_i^j \) are three-dimensional transmission error vectors described in the mesh co-ordinates \( (\chi^i, \sigma^i, \eta^i) \).

### 2.3. Simplified Mesh Force Expression

The assumption of quasi-static state, i.e., limit \( \Omega^* \to 0 \) can be used to formulate the dynamic mesh force on the gear blank as

\[
Q_{\theta^*}(t) = Q_{\theta^*}(t) + Q_{\theta^*}(\delta_{\theta^*}, t) + Q_{\theta^*}(t).
\]

(12)

Since the vibratory motions \( \theta^* \) and \( \delta_{\theta^*} \) are assumed to be small compared to the mean components \( \theta^* \) and \( \mathbf{R}_d \), any products of vibratory components can obviously be neglected. This is desirable at this juncture since equation (10) is non-linear with time and position-varying coefficients. Solutions of such equations are very computationally intensive, especially for systems with multi-meshes. For instance, the third term of equation (5), \( \mathbf{A} \delta(\mathbf{u}_v) \), is a non-linear product of very small components and hence this term can be
effectively neglected, reducing equation (5) to
\[ \delta r_{pi}(t) \approx R_{gm}^i + A_p \Delta u_{pi}^i \theta_m, \]
which can be written in a compact form as follows, where \[ q_m = [R_{gm}^i \ \theta_m]^T \] is the quasi-static generalized co-ordinate of gear \( i \):
\[ \delta r_{pi}(t) \approx [I \ \Delta \hat{u}_{pi}^i]q_m. \] (13)

The term \( \hat{p}_i(t)K\delta \hat{p}_i(t) \) of equations (10) and (11) is non-linear, and time \( (t) \) and position \( (\theta^*) \) dependent. The unit mesh vector \( \hat{p}_i \) can be decomposed into a mean \( \hat{p}_i \) and a time-varying component \( \tilde{p}_i \) as \( \hat{p}_i(t) = \hat{p}_i(0, R_m, \theta^*) + \tilde{p}_i(t) \). Since the time-varying component is very small, it can also be neglected. Thus the above-mentioned term reduces to
\[ \hat{p}_i(\theta^*)K\tilde{p}_i(\theta^*), \]
de where \( i \) is effectively replaced by \( \theta^* \). Substituting this and equation (13) in equation (2) and replacing \( A_p \) by an identity matrix \( I \) since \( \theta_m \) is small, we obtain
\[ Q_{mg}^{\theta^*}(t) = Q_{mg}^{\theta^*}(t) - Q_{mg}^{\theta^*}(t). \] (14)

Here, \( Q_{mg}^{\theta^*}(t) = K_{mm}^{\theta^*}q_m \) is the mesh force on gear \( i \) due to the motion of gear body \( i \) and \( Q_{mg}^{\theta^*}(t) = K_{mm}^{\theta^*}q_m \) is the mesh force on gear \( i \) due to the motion of gear body \( j \). The mesh stiffness terms \( K_{mm}^{\theta^*} \) and \( K_{mm}^{\theta^*} \) are given as
\[ K_{mm}^{\theta^*}(\theta^*) = \int_{\phi} \delta \hat{u}_{pi}(\theta^*)[I_{1 \times 3} \ \tilde{u}_{pi}(\theta^*)] \tilde{p}_i(\theta^*)K_{mm}^{\theta^*} \tilde{p}_i(\theta^*) \]
\[ \times [I_{1 \times 3} \ \tilde{u}_{pi}(\theta^*)] d\sigma, \]
\[ l = i, j. \] (15)

This expression for mesh stiffness is different from the formulation of reference [6] but it is still linear with nominal position-varying coefficients. Both offset \( h^i(\theta^*) \) and contact length \( \Gamma^i(\theta^*) \) can be obtained from the existing gear contact mechanics programs [20, 21]. Also, the stiffness per unit length \( K^i \) can be estimated from such programs. Finally, this expression can further be reduced to a linear time-invariant form by decomposing \( h^i(\theta^*) \) and \( \Gamma^i(\theta^*) \) into mean and \( \theta^* \) varying components as \( h^i(\theta^*) = h^i_m + h^i_{\theta^*}(\theta^*) \) and \( \Gamma^i(\theta^*) = \Gamma^i_m + \Gamma^i_{\theta^*}(\theta^*) \). Now, the \( \theta^* \) varying components can be neglected to give a linear expression for mesh stiffness with position-invariant coefficients. This model may not be accurate since the oscillatory components are not usually negligible. Nonetheless, this model yields an eigenvalue problem which can be very easily solved to gain an insight into the dynamic characteristics of the geared system.

2.4. MASS MATRIX EXPRESSION

The mass matrix expressions developed in reference [6] are adequate for a multi-mesh formulation and are presented here for sake of continuity:
\[ M_{ik}^j = \begin{bmatrix} m_{ik}^{\theta_k \theta_k} & m_{ik}^{\theta_k \theta_1} \\ m_{ik}^{\theta_1 \theta_k} & m_{ik}^{\theta_1 \theta_1} \end{bmatrix}, \]
\[ m_{ik}^{\theta_k \theta_k} = \int \rho \int_{V'} I_{1 \times 3} dV', \]
\[ m_{ik}^{\theta_k \theta_1} = \int \rho \int_{V'} A_i \Delta \hat{u}_{ik}^j dV' G', \]
\[ m_{ik}^{\theta_1 \theta_k} = \int \rho \int_{V'} A_i \Delta \hat{u}_{ik}^j dV' G', \]
\[ m_{ik}^{\theta_1 \theta_1} = \int \rho \int_{V'} A_i A_i \Delta \hat{u}_{ik}^j dV' G'. \] (16a, b, c, d)

2.5. BEARING AND SHAFT STIFFNESS EXPRESSIONS

Since the focus of this study is on the gear mesh dynamics and not on transmissibility through bearings, the rolling element bearings are modelled as simple radial stiffness elements. Each shaft is assumed to be sufficiently thin so that Euler beam theory is applicable. Figure 3 shows a schematic of a combined bearing–shaft–gear model. The bearing stiffness for this configuration is given by
\[ K_b(\theta^*) = \sum_{k=1}^{2} \left[ I_{1 \times 3} \Delta \hat{u}_{ik}^j K_b \Delta \hat{u}_{ik}^j \right]. \] (17)
where $\tilde{u}_b^k$ is an asymmetric matrix formed from $\tilde{u}_b^k$, the position vector of the $k^{th}$ bearing in the geometric co-ordinate; $\hat{n}_{b,3} = [\hat{n}_{b,1}^1 \hat{n}_{b,1}^2 \hat{n}_{b,1}^3]$ is a matrix composed of the unit directional vectors of the three equivalent bearing stiffnesses. The bearing stiffness matrix is given in terms of the scalar radial stiffness values $K_{b1}^k$, $K_{b2}^k$ and $K_{b3}^k$ as follows, where “diag” refers to a diagonal matrix:

$$K_b^k = \text{diag}[K_{b1}^k, K_{b2}^k, K_{b3}^k].$$  \hfill (18)

It is assumed in this paper that the shafts are supported by bearings at both ends. The shaft stiffness matrix $K_{io}^s$ in such a case is symmetric.

$$K_{io}^s = K_{io}^s, \quad \text{Symmetric}$$

where with reference to Figure 3, $K_{ib}^s = 3EI(a+b)(a-b)^2+ab)/a^3b^2$ is the Euler bending stiffness, $K_{io}^s = 3EI(a+b)/ab$ is the rocking stiffness, $K_{ib}^s = 3EI(a^2-b^2)/a^2b^2$ is the rocking–bending coupling stiffness and $K_{io}^s = AE/(a+b)$ is the longitudinal stiffness; $E$ is Young’s modulus, $I$ is the area moment of inertia and $A$ is the cross-sectional area of the shaft. These stiffness terms can also be determined by other computational methods. The combined shaft–bearing stiffness matrix is defined as

$$K_{io}^{sb}(\theta^*) = [K_{io}^{s}(\theta^*) + K_{io}^{b(\theta^*)}]{-1},$$ \hfill (20)

where $\langle -1 \rangle$ implies term-by-term inverse.

2.6. MESH DAMPING MATRIX EXPRESSION

A simplified expression for mesh damping will be employed based on the proportional viscous damping assumption ($k = i, j$):

$$C_m^{uj-k}(\theta^*) = c_bK_m^{ij-k}(\theta^*), \quad Q_m^{uj}(t) = Q_m^{uj-k-1}(t) - Q_m^{uj-k}(t),$$  \hfill (21a, b)

where $c_b$ is a damping proportionality constant, $Q_m^{uj-k}$ is the dissipative mesh force on gear $i$ due to its own vibratory motion and $Q_m^{uj-k}$ is the dissipative force on it due to the vibratory motion of gear $j$. 

Figure 3. A schematic of the bearing–shaft model.
3. Multi-Mesh Formulation

The single gear mesh formulation of reference [6] and as further developed in section 2 will now be extended to multi-mesh, multi-geared systems. Gears can either be connected through mesh and/or by a common shaft. Some of the combinations of such connections are shown in Figure 1. To facilitate analytical and computational developments, we define several subspaces as shown in the figure. A mesh space of any gear \( i \) is defined as a subspace containing all the gears meshing with gear \( i \), for example for configuration 2, \( \mu^i \) is the mesh space of gear \( i \). Similarly, a shaft–bearing subspace contains all the gears that are connected to gear \( i \) by a shaft; e.g., \( \zeta^i \) is the shaft space of gear \( i \) in configuration 3. Note that the shaft subspace is connected to the rigid base via radial-bearing stiffness elements.

The elastic and damping mesh forces on gear \( i \) are the sum of forces from all of the gears in subspace \( \mu^i \):

\[
Q_{mg}^{i}(t) = \sum_{j \in \mu^i} Q_{mg}^{ij}(t), \quad Q_{mn}^{i}(t) = \sum_{j \in \mu^i} Q_{mn}^{ij}(t), \quad Q_{ms}^{i}(t) = \sum_{j \in \mu^i} Q_{ms}^{ij}(t). \quad (22a-c)
\]

Expanding it further by substituting \( Q_n^i = Q_{mg}^{i-1} - Q_{ms}^{i-1} \) and using equations (15) and (21), we obtain

\[
Q_{mg}^{i}(t) = \left( \sum_{j \in \mu^i} K_{m}^{i-1}(\theta^*) \right) q_m(t) - \sum_{j \in \mu^i} (K_{m}^{i-2}(\theta^*) q_m(t)),
\]

\[
Q_{ms}^{i}(t) = \left( \sum_{j \in \mu^i} C_{m}^{i-1}(\theta^*) \right) \dot{q}_m(t) - \sum_{j \in \mu^i} (C_{m}^{i-2}(\theta^*) \dot{q}_m(t)). \quad (23a, b)
\]

A compact form of these equations can be obtained as follows, where \( N_G \) is the total number of gears:

\[
Q_{mg} = [Q_{mg}^{1T} Q_{mg}^{2T} \cdots Q_{mg}^{NG}]^T, \quad Q_{ms} = [Q_{ms}^{1T} Q_{ms}^{2T} \cdots Q_{ms}^{NG}]^T, \quad Q_{mn} = [Q_{mn}^{1T} Q_{mn}^{2T} \cdots Q_{mn}^{NG}]^T,
\]

\[
Q_{mg}(t) = K_{m}(\theta^*) q_m(t), \quad Q_{ms}(t) = C_{m}(\theta^*) \dot{q}_m(t). \quad (24a-f)
\]

Here, the system mesh matrices \( K_m \) and \( C_m \) are given as follows, where \( K_{m}^{i-1} \) and \( C_{m}^{i-1} \) can be obtained from equations (15) and (21):

\[
K_{m,i}(\theta^*) = \sum_{k \in \mu^i} K_{m}^{i-1}(\theta^*) \quad \text{if} \quad i = j
\]

\[
= -K_{m}^{i-1}(\theta^*) \quad \text{if} \quad i \neq j \quad \text{and} \quad j \in \mu^i
\]

\[
= 0_{6 \times 6} \quad \text{if} \quad i \neq j \quad \text{and} \quad j \notin \mu^i; \quad i, j = 1, \ldots, N_G,
\]

\[
C_{m,i}(\theta^*) = \sum_{k \in \mu^i} C_{m}^{i-1}(\theta^*) \quad \text{if} \quad i = j
\]

\[
= -C_{m}^{i-1}(\theta^*) \quad \text{if} \quad i \neq j \quad \text{and} \quad j \in \mu^i
\]

\[
= 0_{6 \times 6} \quad \text{if} \quad i \neq j \quad \text{and} \quad j \notin \mu^i; \quad i, j = 1, \ldots, N_G. \quad (25a, b)
\]

The \( i \)th gear in a multi-geared system is connected to other gears and the rigid base via the corresponding shaft subspace \( \zeta^i \). Thus the combined force transmitted through the shaft and bearings on gear \( i \) is given by

\[
Q_{ms} = K_{ms} \dot{q}_m + \sum_{j \notin \zeta^i} (K_{ms}^{j} q_m), \quad (26)
\]
where \( K_{sb} \) is determined from the influence coefficient calculations between gears \( i \) and \( j \) mounted on the same shaft. Using the notation of section 2, the system shaft–bearing stiffness matrix can be written as

\[
K_{sbi,j}(u_i^*) = K_{io sb}(u_i^*) \quad \text{if} \quad i = j
\]

\[
= -K_{sb}'(\theta^*) \quad \text{if} \quad i \neq j \quad \text{and} \quad j \in \zeta_i
\]

\[
= 0_{6 \times 6} \quad \text{if} \quad i \neq j \quad \text{and} \quad j \notin \zeta_i; \quad i, j = 1, \ldots, N_g.
\] (27)

The system inertia matrix is obtained by assembling the individual mass matrices in a block diagonal form as

\[
M = \text{diag}[M^1 \quad M^2 \quad \cdots \quad M^{N_g}]. \quad (28)
\]

The external vibratory excitation force vector on the gears, say from torque pulsations, can be arranged as

\[
Q = [Q^1 \quad Q^2 \quad \cdots \quad Q^{N_g}]',
\]

to form the system external excitation vector. Equations (23)–(28) and the system vectors are finally assembled to form the dynamic system governing equations for a multi-mesh, multi-geared system as follows; note its dual domain \((\theta^*, t)\) characteristics:

\[
M(\theta^*)\ddot{q}(t) + C_m(\theta^*)\dot{q}(t) + K_m(\theta^*)q(t) + K_{sb}(\theta^*)q(t) = Q_m(t) + Q(t). \quad (29)
\]

4. MODAL ANALYSIS

4.1. EIGENVALUE FORMULATION

Analytical solution of the linear position or time-invariant (LTI) model is obtained by first solving the undamped eigenvalue problem corresponding to equation (29) as

\[
[-\omega^2 \quad I + \Xi]q_r = 0.
\] (30)

where \( \omega_r \) is the \( r \)th natural frequency, \( q_r \) is the \( r \)th eigenvector or mode shape and the system matrix is \( \Xi = M^{-1}(K_m + K_{sb}) \), where the subscript \( o \) denotes mean value. This analytical model is solved numerically to obtain all of the natural frequencies and mode shapes of the transmission system. Since the system properties are assumed to be time-invariant, a finite element model of the quasi-static system could also be constructed by using any general purpose commercial code. We have employed the ANSYS software [22]. A typical finite element model would consist of rigid gears, formed by using eight noded brick elements, flexible shafts described via three-dimensional beam elements and bearings represented by linear stiffness elements. The distributed mesh interface is simulated by creating an array of linear spring elements along the line of action.

4.2. NATURAL MODES OF CONFIGURATION NO. 1

The first example deals with the single gear pair (configuration 1). Since the prior literature [23] on modes of a gear pair is mostly for a unity gear ratio \((m_g = 1)\), results are presented for a non-unity \((m_g > 1)\) pair. Consider the spur gear pair case of Table 1. An analytical model of 12-d.o.f. of Figure 4(a) is constructed and its results for \( \omega_{sr} \) are compared with the predictions of a finite element model. There is an excellent match between these two methods. To understand the natural modes of a non-unity gear pair better, a reduced linear model of 3-d.o.f. is developed next. It is similar to the models described in references [23, 24] for a unity gear pair. A closed form solution for this problem has been found by using a symbolic manipulation code, but it is not included here.
for the sake of brevity. It predicts virtually the same natural frequencies and mode shapes as the 12-d.o.f. model. This 3-d.o.f. model can, therefore, be used for design studies. To illustrate this, we present the natural frequency (\(f_n\)) maps in Figure 5. These are in the dimensionless form 
\[
\frac{f_n}{f_1}\]
where \(f_1\) is the estimate of first natural frequency of the gear pair for a pure torsional mode. The results are being plotted on a log–log scale versus the dimensionless stiffness ratio 
\[
\frac{K_{12}}{K_{sb}}
\]
where \(K_{12}\) is the mesh stiffness and \(K_{sb}\) the combined stiffness of shaft and bearings as viewed by each gear. A horizontal line (of slope 0) implies a pure torsional mode and an inclined straight line (of slope 2 on this scale) is a pure bending mode. Other curves obviously represent coupled torsional-bending modes. Three cases of gear ratio \(m_g\) are presented, including the \(m_g = 1\) case which can be viewed as a reference from which all non-unity \(m_g\) curves may be assessed. Our results for \(m_g = 1\) obviously match those reported earlier in [23, 24].

4.3. NATURAL MODES OF CONFIGURATION 2

The second example deals with an LTI model of a stationary dual mesh, reverse-idler type system containing three gears as shown in Figure 6; other details are given in Table 1. The system parameters, such as contact length, are averaged over the mesh cycle so that the resulting system is time-invariant. Table 3 compares natural frequencies obtained from the analytical procedure developed in this paper and the finite element method (FEM) implemented by using the ANSYS software [22]. The first four modes are of the rigid body motion type, corresponding to the mean gear rotation and the shaft axis of rotation (\(z\)). The fifth mode describes the combination of shaft bending and torsional motions such that
there is no deformation in either of the two mesh stiffnesses. The next five modes correspond to the gear translations due to flexure of three shafts as shown in Figure 6(a). Modes 11–14 represent the rocking motions of two large gear blanks as shown in Figure 6(b). These have been mostly neglected by prior researchers. Mode 15 corresponds to a net deformation within the first gear mesh due to torsional as well as translational motions of the gear blanks. The next two modes represent the rocking motions of the smaller gear blanks. Finally, the eighteenth mode corresponds to a deformation within the second gear mesh. The errors of Table 3 should be viewed with discretion since the finite element method, used as a benchmark here, itself involves computational errors depending on the nature of modelling. Nonetheless, all of these modes are successfully predicted by the analytical model, but in a fraction of time compared with the finite element analysis.

Figure 4. Models for a non-unity spur gear pair (configuration 1): (a) twelve degrees of freedom per gear pair model; (b) three degrees of freedom model.
4.4. NATURAL MODES OF CONFIGURATION 3

The third example deals with a dual mesh system consisting of four spur gears in two planes, as designated by configuration 3 in Figure 1. The dynamic model of the system is shown in Figure 7; see Table 1 for other details. Table 4 compares the natural frequencies yielded by our theoretical model with those predicted by the FEM software. An excellent agreement is again observed. The mode shapes are similar to the example discussed in section 4.3. Two selected bending and rocking mode shapes are shown in Figure 7. This shows that the analytical model predicts accurate natural frequencies of the system when compared with FEM, but more efficiently.

5. FORCED RESPONSE STUDIES

5.1. MULTI-TERM HARMONIC BALANCE METHOD

The dual domain problem of equation (29) can be converted to a single domain problem by assuming time-invariant speed, i.e. $\dot{\theta}^* = \Omega^* t$. The solution of this formulation, which is now of the linear time-varying (LTV) type, can be obtained by using both numerical (direct time domain) and semi-analytical techniques. For our study, Galerkin’s technique or multi-harmonic balance method is chosen as the semi-analytical technique [25, 26, 30] since the excitation is periodic and only the steady state, stable, periodic solution is of interest. Numerical integration is performed only as a confirmation of Galerkin’s procedure which
is briefly discussed here in the context of the problem. To minimize numerical instability, equation (29) can be posed in the following non-dimensionalized form where $\theta^* = \Omega^* t$:

$$q''(\tau) + \tilde{\Omega}^{-1}\Theta(\tau)q'(t) + \tilde{\Omega}^{-2}\Xi(\tau)q(\tau) = \tilde{\Omega}^{-2}F(\tau),$$

where $'$ denotes derivative with respect to $\tau$, $\Omega^* t = n\tau$, $\tilde{\Omega} = \Omega^*/\nu\omega$, $\omega$ is the characteristic frequency and $\nu$ is the subharmonic index. The system matrices are given by

$$\Xi(\tau) = M^{-1}(K_m(\tau) + K_o), \quad \Theta(\tau) = M^{-1}C_m(\tau), \quad F(\tau) = M^{-1}Q(\tau).$$

Figure 6. Selected models of the configuration 2 example shown in Figure 1: (a) fifth mode (shaft-bending type); (b) eleventh mode (rocking type). See Table 3 for a list of natural frequencies.
**Table 2**

Comparison of natural frequencies of 3-d.o.f. and 12-d.o.f. models of a single gear pair (configuration 1) shown in Figure 4

<table>
<thead>
<tr>
<th>Mode</th>
<th>3-d.o.f. system Natural frequencies (Hz)</th>
<th>12-d.o.f. system Natural frequencies</th>
<th>% Error†</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical model (Hz)</td>
<td>Finite element method (Hz)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>656</td>
<td>667</td>
<td>1·6</td>
</tr>
<tr>
<td>2</td>
<td>1126</td>
<td>1130</td>
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</tr>
<tr>
<td>3</td>
<td>1445</td>
<td>1451</td>
<td>0·4</td>
</tr>
<tr>
<td>4</td>
<td>2252</td>
<td>2411</td>
<td>6·6</td>
</tr>
<tr>
<td>5</td>
<td>3367</td>
<td>3601</td>
<td>6·5</td>
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<td>3392</td>
<td>3610</td>
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</tr>
<tr>
<td>8</td>
<td>7745</td>
<td>7851</td>
<td>1·4</td>
</tr>
</tbody>
</table>

† Error, % = 100([analytical − FEM]/FEM).

**Table 3**

Natural frequencies of configuration 2, double mesh geared system (see Table 1 for parameters)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Analytical model (Hz)</th>
<th>Finite element method (FEM)</th>
<th>% Error†</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>83</td>
<td>81</td>
<td>2·5</td>
</tr>
<tr>
<td>6</td>
<td>107</td>
<td>104</td>
<td>2·9</td>
</tr>
<tr>
<td>7</td>
<td>110</td>
<td>107</td>
<td>2·8</td>
</tr>
<tr>
<td>8</td>
<td>110</td>
<td>107</td>
<td>2·8</td>
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<td>151</td>
<td>2·6</td>
</tr>
<tr>
<td>10</td>
<td>202</td>
<td>195</td>
<td>3·6</td>
</tr>
<tr>
<td>11</td>
<td>431</td>
<td>443</td>
<td>2·7</td>
</tr>
<tr>
<td>12</td>
<td>431</td>
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<td>2·7</td>
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<tr>
<td>13</td>
<td>445</td>
<td>463</td>
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</tr>
<tr>
<td>17</td>
<td>1675</td>
<td>1723</td>
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</tr>
<tr>
<td>18</td>
<td>2459</td>
<td>2481</td>
<td>0·9</td>
</tr>
</tbody>
</table>

† Error, % = 100([analytical − FEM]/FEM).
Figure 7. Selected modes of the configuration 3 example shown in Figure 1: (a) eighth mode (shaft-bending and rocking type); (b) sixteenth mode (rocking type). See Table 4 for a list of natural frequencies.

Since the excitation $F(\tau)$, response $q(\tau)$, static transmission error $\delta_{ij}^{(e)}(\tau)$ and system matrices of equation (31a–d) are periodic, these can be expanded in the Fourier series form as

$$
\begin{align*}
F(\tau) &= \hat{F}, & \dot{q}^{(p)}(\tau) &= \hat{D}^{(p)} \hat{a}, & \Theta(\tau) &= \hat{D} \hat{b}, & \Xi(\tau) &= \hat{D} \hat{e}, & \delta_{ij}^{(e)}(\tau) &= \hat{D} \hat{e},
\end{align*}
$$

(32a–e)

where "(p)" denotes the $p$th derivative with respect to $\tau$. The discrete Fourier transform (DFT) matrices are defined along with Fourier coefficient vectors $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ and $\hat{e}$ as follows:

$$
\begin{align*}
\mathcal{F}_{j,1} &= 1, & \mathcal{F}_{j,2k} &= \sin (k \tau_j), & \mathcal{F}_{j,2k+1} &= \cos (k \tau_j), & \\
D^{j,2k} &= D^{j,2k+1} = k, & \\
j &= 1, \ldots, m, & k &= 1, \ldots, n, & m &\geq 2n + 1,
\end{align*}
$$

$$
\begin{align*}
\hat{Q} &= [Q(\tau_1), \ldots, Q(\tau_m)]^T, & \hat{q}^{(p)} &= [q^{(p)}(\tau_1), \ldots, q^{(p)}(\tau_m)]^T, & \\
\hat{a} &= [a_0 \quad \cdots \quad a_{2n}], & \hat{b} &= [b_0 \quad \cdots \quad b_{2n}], & \hat{c} &= [c_0 \quad \cdots \quad c_{2n}], & \\
\hat{d} &= [d_0 \quad \cdots \quad d_{2n}], & \hat{e} &= [e_0 \quad \cdots \quad e_{2n}].
\end{align*}
$$

(33a–l)
The problem can now be posed as a minimization procedure where the residual $R(t)$ to be minimized is obtained from equation (31a) as

$$ R(t) = Iq'(t) + \Omega^{-1}q(t) + \Omega^{-1}\Xi(t)q(t) - \Omega^{-1}F(t) = 0. \quad (34) $$

Writing this in the frequency ($\Omega$) domain by using the harmonic excitation response in the form of $e^{j\Omega t}$ and making use of Kronecker algebra [27], we obtain the following where $\otimes$ implies the Kronecker product:

$$ R(\Omega) = (\Omega^2 \otimes I)a + \Omega^{-1}b + \Omega^{-1}c - \Omega^{-1}d = 0, $$

$$ a = \text{vec} (\hat{a}^T), \quad b = \text{vec} (\hat{b}^T), \quad c = \text{vec} (\hat{c}^T). \quad (35a-d) $$

Several methods can now be applied to solve for the roots of equation (35) such as a simple Newton iteration scheme or a modified Broyden’s procedure [26]; the latter does not need the formation of a cumbersome Jacobean matrix at each iteration. For the Newton’s method however, the values of $\hat{a}$ and $\Omega$ at the $(p+1)$th iteration are given as [26]

$$ \begin{bmatrix} a \\ \Omega \end{bmatrix}^{p+1} = \begin{bmatrix} a \\ \Omega \end{bmatrix}^p + \begin{bmatrix} \frac{\partial R}{\partial a} \\ \frac{\partial R}{\partial \Omega} \end{bmatrix}^{p+1} R^p. $$

### Table 4

<table>
<thead>
<tr>
<th>Mode $r$</th>
<th>Analytical model (Hz)</th>
<th>Finite element method (FEM) (Hz)</th>
<th>% Error†</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
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<td>0</td>
<td>0</td>
</tr>
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<td>91</td>
<td>90</td>
<td>1.1</td>
</tr>
<tr>
<td>4</td>
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</tr>
<tr>
<td>20</td>
<td>1915</td>
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<td>24</td>
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<td>4191</td>
<td>0.1</td>
</tr>
</tbody>
</table>

†Error, % = 100[(analytical − FEM)/FEM].
\[
\frac{\partial \bar{\mathbf{R}}}{\partial \mathbf{a}} = (\mathbf{D}^2 \otimes \mathbf{I}) + \bar{\Omega}^{-1}(\mathbf{D} \mathbf{F}^T + \otimes \mathbf{I})(\mathbf{F} \mathbf{D} \otimes \mathbf{I}) + \bar{\Omega}^{-1}(\mathbf{F}^+ \otimes \mathbf{I}) \mathbf{f} (\mathbf{F} \otimes \mathbf{I}), \quad \mathbf{F}^+ = \mathbf{F}^T (\mathbf{F} \mathbf{F}^T)^{-1} \n\]

\[
\frac{\partial \bar{\mathbf{R}}}{\partial \bar{\Omega}} = 2\bar{\Omega}^{-3} \mathbf{d} - \bar{\Omega}^{-1} \mathbf{b} - 2\bar{\Omega}^{-2} \mathbf{c}, \n\]

\[
\mathbf{I} = \text{diag} \left( \frac{\partial \mathbf{q}(\tau_1)}{\partial \mathbf{q}(\tau_1)}, \ldots, \frac{\partial \mathbf{q}(\tau_m)}{\partial \mathbf{q}(\tau_m)} \right), \n\]

\[
\mathbf{F} = \text{diag} \left( \frac{\partial \mathbf{q}(\tau_1)}{\partial \mathbf{q}(\tau_1)}, \ldots, \frac{\partial \mathbf{q}(\tau_m)}{\partial \mathbf{q}(\tau_m)} \right). \n\]

A mean-square norm is used to determine the convergence of the harmonic balance method as

\[
\Phi^2 = \frac{1}{2}(\mathbf{a}^T \mathbf{D} \mathbf{a}) - \frac{1}{4} \mathbf{a}^T \mathbf{a}, \quad \Phi^2 = \frac{1}{2}(\mathbf{a}^T \mathbf{D} \mathbf{a}), \quad (37) \n\]

where the subscript \( l \) denotes the \( l \)-th generalized co-ordinate. In a geared transmission system, the frequency contents of the vibro-acoustic signals are related to both shaft and gear dynamics. For instance, the fundamental frequencies of interest for a non-unity gear system, the frequency contents of the vibro-acoustic signals are related to both shaft and modulation phenomenon [28] gives rise to sidebands which are given by

\[
A \text{mean-square norm is used to determine the convergence of the harmonic balance method as}
\]

A mean-square norm is used to determine the convergence of the harmonic balance method as

\[
\Phi^2 = \frac{1}{2}(\mathbf{a}^T \mathbf{D} \mathbf{a}) - \frac{1}{4} \mathbf{a}^T \mathbf{a}, \quad \Phi^2 = \frac{1}{2}(\mathbf{a}^T \mathbf{D} \mathbf{a}), \quad (37) \n\]

where the subscript \( l \) denotes the \( l \)-th generalized co-ordinate. In a geared transmission system, the frequency contents of the vibro-acoustic signals are related to both shaft and gear dynamics. For instance, the fundamental frequencies of interest for a non-unity gear pair are the shaft frequencies \( f_s \) and \( f_m \) and their harmonics \( mf_s, mf_m \). The next frequencies of interest are the gear mesh frequency \( f_m \) and its harmonics \( nf_m \). Also, the modulation phenomenon [28] gives rise to sidebands which are given by \( nf_m \pm mf_s \), where \( m = 0, 1, 2, \ldots, n = 1, 2, 3, \ldots \). Thus, the periodic mesh stiffness and the static transmission error can be expanded in Fourier series forms which are constructed as follows from equations (32d, e):

\[
\mathbf{z}(\tau) = \sum_{i=1}^{2} \sum_{m=0}^{2} \left( e_{b,2,2m-1} \cos (2\pi mf_s \tau) + e_{c,2,2m} \sin (2\pi mf_s \tau) \right) + \sum_{i=1}^{2} \sum_{m=0}^{2} \left( e_{a,2,2m-1} \cos (2\pi nf_m \tau) + e_{b,2,2m} \sin (2\pi nf_m \tau) \right) \n\]

\[
\mathbf{a}_m(\tau) = \sum_{i=1}^{2} \sum_{m=0}^{2} \left( e_{b,2,2m-1} \cos (2\pi mf_s \tau) + e_{c,2,2m} \sin (2\pi mf_s \tau) \right) + \sum_{i=1}^{2} \sum_{m=0}^{2} \left( e_{a,2,2m-1} \cos (2\pi nf_m \tau) + e_{b,2,2m} \sin (2\pi nf_m \tau) \right), \n\]

\[
k = (2n + 1)n + (m + 1), \quad (38a–c) \n\]

where \( n_s \) is the number of shaft frequency harmonics and \( n_m \) is the number of mesh frequency harmonics included in the multi-term harmonic balance analysis. An analytical expansion of the response terms like equation (38) is not considered here since this can be very cumbersome, especially for multi-mesh systems. Therefore, all such expansions are accomplished numerically. Obviously, the accuracy of the method will depend on the number of harmonic terms included in the analysis, and, consequently, the error associated with this method is primarily due to the truncation of shaft and mesh harmonics, in addition to the numerical tolerance error for convergence criterion.

5.4. NUMERICAL INTEGRATION

Direct numerical integration will be briefly discussed next, again keeping our physical system in context. The governing equations (29) can be modified to a state space form as

\[
\mathbf{X}(\tau) = \begin{bmatrix} \mathbf{q}(\tau) \\ \dot{\mathbf{q}}(\tau) \end{bmatrix}, \quad \dot{\mathbf{X}}(\tau) = \begin{bmatrix} \mathbf{0} \\ \bar{\Omega}^{-2} \mathbf{F} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\bar{\Omega}^{-1} \mathbf{I} \end{bmatrix} \mathbf{X}(\tau). \quad (39a, b) \n\]
This reduced first order form is solved by using a modified fourth–fifth order Runge–Kutta method as suggested by Lee et al. [25]. Since we are interested only in the steady state periodic solutions, the integrator must be run for a sufficiently long time to ensure that the transients due to initial conditions die down. This makes the simulation very time consuming.

5.3. RESULTS FOR CONFIGURATION 1

The example case chosen to illustrate the methodology is the single non-unity spur gear pair (configuration 1) with time-varying mesh stiffness \( K_{ij}(t) \). The system properties are given in Table 1. The mesh stiffness \( K^{ij} \) is formulated using equation (15), where offset \( h^{ij}(t) \) is set to zero and contact lengths \( \Gamma^{ij}(t) \) are obtained as a function of position \( \theta^* \) during the mesh cycle from the contact mechanics (LDP) software [20, 21], as shown in Figure 8. The angular position has been normalized in all of the plots such that \( \theta_{\text{norm}}^* = 1.0 \) corresponds to one full cycle of the pinion rotation and the frequency has been normalized such that \( \Omega_{\text{norm}}^* = 1.0 \) denotes the nominal pinion rotational frequency. As observed from the frequency spectrum, the contact length is composed of shaft frequency \( f_s \), mesh frequency \( f_m \) and their harmonics. The shaft frequency term is mainly due to an indexing error that was intentionally introduced in the pinion. In this example, the periodic
excitation is formulated by using equation (11) where the static transmission error (STE) \( \delta_{ij} \) is again calculated from the LDP software [20, 21]. Predictions are shown in Figure 9. The spectrum of \( \delta_{ij} \) shows that it is also composed of shaft frequency, mesh frequency and their harmonics.

As discussed in sections 5.1 and 5.2, two solution techniques are used to analyze the resulting LTV formulation: numerical integration and multi-term harmonic balance method. Numerical integration has been primarily used to verify results from the harmonic balance analysis which is used to generate both time and frequency domain response characteristics. The damping proportionality constant \( c_d \) is arbitrarily set to 0.0001 and the shaft speed of 30 Hz for this case is chosen such that none of the shaft or mesh harmonics is near any of the system natural frequencies. The numbers of harmonic terms used for the harmonic balance are \( n_s = 2 \) and \( n_m = 2 \), where \( n_s \) and \( n_m \) are described by equation (38).

Figures 10–15 show comparisons between the two techniques. Figure 10 shows the frequency or order domain spectrum and time history of translational motion \( Y_i \) of the first gear. Figures 11(a–c) show similar plots for the related rotational motion \( \theta_i \). As can be seen from the cyclic plots, results from the harmonic balance method \( (n_s = 2 \) and \( n_m = 2 \)) and the numerical integration disagree slightly. However, such discrepancies are significantly reduced when the number of harmonics included in the harmonic balance analysis is increased to \( n_s = 4 \) and \( n_m = 3 \), as evident from Figures 12 and 13. Figures 14 and

![Figure 9](image-url)

**Figure 9.** Static transmission error predictions for configuration 1: (a) frequency (or order domain) spectrum; (b) cyclic variations; (c) expanded view of (b) over two cycles. Here, \( \theta^{*} \) and \( \Omega^{*} \) have been normalized such that \( \theta_{\text{norm}}^{*} = 1.0 \) corresponds to full cycle of the pinion rotation and \( \Omega_{\text{norm}}^{*} = 1.0 \) denotes the nominal pinion rotational frequency.
Figure 10. Translational displacement ($Y_1^s$) predictions for configuration 1 with shaft speed of 30 Hz. Number of harmonic terms included in the harmonic balance method: $n_i = 2$, $n_m = 2$. (a) Frequency (or order domain) spectrum, (b) cyclic variation or time history, (c) expanded view of (b). ——, numerical integration; $x$ (part a) or - - - - (parts b and c), multi-term harmonic balance method.

15 show plots of $Y_1^s$ and $\theta_1^s$ when the shaft speed $\Omega^*$ is reduced to 24.6 Hz and the damping proportionality constant $c_d$ is decreased to 0.000 01. At this speed, the first gear mesh frequency $f_m$ is equal to the first gear mesh natural frequency ($\sim 658$ Hz from Table 2). Observe that the sidebands around $f_m$, $2f_m$, and $3f_m$ do not seem to match between the predictions of the harmonic balance and numerical integration methods. This is due to an insufficient frequency resolution while calculating the Fourier transform of the time domain predictions obtained from numerical integration. Since these are lower than the primary harmonic frequency by at least an order of magnitude, they have an insignificant contribution to the overall response which explains why virtually identical time signals are presented by both methods, as evident from Figures 14(c) and 15(c).

6. COMPARISON OF LTV AND LTI MODELS

6.1. EXAMPLE CASE AND PROCEDURE

A single non-unity gear pair (configuration 1) is again chosen for a detailed forced response study using both LTI and LTV models. Each model consists of 12 degrees of freedom as shown in Figure 4(a). For this example case, it is assumed that the excitation is only due to the internal mesh force associated with the transmission error and is given
by $Q_{im}^j(t) = K_{im}^j(t)\delta_{im}^j(t)$. The transmission error and the contact length variations are similar to those used in section 5, as shown in Figures 8 and 9. These have been obtained using the LDP software [20, 21] which predicts a scalar value of the transmission error along the line of action ($\mathbf{p}_b$ of Figure 2). Since the spur gears are considered in this example, the internal excitation is only along four degrees of freedom ($Y_1^G$, $Y_2^G$, $\theta_1^Z$, $\theta_2^Z$). For the sake of brevity, we will, however, study only one response, say translation $Y_1^G$, to compare the LTI and LTV models. In this study, the mean torque on the system is 1356 N-m (1000 in-lb) and the speed $\Omega^*$ is varied from 1 to 1800 Hz (60–108 000 rpm). A normal mode expansion technique is used to obtain the forced response of the LTI model. All 12 modes of Table 2 are used and the excitation $Q_{im}^j(t) = K_{im}^j(t)\delta_{im}^j(t)$ is obtained by taking a mean value of $T^i$, shown in Figure 8, and by using $K_{im}^j = K^i T^i$, where $K^i$ is the scalar mesh stiffness per unit length as defined earlier in section 2.3. At each speed, the response $q^i$ is calculated in frequency domain and then the root-mean-square (r.m.s.) value $q_{im}^i$ is obtained by summing up the response over the frequency range of interest. Then a speed map $q_{im}^i(\Omega^*)$ is constructed for a given torque load of the system. The multi-term harmonic balance method is used to calculate the response of the LTV system. The shaft speed is varied from 1 to 1800 Hz and the static transmission error is assumed to be uniform over this range. The response $q^i$ is obtained at each speed from the harmonic balance method and is given as a function of frequency. Further, a root-mean-square value $q_{m}^i$ is calculated at each speed by summing up all of the frequency components of the

![Figure 11. Rotational displacement ($\theta_i^Z$) predictions for configuration 1 with shaft speed of 30 Hz. Number of harmonic terms included in the the harmonic balance method: $n_s = 2$, $n_m = 2$. (a) Frequency (or order domain) spectrum, (b) cyclic variation or time history, (c) expanded view of (b). Key as in Figure 10.](image-url)
system variables. Four specific cases, as listed in Table 5, are studied. These cases are conceptually similar to the prior study conducted by Kahraman and Singh [29] for a single and three-degrees-of-freedom non-linear geared systems of unity gear ratio. Unlike the previous work, both shaft and gear mesh predictions are considered here in the context of a 12-degrees-of-freedom linear model of a non-unity gear pair in order to gain an improved understanding of the system behavior over the entire range of operating speed. Only stable, periodic solutions are of interest here.

6.2. RESULTS

6.2.1. Case I

Figure 16(a) shows the speed map of the response $Y_{1,G_{rms}}$ for a gear pair with harmonically varying transmission error $\delta_{in}(t)$ and the mesh stiffness $K_{in}(t)$ for the LTV model, both containing only the mesh harmonic term such as $\cos(2\pi f_n t)$ in equation (38). Of course, the corresponding LTI model has a time-invariant mesh stiffness. It is clear that for the LTV model, the excitation force $Q_{in}(t) = K_{in}(t)\delta_{in}(t)$ of equation (11) is a combination of the two time-varying expressions. A term-by-term product for $Q_{in}$ yields a term such as $\cos(2\pi f_n t) \cos(2\pi f_n t)$, which upon simplification gives superharmonic terms $\cos(4\pi f_n t)$, $\cos(6\pi f_n t)$, etc. In this and subsequent figures, the primary resonant speed range is from 24 to 54 Hz (1440–3240 rpm) since the mesh harmonic term ($\cos(2\pi f_n t)$) excites the three resonances of the system ($f_{in}, f_{nk}, f_{nk}$) corresponding to $Y_{1,G}$.  

Figure 12. Translational displacement ($Y_{1,G}$) predictions for configuration 1 with shaft speed of 30 Hz. Number of harmonic terms included in the harmonic balance method: $n_s = 4$, $n_m = 3$. (a) Frequency (or order domain) spectrum, (b) cyclic variation or time history, (c) expanded view of (b). Key as in Figure 10.
$Y_G$, $\theta_2$. Note that these three natural frequencies coincide with $f_{n1}$, $f_{n2}$ and $f_{n3}$ of the three-degrees of freedom model of Figure 4(b) and Table 2. At speeds below 24 Hz (1440 rpm), superharmonic resonances occur due to the presence of harmonics of the mesh frequency such as terms $(\cos(4\pi f_m t))$ and $(6\pi f_m t)$ in the excitation as well as the mesh stiffness. At speeds above 54 Hz (3240 rpm), subharmonic resonances occur due to the presence of shaft harmonics ($nf_i$) which now coincide with the system natural frequencies. For this case, observe that the LTI analysis, as expected, does not exhibit any resonant peaks in the sub- and superharmonic resonant speed regions. Also, the absence of any shaft frequency components in the LTV analysis results in a zero response over the subharmonic speed region. Table 6 lists the r.m.s. value $Y_{G, rms}$ at selected resonant speeds for LTI and LTV models. The first speed is $\Omega^* = 8.14$ Hz when the third harmonic of the mesh excitation frequency ($3f_m$) approaches the second natural frequency $f_{n2}$. Understandably, $Y_{G, rms}$ vanishes for the LTI model at this speed as excitation does not contain the $3f_m$ frequency component. However, the LTV excitation does contain this frequency component (see Table 5) and consequently a non-zero response is predicted. At a rotational speed of $\Omega^* = 53.5$ Hz, excitation at the mesh frequency ($f_m$) approaches the fourth system natural frequency $f_{n4}$. This component is present in both the LTI and the LTV model, so both predict non-zero responses. The third rotational speed for comparison, $\Omega^* = 660$ Hz, is chosen such that now the shaft frequency ($f_i$) approaches $f_{n2}$. Since the shaft frequency

![Figure 13](image)

Figure 13. Rotational displacement ($\theta_1^*$) predictions for configuration 1 with shaft speed of 30 Hz. Number of harmonic terms included in the harmonic balance method: $n_i = 4$, $n_{m} = 3$. (a) Frequency (or order domain) spectrum, (b) cyclic variation or time history, (c) expanded view of (b). Key as in Figure 10.
component is not present in either LTI and LTV models, the response vanishes in both cases.

6.2.2. Case II

Figure 16(b) shows the speed map of $Y_{\text{rms}}$ for the LTI and LTV cases where the transmission error $\delta_{\text{rms}}(t)$ varies periodically with mesh frequency terms $\cos(2\pi mf_n t)$, shaft frequency terms $\cos(2\pi nf s t)$ and sideband frequency terms $\cos(2\pi mf_n \pm nf_s t)$. But the mesh stiffness $K_m(t)$ for the LTV model has only the mesh frequency term $\cos(2\pi mf_n t)$, like case I. Again, the mesh stiffness is obviously constant for the LTI model. Now apart from peaks being excited by the mesh and shaft frequency components of the excitation, the presence of the strong shaft frequency terms in the transmission error expressions gives rise to a number of sideband frequency terms given by $\cos(2\pi mf_n \pm nf_s t)$ around the resonant peaks, where $m$ and $n$ are defined by equation (38). The nature of such sideband structures has been recently described by Blankenship and Singh [28]. Both LTI and LTV models predict peaks in the primary, sub- and superharmonic resonant speed ranges, and predictions from each model are similar. This can be explained by the fact that the frequency composition of responses from either model is dominated by the excitation, and thus they are similar but do not have identical amplitudes, as shown in Table 6 for selected speeds.
Figure 15. Rotational displacement ($\theta_1$) predictions for configuration 1 with shaft speed of 24.6 Hz. Number of harmonic terms included in the harmonic balance method: $n_s = 4$, $n_m = 3$. (a) Frequency (or order domain) spectrum, (b) cyclic variation or time history, (c) expanded view of (b). Key as in Figure 10.

6.2.3. Case III

Figure 17(a) shows the speed map of $Y_{G\text{rms}}$ for a case where the transmission error $\delta_{G\text{rms}}^i(t)$ contains only one frequency term as in case I. But the stiffness $K_{G\text{rms}}^i(t)$ for the LTV analysis varies periodically with mesh frequency harmonic terms $\cos(2\pi mf_m t)$, shaft frequency

<table>
<thead>
<tr>
<th>Case</th>
<th>$\delta_{G\text{rms}}^i(t)$ (LTI, LTV)</th>
<th>$K_{G\text{rms}}^i(t)$ (LTV)</th>
<th>Response $X_i$, $Y_i$, $Z_i$, $\theta_i$, $\theta'_i$, $\theta''_i$, $i = 1, 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$f_m$ Harmonic</td>
<td>$f_m$ Harmonic</td>
<td>$f_m$ Harmonic Periodic $mf_m$</td>
</tr>
<tr>
<td>II</td>
<td>$nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic</td>
<td>$f_m$ Harmonic $nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic $nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$f_m$ Harmonic</td>
<td>$nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic</td>
<td>$f_m$ Harmonic $nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic $nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic</td>
</tr>
<tr>
<td>IV</td>
<td>$nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic</td>
<td>$nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic</td>
<td>$nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic $nf_i$, $mf_m$, $mf_m \pm nf_i$ Periodic</td>
</tr>
</tbody>
</table>
Figure 16. Speed maps of the rms response $Y_{1,\text{rms}}$ of configuration 1. (a) Case I of Table 5, (b) Case II of Table 5. ——, linear time-varying (LTV) model; · · · · ·, linear time-invariant (LTI) model.

<table>
<thead>
<tr>
<th>Speed $\Omega^*$ (Hz)</th>
<th>Model/case</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.14 (excitation)</td>
<td>LTV</td>
<td>0.0</td>
<td>2.745</td>
<td>0.379</td>
<td>66.77</td>
</tr>
<tr>
<td>$3f_n \rightarrow f_n$</td>
<td>LTI</td>
<td>0.0</td>
<td>1.987</td>
<td>0.0</td>
<td>1.987</td>
</tr>
<tr>
<td>53.5 (excitation)</td>
<td>LTV</td>
<td>1.833</td>
<td>1.646</td>
<td>2.976</td>
<td>22.43</td>
</tr>
<tr>
<td>$f_n \rightarrow f_n$</td>
<td>LTI</td>
<td>1.624</td>
<td>1.559</td>
<td>1.624</td>
<td>1.559</td>
</tr>
<tr>
<td>660.0 (excitation)</td>
<td>LTV</td>
<td>0.0</td>
<td>6.499</td>
<td>0.379</td>
<td>88.59</td>
</tr>
<tr>
<td>$f_i \rightarrow f_n$</td>
<td>LTI</td>
<td>0.0</td>
<td>5.032</td>
<td>0.0</td>
<td>5.032</td>
</tr>
</tbody>
</table>

$^\dagger f_n$ is the $n^{th}$ natural frequency of the 12-d.o.f. LTI system, as listed in Table 4.
terms \( \cos (2\pi nf_t t) \) and sideband terms \( \cos (2\pi (mf_m \pm nf_s) t) \). Since a constant time-averaged mesh stiffness value is used for the LTI analysis, this case is similar to that of case I and it can be verified by comparing Figures 16(a) and 17(a). However, the LTV analysis now predicts peaks in the primary, sub- and superharmonic resonant speed regions, as explained earlier in this section. Some sidebands are also predicted due to the presence of both mesh and shaft frequency components. For this case, the LTI model seems to be inadequate especially at higher speeds.

6.2.4. Case IV

Figure 17(b) shows the speed map of \( Y_{G,1G} \), for the case when both transmission error \( \delta_{cm}(t) \) and mesh stiffness \( K_{cm}(t) \) vary periodically and have terms corresponding to the mesh frequency harmonics \( \cos (2\pi mf_m t) \), shaft frequency harmonics \( \cos (2\pi nf_s t) \) and sideband frequency terms \( \cos (2\pi (mf_m \pm nf_s) t) \). Again, a constant time-averaged mesh stiffness is used for the LTI model and this case is similar to that of case II as can be verified from Figures 16(b) and 17(b) and Table 6. It is obvious from these results that over the high speed ranges (mesh subharmonic regimes) the LTI predictions are accurate. However, the
LTI predictions are invariably lower, by as much as two orders of magnitude, than the LTV predictions in the primary and superharmonic resonant speed ranges. This is due to the fact that at lower speeds, the gear pair exhibits parametric resonances. Also, the LTI model does not predict those sidebands which are associated with the parametric excitation phenomenon.

7. CONCLUSIONS

The analysis of multi-mesh transmissions is indeed complicated since the effects of contact mechanics, rigid body motions, elastic deformations, etc. must be considered simultaneously. Obviously, a single paper cannot address all of them in a complete and satisfying manner. Consequently, this paper should be viewed as a first report on a comprehensive research project. Three main contributions, beyond the prior work [6, 23], emerge which will guide us and other researchers in future work. First, new and useful extensions to the multi-body dynamics framework have been developed which are suitable and yet computationally efficient for multi-mesh geared systems. Second, the strategy of constructing steady state periodic solutions to the resulting linear position or time-varying (LTV) formulations by using the multi-term harmonic balance is very promising; these include both low (shaft) and high (mesh) frequency terms. Third, new results for a non-unity gear pair have been generated by using LTI and LTV models, including a generalized natural frequency graph and r.m.s. response versus speed maps. Both sideband structures and sub- and superharmonic resonant regimes have been predicted successfully. Various simplifications and approximations seem to be valid since analytical predictions match well with those yielded by the commonly used numerical solvers such as the finite element method and the direct time domain integration. Ongoing research is focused on the inclusion of elastic characteristics of gear blanks. Subsequently, components of the planetary gear drives will be modelled. Parallel efforts will also be directed towards an experimental validation of the methodology.

ACKNOWLEDGMENT

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REFERENCES

APPENDIX A: LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>rotational transformation matrix</td>
</tr>
<tr>
<td>$A_s$</td>
<td>cross-section area of shafts</td>
</tr>
<tr>
<td>$a, b$</td>
<td>shaft lengths</td>
</tr>
<tr>
<td>$c$</td>
<td>damping matrix</td>
</tr>
<tr>
<td>$d$, $e$, $f$, $g$, $h$, $i$, $j$</td>
<td>Fourier coefficient vectors</td>
</tr>
</tbody>
</table>
c_d  damping proportionality constant
E   Young’s modulus
F   force vector
\mathcal{F} discrete Fourier transform (DFT) matrices
f   frequency (Hz)
\nu f   natural frequency (Hz)
G   Euler parameter matrix
h   contact line offset
I   identity matrix
i   gear number
K   scalar stiffness value
\mathbf{K} stiffness matrix
\mathcal{D} Fourier differentiation matrix
M   inertia matrix
m   inertia sub-matrices
m_n harmonic order
m_o gear speed ratio
N_o number of gears
\hat{a} bearing position vector
P   a point in the line of contact
p   net transmission error
\mathbf{Q} generalized force vector
q   generalized co-ordinate vector
\mathbf{R} generalized translational co-ordinate
\mathcal{R} residue
\mathbf{r} vector position of a point in the geometric co-ordinate
T   torque vector
t   time
\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q} unit vectors along the mesh co-ordinates
\mathbf{X} state space system variable
\mathcal{X}, \mathcal{Y}, \mathcal{Z} co-ordinate system
\Gamma contact length
\delta displacement
\delta small change
\zeta shaft subspace
\Theta damping matrix
\Theta generalized rotational co-ordinate
\Theta nominal angular position
\mu mesh space
\nu subharmonic index
\Xi system matrix
\xi gear diameter ratio
\rho density
\tau non-dimensionalized time
\Phi mean square norm
\Phi' gear diameter
\chi, \gamma, \phi mesh co-ordinate system
\chi, \sigma, \eta helical mesh co-ordinate system
\Psi helix angle
\Omega rotational speed (rpm or Hz)
\omega_o natural frequency (rad/s)

Subscripts
T transpose
i, j gear numbers
p order of differentiation
* nominal value
< -1 term-by-term inverse
### Subscripts
- $b$: base
- $G$: body co-ordinate system
- $g$: rigid body blank motion
- $G_m$: geometric co-ordinate system
- $k$: bearing number
- $m$: dynamic
- $md$: dynamic (dissipative)
- $me$: dynamic (elastic)
- $o$: mean
- $q$: generalized co-ordinates
- $R$: linear
- $s$: shaft
- $sB$: shaft bending
- $sR$: shaft rocking
- $sT$: shaft longitudinal
- $b$: bearing
- $r$: modal index
- $rms$: root mean square
- $x, y, z$: co-ordinates
- $\theta$: angular
- $e$: transmission error
- $\sigma$: linear position along the line of contact
- $\sim$: asymmetric matrix

### Abbreviations
- FEM: finite element method
- LTI: linear time-invariant
- LTV: linear time varying
- STE: static transmission error
- diag: diagonal matrix
- vec: vectorize