Structurally transmitted dynamic power in discretely joined damped component assemblies

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In this paper, a formulation for computing the vibration transmitted among components joined at discrete locations is presented. The formulation is based on a component modal synthesis approach. Natural frequencies and modes of individual components subject to specific boundary conditions at joint locations are used to compute the modal properties of assemblies from a variational formulation. Lagrange multipliers are used to enforce constraints that motions of components be identical at joint locations, and the assembly forced response is obtained from the normal mode method. Since the Lagrange multipliers used to enforce the motion constraints are equal to the interfacial forces, evaluating these from the assembly response provides an efficient means for computing the dynamic forces and moments at joints and the mechanical power transmitted among components. The formulation is initially developed for proportionally damped assemblies. Because localized damping at joint locations is often the most significant source of damping in component assemblies, the modal synthesis and vibration transmission formulations are subsequently extended to structures with arbitrary viscous damping. Simple illustrative examples of proportionally damped and nonproportionally damped assemblies of lumped parameter systems are included.

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INTRODUCTION

Vibration is transmitted among components in machine assemblies due to the work done by forces and moments acting at the joints where components are connected, and may be quantified by the mechanical power transmitted from one component to another. In this paper, a formulation for computing the vibration transmitted among components in discretely joined assemblies in terms of the modal properties of individual components is presented. Because localized friction at joint locations is often the dominant source of damping in assembled structures, special attention is focused on transmitted vibration in assemblies of components with general viscous damping.

Dynamic power transmission in vibrating assemblies has been addressed previously by broad frequency band[1–3] and narrow frequency band[4–26] analysis methods. Broadband methods such as Statistical Energy Analysis[1] and Asymptotic Modal Analysis[2] have more frequently been applied. These techniques were developed to predict high-frequency vibration in systems of components having large numbers of participating modes, and yield only a single temporally, spectrally, and spatially averaged vibration level for each component. Such methods are of limited use if one or more components has only a few participating modes in the frequency range of interest, and offer no information for determining which choice of joint locations might best reduce the vibration transmitted to one or more components.

Narrow-band methods, which include the method proposed in this paper, have also been used to compute transmitted vibration. Several previous studies considered the power transmission associated with flexural and longitudinal vibrations in assemblies of beams and rods.[4–11] Most of these were based on wave propagation ideas,[5–11] and several were limited to connections among infinite or semi-infinite bodies.[6–9] Other investigators applied similar methods to assemblies of flat plates,[12–18] and additional investigations[19–25] addressed problems associated with the power transmitted from vibrating rigid bodies to flexible foundations through compliant mounts. In none of these studies were the effects of complex-valued component modes considered. Chen and Soong,[26] however, examined the power transfer between two single degree of freedom oscillators joined through a compliant and dissipative connection.

Although these narrow-band analysis methods have been successfully applied to specific problems, none seems sufficiently general to address problems of structurally transmitted vibration in damped assemblies of components of finite size and arbitrary shape. Consequently, the purpose of this paper is to present a new formulation which affords such general utility. The formulation is based on a modal synthesis approach, which is a significant departure from previous investigations of transmitted vibration. While several different modal synthesis formulations have been proposed,[27] the one used in this study is similar to that used by Min *et al.*,[28,29] which is based on ideas originally developed by Hurty,[30] but with the Lagrangian function of the assembly expressed in terms of component modal properties and motion constraints at the joints enforced by Lagrange multipliers as proposed originally by Dowell.[31,32] Unfortunately, the synthesis procedure of Min *et al.*[28,29] can be applied only to undamped assemblies. To overcome this limitation, an extended modal synthesis formulation which permits the modal properties of assemblies of generally damped components to be determined is also developed. Literature in the area of modal synthesis methods for damped component assemblies is rather

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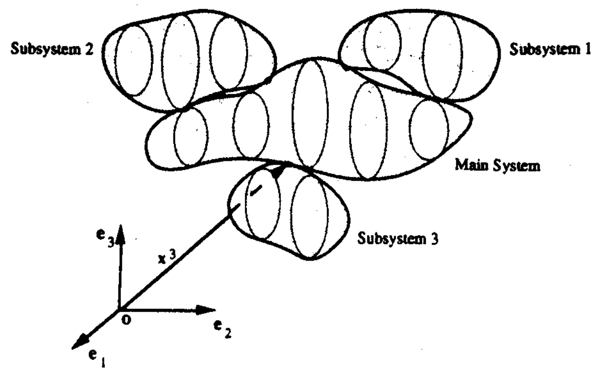

FIG. 1. System of vibrating components joined at discrete points to form an assembly.

sparse. Hallquist and Snyder[33] extended ideas proposed by Dowell[31] to systems with viscous damping. Dowell and his collaborators[34–38] investigated the application of component modal analysis methods to nonconservative and nonlinear systems. Their analyses were restricted to cases of undamped beams attached to damped single degree of freedom oscillators, however, and are not readily applicable to problems involving assemblies of damped components of arbitrary shape.

In this paper, the formulation is first developed for proportionally damped component assemblies, and is subsequently extended to component assemblies with general viscous damping. Examples of simple assemblies of lumped parameter oscillators are included to illustrate its application.

I. MODAL SYNTHESIS FOR UNDAMPED ASSEMBLIES

Consider a vibrating assembly of $R+1$ components as shown in Fig. 1. Let one of the components arbitrarily be denoted as the "main" system, and let the other components be denoted as "sub" systems. The subsystems may be connected to the main system through joints at any number of discrete points, but may not be connected to each other. Ways to circumvent this assumption are discussed later. Let $\mathscr{H}^0$ and $\mathscr{H}^r$ be the Hilbert spaces for motions of the main system and subsystem r, respectively. It is assumed that the assembly is a linear system, and it is further assumed in this section to be proportionally damped so that its modes are real valued. The rth subsystem is connected to the main system at the J^r points $[\mathbf{x}_1^r,\mathbf{x}_2^r,\ldots,\mathbf{x}_{J^r}^r]$. The vector $\mathbf{d}_j^r$ is the displacement of the joint at $\mathbf{x}_j^r$ caused by its connection to the main system. The dimension of $\mathbf{d}_j^r$ may be as great as six, but also may be smaller if the corresponding joint constrains only some displacement components. A discrete joint may, in general, transmit both translational and rotational motions. To clarify the present development, however, only connections which transmit only translational motions will be considered explicitly.

Consider the free vibration of the undamped components separately. The displacement $\mathbf{q}^0$ of the main system with respect to the inertial reference frame $\mathbf{o}$ satisfies

$$\mathbf{M}^0\ddot{\mathbf{q}}^0+\mathbf{K}^0\mathbf{q}^0=\mathbf{0}. \tag{1}$$

The self-adjoint mass and stiffness operators $\mathbf{M}^0$ and $\mathbf{K}^0$ are those appropriate for the main system when all joint locations $\mathbf{x}_j^r$ are unconstrained, but other main system boundary conditions are enforced. Note that $\mathbf{q}^0$ may be either a vector of continuous functions or discrete values, depending on the nature of $\mathscr{H}^0$. The eigenvalue problem associated with Eq. (1) is solved to yield eigenvalues ξ_i^2 and concomitant modes $\boldsymbol{\phi}_i$. The main system modes are scaled to be orthonormal with respect to $\mathbf{M}^0$ so that $\langle\boldsymbol{\phi}_i,\mathbf{M}^0\boldsymbol{\phi}_j\rangle=\delta_{ij}$ and $\langle\boldsymbol{\phi}_i,\mathbf{K}^0\boldsymbol{\phi}_j\rangle=\xi_i^2\delta_{ij}$, where the inner product is that associated with $\mathscr{H}^0$ and δ_{ij} is the Kronecker delta.

Similarly, the displacement $\mathbf{q}^r$ of subsystem r satisfies

$$\mathbf{M}^r\ddot{\mathbf{q}}^r+\mathbf{K}^r\mathbf{q}^r=\mathbf{0}, \tag{2}$$

where $\mathbf{M}^r$ and $\mathbf{K}^r$ are self-adjoint subsystem mass and stiffness operators corresponding to all subsystem joint displacements locations set to zero. Each of the R subsystem eigenvalue problems is solved to yield eigenvalues $(\nu_k^r)^2$ and mode functions $\boldsymbol{\psi}_k^r$ scaled such that $\langle\boldsymbol{\psi}_k^r,\mathbf{M}^r\boldsymbol{\psi}_j^r\rangle=\delta_{kj}$ and $\langle\boldsymbol{\psi}_k^r,\mathbf{K}^r\boldsymbol{\psi}_j^r\rangle=(\nu_k^r)^2\delta_{kj}$, where the inner product is that associated with $\mathscr{H}^r$.

Observe that the sets $\{\boldsymbol{\phi}_i\}$ and $\{\boldsymbol{\psi}_k^r\}$, respectively, form bases for $\mathscr{H}^0$ and $\mathscr{H}^r$. Hence, it is reasonable to suppose that the motion of the assembled structure may be expressed as a linear combination of the mode functions of its components. The steady harmonic motion of the main system after the components are assembled is

$$\mathbf{q}^0=\sum_{i=1}^{N^0} a_i\boldsymbol{\phi}_i e^{-j\Omega t}, \tag{3}$$

where N^0 is the number of main system modes included in the series. If the main system component has a large number of modes, it may not be practical to include all of them in Eq. (5). In such cases, the main system motion is projected on a subspace of $\mathscr{H}^0$ spanned by the N^0 modes included, and some error may be expected due to the incomplete nature of this set.

The motions of the subsystems after assembly require more careful consideration. Since $\{\boldsymbol{\psi}_k\}$ span the deformation of subsystem r relative to fixed conditions at the joint locations, the motion of the subsystems may be obtained by superimposing upon these additional deformations solely due to displacements imposed at the connection points. To illustrate, suppose that the motion of a joint at $\mathbf{x}_j^r$ is $\mathbf{d}_j^r$. Let a static displacement of unit magnitude corresponding to the lth component of $\mathbf{d}_j^r$ be imposed on subsystem r, while all other joint displacements are zero. If $\mathbf{g}_{jl}$ is the lth component of the static influence function relating internal displacements of subsystem r to external displacements at $\mathbf{x}_j^r$, then the deformation of subsystem r due to the imposed unit joint displacement will be precisely $\mathbf{g}_{jl}^r$. If a more general field of external joint displacements were applied, their effects would be superimposed and the subsystem displacement would be

$$\mathbf{q}^r=\mathbf{G}^r\mathbf{d}^r, \tag{4}$$

where $\mathbf{d}^r = [\mathbf{d}^{r_1T},\mathbf{d}^{r_2T},\ldots,\mathbf{d}^{r_{J^r}T}]^T$ is the vector of all displacements constrained by joints to the main system and the columns of $\mathbf{G}^r$ are the static influence functions $\mathbf{g}_{jl}^r$. To obtain the dynamic motion of subsystem r when the components are assembled, the motions with respect to fixed joint conditions and the motions due to joint displacements are superimposed. Hence,

$$\mathbf{q}^r = \left[\sum_{k=1}^{N^r} b_k^r \boldsymbol{\psi}_k^r + \mathbf{G}^r \mathbf{d}^r \right] e^{-j\Omega t}, \tag{5}$$

where N^r is the number of modes of subsystem r included in the expansion. Mathematically, the union of the set of subsystem mode functions $\{\boldsymbol{\psi}_k^r\}$ and the set of static influence functions $\{\mathbf{g}_{jl}^r\}$ is used as a basis for the subsystem motion after assembly. For the remainder of this development, the ubiquitous term $e^{-j\Omega t}$ will be omitted with the reader's understanding that its presence is implied.

The kinetic energy of the vibrating assembled structure is given by the sum of the kinetic energies of its components, $T = T^0 + \Sigma_{r=1}^R T^r$. Using the kinematic relations (3) and (5),

$$T^0 = \frac{1}{2} \langle \dot{\mathbf{q}}^0, \mathbf{M}^0 \dot{\mathbf{q}}^0 \rangle = \frac{1}{2} \Omega^2 \sum_{i=1}^{N^0} a_i^2 \tag{6}$$

$$T^r = \frac{1}{2} \langle \dot{\mathbf{q}}^r, \mathbf{M}^r \dot{\mathbf{q}}^r \rangle = \frac{1}{2} \Omega^2 \left(\sum_{k=1}^{N^r} (b_k^r)^2 + 2 \sum_{k=1}^{N^r} b_k^r \langle \boldsymbol{\psi}_k^r, \mathbf{M}^r \mathbf{G}^r \mathbf{d}^r \rangle + \langle \mathbf{G}^r \mathbf{d}^r, \mathbf{M}^r \mathbf{G}^r \mathbf{d}^r \rangle \right). \tag{7}$$

The potential energy for the assembly is also the sum of the potential energies of its components, $U = U^0 + \Sigma_{r=1}^R U^r$, and is computed in a similar fashion. For the main system,

$$U^0 = \frac{1}{2} \langle \mathbf{q}^0, \mathbf{K}^0 \mathbf{q}^0 \rangle = \frac{1}{2} \sum_{i=1}^{N^0} a_i^2 \xi_i^2. \tag{8}$$

The potential energy of the rth subsystem may be considered as the sum of the work done to deform the subsystem into a particular configuration relative to fixed joint conditions ($\mathbf{d} = \mathbf{0}$) and the additional work done on the subsystem by the forces acting at the joints ($\boldsymbol{\psi} = \mathbf{0}$). Hence,

$$U^r = (U^r)_{\mathbf{d}=\mathbf{0}} + (U^r)_{\boldsymbol{\psi}=\mathbf{0}}, \tag{9a}$$

$$(U^r)_{\mathbf{d}=\mathbf{0}} = \frac{1}{2} \langle (\mathbf{q}^r)_{\mathbf{d}=\mathbf{0}}, \mathbf{K}^r (\mathbf{q}^r)_{\mathbf{d}=\mathbf{0}} \rangle = \frac{1}{2} \sum_{k=1}^{N^r} (b_k^r)^2 \nu_k^r, \tag{9b}$$

$$(U^r)_{\boldsymbol{\psi}=\mathbf{0}} = \frac{1}{2} (\mathbf{d}^r)^H \mathbf{S}^r \mathbf{d}^r. \tag{9c}$$

The operator $\mathbf{S}^r$ denotes the external stiffness matrix for subsystem r which relates forces and displacement at the joint locations. The self-adjoint property of the subsystem stiffness operator $\mathbf{K}^r$ guarantees that $\mathbf{S}^r$ is also self-adjoint.

The displacements of the joints connecting the main system and subsystems occur in the expressions for both $\mathbf{q}^0$ and $\mathbf{q}^r$. Consequently, constraints on the joint motions $\mathbf{d}^r$ are necessary. The constraint that is used requires that the motions of the main system and subsystems be identical at the joint locations. This is expressed mathematically as

$$f^r = \sum_{i=1}^{N^0} a_i \mathbf{L}^r \boldsymbol{\phi}_i - \mathbf{d}^r = \mathbf{0}, \tag{10}$$

where the operator $\mathbf{L}^r$ is such that it evaluates the main system modes at the points where subsystem r is joined so that $\mathbf{L}^r \boldsymbol{\phi} = [\boldsymbol{\phi}(\mathbf{x}_1^r)^T, \ldots, \boldsymbol{\phi}(\mathbf{x}_{J^r}^r)^T]^T$.

Equations of motion for free vibration of the assembled structure are obtained from Lagrange's equation with Lagrange multipliers used to enforce the constraint conditions of Eq. (10). The vector of generalized coordinates is $\mathbf{y} = [a_1, \ldots, a_{N^0}, b_1^1, \ldots, b_{N^R}^R, d_{11}^1, \ldots, d_{JR}^R]^T$. Since the components of $\mathbf{y}$ are time invariant, the appropriate form of Lagrange's equation is

$$\frac{\partial L}{\partial y_i} + \sum_{r=1}^{R} \sum_{j=1}^{J^r} \lambda_j^r \frac{\partial f_j^r}{\partial y_i} = 0, \tag{11}$$

where $L = T - U$ is the Lagrangian function, λ_j^r is the Lagrange multiplier associated with the jth component of the rth constraint Eq. (10), and J^r is the dimension of $\mathbf{d}^r$.

If the equations resulting from application of Lagrange's equation (11) and the constraint equations (10) are manipulated to eliminate all d_j^r and λ_j^r, then the assembly eigenvalue problem

$$\mathbf{A}\mathbf{x} = \omega^2 \mathbf{B}\mathbf{x} \tag{12}$$

is obtained, where $\mathbf{x} = [a_1, \ldots, b_{N^R}^R]^T$ is the vector of unknown component modal expansion coefficients and $\mathbf{A}$ and $\mathbf{B}$ are self-adjoint matrices with nonzero coefficients

$$A_{ij} = \xi_i^2 \delta_{ij} + \sum_{r=1}^{R} (\mathbf{L}^r \boldsymbol{\phi}_j)^T \mathbf{S}^r \mathbf{L}^r \boldsymbol{\phi}_i, \quad i,j \in \mathcal{I}\,[1, N^0] \tag{13a}$$

$$A_{ij} = (\nu_i^r)^2 \delta_{ij}, \quad i,j \in \mathcal{I}\,[N^0 + 1, N^0 + \Sigma N^r] \tag{13b}$$

$$B_{ij} = \delta_{ij} + \sum_{r=1}^{R} \langle \mathbf{G}^r \mathbf{L}^r \boldsymbol{\phi}_j, \mathbf{M}^r \mathbf{G}^r \mathbf{L}^r \boldsymbol{\phi}_i \rangle, \quad i,j \in \mathcal{I}\,[1, N^0] \tag{13c}$$

$$B_{ij} = \delta_{ij}, \quad i,j \in \mathcal{I}\,[N^0 + 1, N^0 + \Sigma N^r] \tag{13d}$$

$$B_{ij} = \langle \boldsymbol{\psi}_k^r, \mathbf{M}^r \mathbf{G}^r \mathbf{L}^r \boldsymbol{\phi}_i \rangle, \quad i \in \mathcal{I}\,[1, N^0], \quad j \in \mathcal{I}\,[N^0 + 1, N^0 + \Sigma N^r], \quad k = j - \sum_{n=0}^{N^{r-1}} N^n. \tag{13e}$$

Equation (12) yields $(N^0 + \Sigma N^r)$ eigenvalues ω_i^2 and eigenvectors $\mathbf{x}_i$. The frequencies ω_i are the undamped natural frequencies of the assembled structure, and the concomitant modes are obtained from the eigenvectors $\mathbf{x}_i$. The kinematic relations of Eqs. (3) and (5) are used to compute the assembly mode functions. The matrix $\mathbf{U}$ may be defined such that $\mathbf{U} = \text{diag}[[\boldsymbol{\Phi}][\boldsymbol{\Psi}^1] \cdots [\boldsymbol{\Psi}^R]]$, where $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{N^0}]$ and

$\Psi^r = [\psi_1^r, \ldots, \psi_{N^r}^r]$ are matrices of component modes. Let the assembly displacement vector be $\mathbf{q}=[(\mathbf{q}^0)^T, (\mathbf{q}^1)^T, \ldots, (\mathbf{q}^R)^T]^T$. Then, the assembly mode functions $\mathbf{v}$ corresponding to $\mathbf{q}$ are

$$\mathbf{v}_i=[\mathbf{I}+\bar{\mathbf{G}}\bar{\mathbf{L}}]\mathbf{U}\mathbf{x}_i, \tag{14}$$

where $\mathbf{I}$ is an identity operator and

$$\bar{\mathbf{G}}=\mathrm{diag}[[\mathbf{0}] \quad [\mathbf{G}^1] \quad \cdots \quad [\mathbf{G}^R]], \tag{15}$$

$$\bar{\mathbf{L}}=\begin{bmatrix} [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{L}^1] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ \vdots & \vdots & \ddots & [\mathbf{0}] \\ [\mathbf{L}^R] & [\mathbf{0}] & \cdots & [\mathbf{0}] \end{bmatrix}. \tag{16}$$

The domains of the main system and subsystems intersect only at their boundaries. Hence, the components share no common mass, and the structure of $\mathbf{q}$ insures that the assembly mass operator has the form $\mathbf{M}=\mathrm{diag}[[\mathbf{M}^0][\mathbf{M}^1]\cdots[\mathbf{M}^R]]$. Consequently, if the coupled system stiffness operator is self-adjoint, then the coupled system eigenvectors $\mathbf{v}_i$ will be orthogonal with respect to $\mathbf{M}$, and may be scaled to be orthonormal with respect to $\mathbf{M}$ so that $\langle \mathbf{v}_i, \mathbf{M}\mathbf{v}_j \rangle = \delta_{ij}$.

The reader may recall that the formulation permits connections between the main system and the subsystems, but disallows connections among subsystem components. This does not, however, limit the utility of the formulation in cases of assemblies with interconnected subsystems, because the modal synthesis procedure may be applied recursively. The modal synthesis procedure first may be applied to a subset of components with connections only between the arbitrarily designated main system and subsystem components. In a subsequent analysis step, the entire assembly analyzed in the first step may be considered the main system, and some of the remaining components are considered subsystems. This procedure may be repeated as many times as necessary, with only connections between the main system and subsystems existing at each step of the analysis, and the modal properties of the entire assembly obtained from the final step.

II. DYNAMIC POWER TRANSMISSION AMONG COMPONENTS

Because the set $\{\mathbf{v}_i\}$ forms an orthonormal basis for the motion of the assembled structure, the normal mode method may be used to compute its forced response. Although the assembly stiffness matrix is not explicitly known, its self-adjoint property ensures that $\langle \mathbf{v}_i, \mathbf{K}\mathbf{v}_j \rangle = \omega_i^2 \delta_{ij}$. If $\mathbf{f}e^{-j\Omega t}$ is the assembly force vector, then the equation of motion for forced vibration is

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{f}e^{-j\Omega t}. \tag{17}$$

Under the transformation $\mathbf{q}(t)=\mathbf{V}\boldsymbol{\eta}e^{-j\Omega t}$, where $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_{N^0+\Sigma N^r}]$ is the matrix of assembly modes, Eq. (17) becomes

$$[\boldsymbol{\Lambda}-\Omega^2\mathbf{I}]\boldsymbol{\eta}=\mathbf{V}^T\mathbf{f}, \tag{18}$$

where $\boldsymbol{\Lambda}=\mathrm{diag}[\omega_1^2 \quad \omega_2^2 \cdots \omega_{N^0+\Sigma_r N^r}^2]$. If modal damping ζ_i is introduced, the modal participation factors η_i are

$$\eta_i(\Omega)=\frac{\mathbf{v}_i^T\mathbf{f}}{\omega_i^2-\Omega^2+j2\zeta_i\Omega\omega_i}. \tag{19}$$

The forced response of the assembly may be used to compute the mechanical power transmitted among components through joints. The present formulation is particularly well suited for this task, since the forces transmitted at the joints are precisely the Lagrange multipliers used to enforce the constraint conditions. In terms of $\mathbf{q}(\Omega)=\mathbf{V}\boldsymbol{\eta}(\Omega)$, the Lagrange multipliers for subsystem r are

$$\boldsymbol{\lambda}^r(\Omega)=\Omega^2(\mathbf{G}^r)^T\mathbf{M}^r\mathbf{q}^r-\mathbf{S}^r\mathbf{L}^r\mathbf{q}^0, \tag{20}$$

where $\boldsymbol{\lambda}^r$ is the vector of Lagrange multipliers associated with subsystem r.

The power transmitted among the components is computed using the corresponding joint velocities. The total power W_{in} injected into the assembly is

$$W_{\text{in}}(\Omega)=\tfrac{1}{2}\,\mathrm{Re}(j\Omega\mathbf{q}^T\mathbf{f}^*), \tag{21}$$

and the total power W^r transmitted from the main system to subsystem r is

$$W^r(\Omega)=\tfrac{1}{2}\,\mathrm{Re}[-j\Omega(\mathbf{L}^r\mathbf{q}^0)^T(\boldsymbol{\lambda}^r)^*]. \tag{22}$$

By considering cases when all components of $\boldsymbol{\lambda}^r$ are set to zero except those associated with a particular joint, the vibration transmitted through a particular structural path may be examined.

III. VIBRATION TRANSMISSION IN ASSEMBLIES OF GENERALLY DAMPED COMPONENTS

Of the assumptions involved in the development of the Secs. I and II, the requirement that the assembly modes be real-valued is particularly restrictive. It is not uncommon for relatively large localized damping to exist at joint locations in component assemblies due to interfacial friction. Since it is likely that this additional localized damping affects the mechanical power transmitted, it is necessary to extend the formulation of the previous sections to the case of complex-valued component and assembly modes.

Consider again the assembly of Fig. 1. The assumptions of Sec. I are again imposed, except that the assembly modes are no longer required to be real valued. The central idea of the extended modal synthesis formulation for damped component assemblies is the same as for the undamped formulation: The motion of the assembly is expressed in terms of the basis of individual component modes and static influence functions. In the formulation for damped component assemblies, the *undamped* component modes corresponding to the eigenvalue problems associated with Eqs. (1) and (2) are once again used, with free and fixed conditions imposed at the joint locations of the main system and subsystems, respectively. The kinematic relations for the main system and subsystem motions in terms of component modes are

$$\mathbf{q}^0=\sum_{i=1}^{N^0} a_i(t)\boldsymbol{\phi}_i \tag{23}$$

$$\mathbf{q}^r=\sum_{k=1}^{N^r} b_k^r(t)\boldsymbol{\psi}_k^r+\mathbf{G}^r\mathbf{d}^r(t), \tag{24}$$

where the functions $a_i(t)$, $b_k^r(t)$, and $\mathbf{d}^r(t)$ are unknown. An approach similar to that of the undamped modal synthesis formulation is applied. Since undamped component modes are used, the kinetic and potential energies of the main system and subsystems are given by Eqs. (6), (7), (8), and (9). Because damping is present, the additional work done by dissipative forces and motions also must be considered. To do this, a Rayleigh dissipation function[39] is defined for each component. For the main system, the dissipation function is

$$D^0=\frac{1}{2}\sum_{i=1}^{N^0}\sum_{j=1}^{N^0} \dot{a}_i(t)\dot{a}_j(t)\boldsymbol{\phi}_i^T\mathbf{C}^0\boldsymbol{\phi}_j, \tag{25}$$

where $\mathbf{C}^0$ is the viscous damping matrix for the main system. Observe that unless the main system is proportionally damped, the main system modes $\boldsymbol{\phi}$ are not orthogonal with respect to $\mathbf{C}^0$, and all terms in Eq. (25) are, in general, nonzero. The dissipation function for each of the subsystems consists of two parts, analogous to those of the subsystem potential energy. The first is the energy which is dissipated within the subsystem when the joint locations are held fixed, and the second is the additional work which is dissipated by the forces acting at the joints. If $\boldsymbol{\Gamma}^r$ is the external damping operator for subsystem r which relates dissipative forces and velocities at the joint locations, then the dissipation function for subsystem r is

$$D^r=\frac{1}{2}\left\{\sum_{k=1}^{N^r}\sum_{j=1}^{N^r} \dot{b}_k^r(t)\dot{b}_j^r(t)(\boldsymbol{\psi}_k^r)^T\mathbf{C}^r\boldsymbol{\psi}_j^r+(\dot{\mathbf{d}}^r)^T\boldsymbol{\Gamma}^r\dot{\mathbf{d}}^r\right\}, \tag{26}$$

where $\mathbf{C}^r$ is the overall viscous damping matrix for subsystem r. The total dissipation function of the assembly is the sum of those of its components, $D=D^0+\Sigma_{r=1}^R D^r$.

A modified form of Lagrange's equation which includes dissipation is used to obtain the equations of motion for the assembly. The modified Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right)-\frac{\partial L}{\partial y_i}+\frac{\partial D}{\partial \dot{y}_i}-\sum_{r=1}^{R}\sum_{j=1}^{J^r}\lambda_j^r\frac{\partial f_j^r}{\partial y_i}=0. \tag{27}$$

The constraints imposed on Eq. (27) are again those of Eq. (10), which are enforced using Lagrange multipliers. The generalized coordinate vector $\mathbf{y}$ is also the same as for the undamped case, except that the component modal expansion coefficients are unknown functions of time rather than constants. To obtain an eigenvalue problem for the assembly in terms of the modes of its components, the equations resulting from application of Lagrange's equation and the constraint equation are manipulated to eliminate all $\mathbf{d}^r(t)$ and $\lambda^r(t)$. Further, the unknown component modal participation functions are assumed to have the exponential forms $a_i(t)=\alpha_i e^{st}$ and $b_k^r(t)=\beta_k^r e^{st}$, where s is an unknown complex variable. This yields the damped assembly eigenvalue problem

$$[\mathbf{A}+s\mathbf{D}+s^2\mathbf{B}]\mathbf{x}=\mathbf{0}, \tag{28}$$

where $\mathbf{x}=[\alpha_1,\ldots,\beta_J]^T$. The [illegible] again defined by Eq. (13) and the matrix $\mathbf{D}$ has non-zero coefficients

$$D_{ij}=\boldsymbol{\phi}_j^T\mathbf{C}^0\boldsymbol{\phi}_i+\sum_{r=1}^{R}\boldsymbol{\phi}_j^T(\mathbf{L}^r)^T\boldsymbol{\Gamma}^r\mathbf{L}^r\boldsymbol{\phi}_i, \quad i,j\in\mathcal{N}[1,N^0] \tag{29a}$$

$$D_{ij}=(\boldsymbol{\psi}_i^r)^T\mathbf{C}^r\boldsymbol{\psi}_j^r,$$
$$i,j\in\mathcal{N}\left[N^0+\sum_{n=1}^{r-1}N^n, N^0+\sum_{n=1}^{r}N^n\right]. \tag{29b}$$

Equation (28) is a quadratic matrix eigenvalue problem which cannot be solved using conventional solution methods for linear eigenvalue problems. It can, however, be transformed into an equivalent linear eigenvalue problem by defining a new unknown vector $\mathbf{y}=s\mathbf{x}$. Since $\mathbf{B}$ is positive definite and, consequently, invertible, Eq. (28) becomes

$$\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1}\mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}=s\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}, \tag{30}$$

which is the desired linear eigenvalue problem. The eigenvalues $s_i = \zeta_i\omega_i \pm j\omega_i\sqrt{1-\zeta_i^2}$ contain the natural frequencies and damping ratios of the assembly. The corresponding assembly modes are obtained from the eigenvectors $\mathbf{x}_i$ using Eq. (14). Although the component modes used to compute the assembly modes are real valued, the eigenvectors of modal expansion coefficients $\mathbf{x}_i$ are, in general, complex valued, so that assembly modes $\mathbf{v}_i$ are also complex valued.

Since a full set of assembly eigenvalues and eigenvectors have been obtained, a transformation similar to the normal mode method used for the proportionally damped system case is sought to uncouple the equations of motion of the assembly. The assembly modes $\mathbf{v}_i$ correspond to the forced equation of motion

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{f}, \tag{31}$$

where $\mathbf{C}$ is the assembly damping operator. A state-space method originally proposed by Frazer *et al.*[40] is often used to obtain a modal transformation to diagonalize such systems. If the new vector of unknowns $\mathbf{y}(t)=[\dot{\mathbf{q}}^T(t)\quad \mathbf{q}^T(t)]^T$ is introduced, then Eq. (31) becomes

$$\bar{\mathbf{M}}\dot{\mathbf{y}}+\bar{\mathbf{K}}\mathbf{y}=\bar{\mathbf{f}}, \tag{32}$$

where

$$\bar{\mathbf{M}}=\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix},\quad \bar{\mathbf{K}}=\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix},\quad \bar{\mathbf{f}}=[\mathbf{0}^T\quad \mathbf{f}^T]^T. \tag{33}$$

Associated with Eq. (32) is the eigenvalue problem

$$p\bar{\mathbf{M}}\mathbf{z}+\bar{\mathbf{K}}\mathbf{z}=\mathbf{0}, \tag{34}$$

where p is the eigenvalue parameter and $\mathbf{z}$ is a state-space eigenvector corresponding to $\mathbf{y}$. The eigenvectors $\mathbf{z}_i$ may be scaled so that they have the properties $\mathbf{z}_i^T\bar{\mathbf{M}}\mathbf{z}_j=\delta_{ij}$ and $\mathbf{z}_i^T\bar{\mathbf{K}}\mathbf{z}_j=-p_i\delta_{ij}$, which permit Eq. (31) to be diagonalized. It would be convenient to use this approach to compute the forced response of the assembly, but the relationships between eigenvalues s_i and p_i and eigenvectors $\mathbf{v}_i$ and $\mathbf{z}_i$ first must be established. To do this, observe that the eigenvalues and eigenvectors obtained from the modal synthesis procedure satisfy

$$[s_i^2\mathbf{M}+s_i\mathbf{C}+\mathbf{K}]\mathbf{v}_i=\mathbf{0}. \tag{35}$$

The eigenvectors associated with the state-space eigenvalue problem of Eq. (34) have the structure $\mathbf{z}_i=[p_i\boldsymbol{\theta}_i^T \quad \boldsymbol{\theta}_i^T]^T$, and are known to occur in complex conjugate pairs.[41] Substituting this form into Eq. (34),

$$p_i\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}\begin{bmatrix}p_i\boldsymbol{\theta}_i\\ \boldsymbol{\theta}_i\end{bmatrix}+\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\begin{bmatrix}p_i\boldsymbol{\theta}_i\\ \boldsymbol{\theta}_i\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \tag{36}$$

which implies that

$$[p_i^2\mathbf{M}+p_i\mathbf{C}+\mathbf{K}]\boldsymbol{\theta}_i=\mathbf{0}. \tag{37}$$

Comparing Eqs. (37) and (35), it is clear that $p_i=s_i$ and $\mathbf{v}_i=\boldsymbol{\theta}_i$, which are the desired relationships. Consequently, the forced vibration of the assembly may be obtained from the state-space approach. The state-space modal matrix $\mathbf{Z}$ is constructed from the assembly modal matrix $\mathbf{V}$ such that

$$\mathbf{Z}=\begin{bmatrix}\mathbf{VS} & \mathbf{V}^*\mathbf{S}^*\\ \mathbf{V} & \mathbf{V}^*\end{bmatrix}, \tag{38}$$

where $\mathbf{S}=\mathrm{diag}[s_1,s_2,\ldots,s_{N^0+\Sigma N^r}]$. Because s_i and $\mathbf{v}_i$ occur in complex conjugate pairs, the matrices $\mathbf{V}$ and $\mathbf{S}$ each contain only one-half of each pair, and the conjugate matrices $\mathbf{V}^*$ and $\mathbf{S}^*$ contain the other half. Under the transformation $\mathbf{y}=\mathbf{Z}\boldsymbol{\eta}$, the system of equation (32) is uncoupled. If the state-space harmonic force $\bar{\mathbf{f}}(t)=\bar{\mathbf{f}}e^{-j\Omega t}$ is applied, then modal participation factors η_i are given by

$$\eta_i(\Omega)=\mathbf{z}_i^T\bar{\mathbf{f}}/(j\Omega-s_i). \tag{39}$$

From the modal participation factors, the state-space response $\mathbf{y}(t)$ may be evaluated, and the assembly response $\mathbf{q}(t)$ may be obtained from $\mathbf{y}(t)$. The Lagrange multipliers representing the interfacial forces for the damped assembly are

$$\boldsymbol{\lambda}^r(\Omega)=\Omega^2(\mathbf{G}^r)^T\mathbf{M}^r\mathbf{q}^r-(j\Omega\boldsymbol{\Gamma}^r+\mathbf{S}^r)\mathbf{L}^r\mathbf{q}^0. \tag{40}$$

The dynamic power transmitted through the joints is then computed from Eq. (22), just as it was for proportionally damped assemblies.

One detail remains which must be addressed. In order to scale the state-space mode vectors $\mathbf{z}_i$ to be orthonormal with respect to $\bar{\mathbf{M}}$, either $\bar{\mathbf{M}}$ or $\bar{\mathbf{K}}$ must be known. The block diagonal assembly mass operator $\mathbf{M}$ is known, but the assembly stiffness operator $\mathbf{K}$ and damping operator $\mathbf{C}$ are not known explicitly. The stiffness operator $\mathbf{K}$ may be obtained from the natural frequencies and modes of the undamped assembly, however. If $\hat{\mathbf{V}}$ is the matrix of normalized undamped assembly modes, then the assembly stiffness operator is given by

$$\mathbf{K}=\mathbf{M}\hat{\mathbf{V}}\boldsymbol{\Lambda}\hat{\mathbf{V}}^T\mathbf{M}, \tag{41}$$

where $\boldsymbol{\Lambda}=\mathrm{diag}[\omega_1^2,\ldots,\omega_{N^0+\Sigma_r N^r}^2]$ is the matrix of undamped assembly eigenvalues. Hence, the state-space stiffness operator $\bar{\mathbf{K}}$ may be evaluated, and the orthonormality condition $\mathbf{z}_i^T\bar{\mathbf{K}}\mathbf{z}_j=-p_i\delta_{ij}$ may be used to correctly scale the state-space modes.

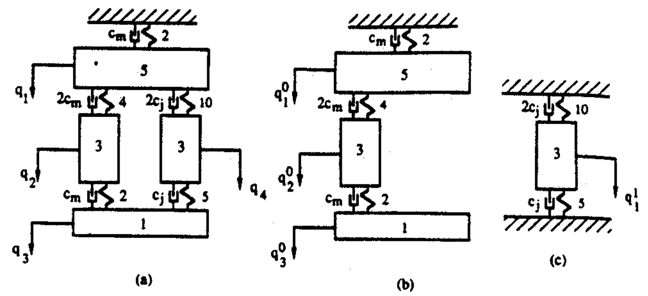

FIG. 2. Four degree of freedom lumped parameter system considered as example. (a) Assembled system. (b) Two dof "main" system. (c) Single dof "sub" system.

The constraint conditions enforced at the joints correspond to ideal joints which permit no relative motion of joined components at the connection points. Consequently, if one wishes to use the extended formulation to examine vibration transmission in assemblies with significant localized damping at the joints, the additional damping must be included in the component models of the subsystems. An example in the following section illustrates these ideas.

IV. EXAMPLE CASES

A. Proportionally damped lumped parameter system

Consider the four degree-of-freedom (dof) lumped parameter system shown in Fig. 2(a), with all damping coefficients set to zero. Let the system be considered as a three dof "main" system with a single dof "sub" system attached by two joints. The main system, shown in Fig. 2(b) with damping set to zero, has mass and stiffness matrices

$$\mathbf{M}^0=\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}\ \mathrm{kg},\qquad \mathbf{K}^0=\begin{bmatrix}6 & -4 & 0\\ -4 & 6 & -2\\ 0 & -2 & 2\end{bmatrix}\ \mathrm{N/m}. \tag{42}$$

The subsystem is shown in Fig. 2(c), with damping set to zero. Observe that zero displacement conditions are imposed

TABLE I. Component and assembly modal properties.

System	Natural frequency (rad/s)	Eigenvector
Main system [Fig. 3(b)]	0.4439	$[0.293,0.367,0.408]^T$
	1.2583	$[-0.302,0.145,0.694]^T$
	1.8492	$[0.152,-0.412,0.594]^T$
Sub system [Fig. 3(c)]	2.2361	$[0.577]$
Assembled system [Fig. 3(a)]	0.3968	$[0.272,0.307,0.303,0.292]^T$
	1.5839	$[0.154,-0.482,0.059,0.246]^T$
	2.2159	$[0.296,0.003,-0.606,-0.255]^T$
	3.1022	$[0.120,-0.085,0.733,-0.351]^T$

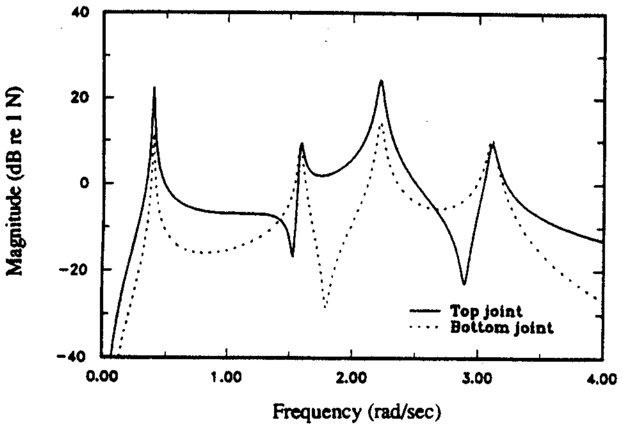

FIG. 3. Joint force magnitudes for proportionally damped four dof system example.

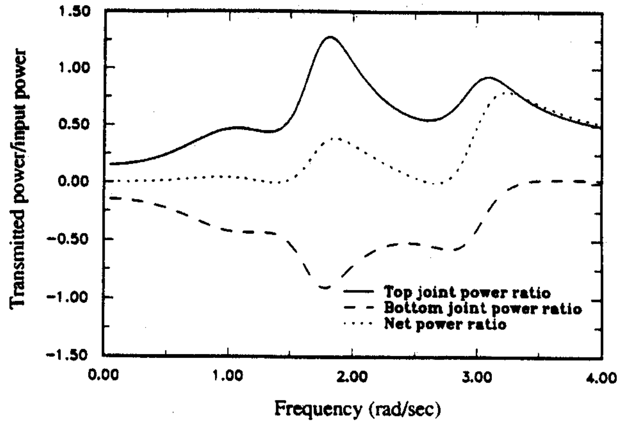

FIG. 5. Joint power transmission ratios for proportionally damped four dof system example.

at the joint locations. The mass and stiffness for the subsystem are $\mathbf{M}^1=3$ kg and $\mathbf{K}^1=15$ N/m. The eigenvalues and normalized eigenvectors of the individual components appear in Table I. The subsystem static influence function and external stiffness matrices are computed by imposing unit displacements at each joint location and evaluating the internal displacements and external forces. For this example,

$$\mathbf{G}^1=\left[\tfrac{2}{3}\quad \tfrac{1}{3}\right],\quad \mathbf{S}^1=\begin{bmatrix} \frac{10}{3} & -\frac{10}{3} \\ -\frac{10}{3} & \frac{10}{3} \end{bmatrix}\ \text{N/m}. \tag{43}$$

The constraint equation is

$$\mathbf{f}^1=\sum_{i=1}^{3} a_i \mathbf{L}^1 \boldsymbol{\phi}_i - \mathbf{d}^1 = \mathbf{0}, \tag{44}$$

with the selection operator $\mathbf{L}^1$ given by

$$\mathbf{L}^1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{45}$$

The modal synthesis procedure of Sec. I was applied to obtain natural frequencies and modes for the assembly. These are also listed in Table I, where the eigenvectors have been scaled to be orthonormal with respect to the assembly mass matrix. Conventional modal analysis of the assembled four dof system yielded modal properties identical to those of Table I.

The forced response and dynamic power transmission were computed using the formulation of Sec. II. The magnitudes of the interfacial forces at the joints for the excitation force $\mathbf{f}=[1\quad 0\quad 0\quad 0]^T$ with uniform harmonic content over the entire frequency range and damping ratios $\zeta=0.01$ assigned to all assembly modes appear in Fig. 3. The joint forces exhibit peaks in the neighborhoods of the assembly natural frequencies, and may be greater than or less than the exciting force at a particular frequency. The magnitudes of the input power and the power transmitted from the main system to the subsystem are shown in Fig. 4. Virtually all of the input and transmitted power is concentrated in narrow bands near the assembly natural frequencies. The ratio of transmitted power to input power is shown in Fig. 5. Observe that the subsystem receives mechanical power from the main system through the top joint and transmits power back to the main system through the bottom joint over almost the entire frequency range. The power transmitted through the subsystem is greater than the input power in the neighborhood of $\Omega=1.8$ rad/s. The ratio of net power transmitted to the subsystem to input power is always less than one, however,

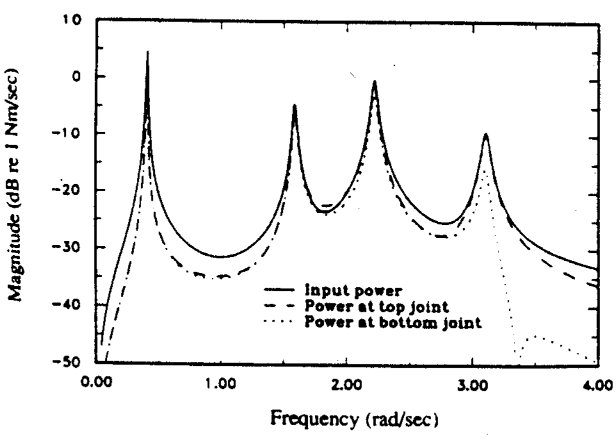

FIG. 4. Input and transmitted dynamic power for proportionally damped four dof system example.

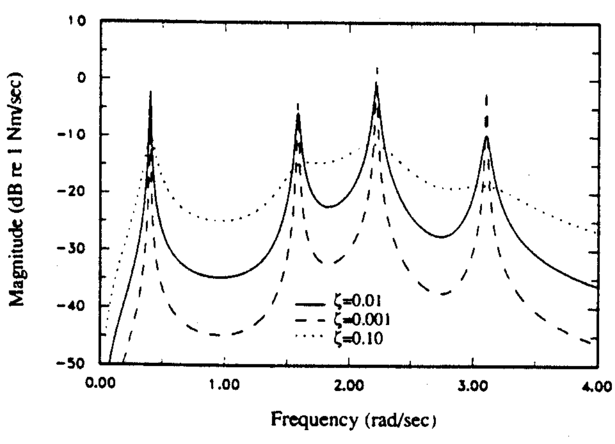

FIG. 6. Effect of assembly damping ratio on power transmitted through the top joint for proportionally damped four dof system example.

TABLE II. Assembly damping ratios for damped four dof system.

Modal index, i	Damping ratio ζ_i, $c_j=0.125$ N/(m/s)	Damping ratio ζ_i, $c_j=0.500$ N/(m/s)	Damping ratio ζ_i, $c_j=1.000$ N/(m/s)
1	0.0050	0.0054	0.0059
2	0.0198	0.0258	0.0327
3	0.0277	0.0894	0.1716
4	0.0388	0.1367	0.2690

as is thermodynamically necessary since the subsystem is not excited directly.

The effect of assembly modal damping ratio on power transmission for the same exciting force was examined. Damping ratios of 0.10, 0.01, and 0.001 were assigned to all assembly modes. The magnitudes of transmitted power through the top joint for these cases are shown in Fig. 6. Increasing damping ratios tends to reduce the power transmitted to the subsystem near the assembly natural frequencies and increase the power transmitted away from the resonant peaks. The power transmitted through the bottom joint and the input power exhibit similar trends.

B. Nonproportionally damped lumped parameter system

Another example of a four dof lumped parameter system with nonproportional damping is considered. This system, shown in Fig. 2(a), has viscous damping distributed over the main system of Fig. 2(b) in proportion to the main system stiffness matrix $\mathbf{K}^0$. Localized damping at the joint locations is included in the subsystem model, as indicated in Fig. 2(c). For this example, the main system damping parameter $c_m=0.05$ N/(m/s) is used. Since the motivation for the extended formulation of Sec. III was to permit the effects of relatively large localized joint damping to be examined, the subsystem joint damping parameter c_j is chosen to be relatively large compared to c_m. For this example, cases of $c_j=0.125$ N/(m/s), $c_j=0.500$ N/(m/s), and $c_j=1.000$ N/(m/s) are considered. Note that the case of $c_j=0.125$ N/(m/s) corresponds to stiffness proportional damping for the entire assembly.

Modal properties of the undamped components are again found in Table I. The subsystem static influence function matrix, external stiffness matrix, and the equation of constraint are the same as they were for the previous example. The main system and subsystem damping matrices are, respectively,

$$\mathbf{C}^0=\begin{bmatrix} 3c_m & -2c_m & 0 \\ -2c_m & 3c_m & -c_m \\ 0 & -c_m & c_m \end{bmatrix} \text{ N/(m/s)},$$
$$\mathbf{C}^1=[3c_j] \text{ N/(m/s)}. \tag{46}$$

The subsystem external damping matrix relating damping forces and velocities at the joint locations is

$$\mathbf{\Gamma}^1=\begin{bmatrix} \bar{c} & -\bar{c} \\ -\bar{c} & \bar{c} \end{bmatrix}, \text{ where } \bar{c}=[(1/2c_j)+(1/c_j)]^{-1}. \tag{47}$$

The extended formulation of Sec. III was applied to determine the modal properties of the assembly. The undamped natural frequencies ω_i obtained were the same as those found for the proportionally damped case appearing in Table I. Modal damping ratios, listed in Table II, were different for each mode and significantly greater for the higher frequency assembly modes. The real-valued assembly eigenvectors of Table I were again obtained for the proportionally damped case of $c_j=0.125$ N/(m/s), but the eigenvectors for the other two joint damping cases were complex valued. Assembly eigenvectors for the case of $c_j=1.000$ N/(m/s) appear in Table III, where the eigenvectors have been scaled to be orthonormal with respect to the state-space mass matrix $\bar{\mathbf{M}}$ and then rotated so that the first component of each is real. While the imaginary parts of the eigenvectors are often smaller than the real parts, they are not negligible, particularly for the higher frequency modes. Conventional state-space modal analysis of the damped assembly was also performed, and the natural frequencies, damping ratios, and eigenvectors obtained were found to be virtually identical to those found from the extended damped modal synthesis formulation in all cases.

The forced response of the assembly was computed for the same exciting force applied in the previous example. The interfacial force magnitudes for $c_j=1.000$ N/(m/s) are shown in Fig. 7. Note that the transmitted force magnitudes again may be greater than or less than that of the exciting force at

TABLE III. Assembly eigenvectors for damped four dof system, $c_j=1.000$ N/(m/s).

Mode 1 $\omega_1=0.3968$ rad/s	Mode 2 $\omega_2=1.5839$ rad/s	Mode 3 $\omega_3=2.2159$ rad/s	Mode 4 $\omega_4=3.1022$ rad/s
$0.3058+j0.0000$	$0.0900+j0.0000$	$0.1397+j0.0000$	$0.0497+j0.0000$
$0.3442-j0.0006$	$-0.2717+j0.0161$	$0.0063+j0.0050$	$-0.0257+j0.0208$
$0.3398-j0.0017$	$0.0378+j0.0135$	$-0.2947-j0.0343$	$0.2928-j0.0382$
$0.3274-j0.0013$	$0.1328-j0.0212$	$-0.1229+j0.0010$	$-0.1516-j0.0096$

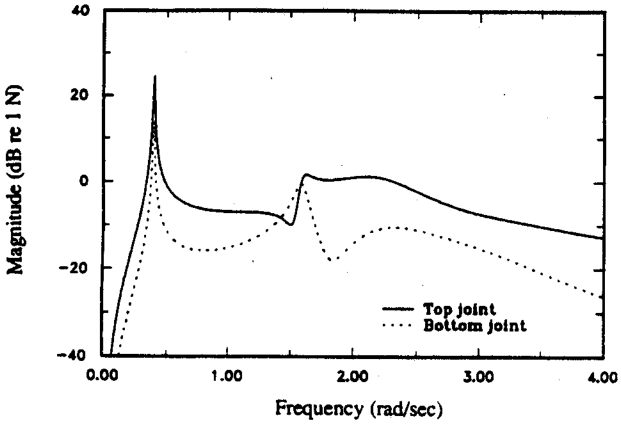

FIG. 7. Joint force magnitudes for nonproportionally damped four dof system example with $c_j = 1.00$.

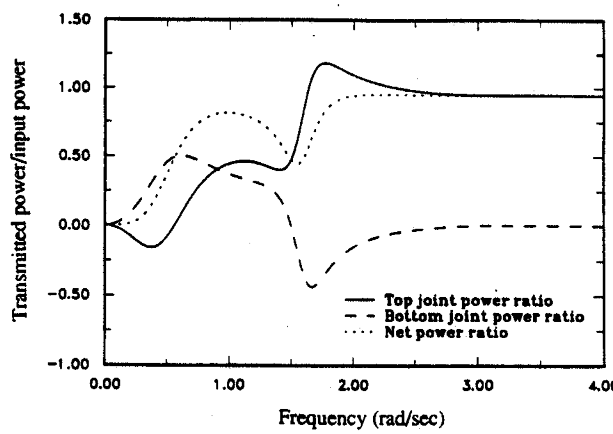

FIG. 9. Joint power transmission ratios for nonproportionally damped four dof system example with $c_j = 1.00$.

a particular frequency, and that the forces of Fig. 7 differ significantly from those of Fig. 3. Input and transmitted power magnitudes for the same joint damping are shown in Fig. 8. These results are also quite different from those of the proportionally damped case shown in Fig. 4. Observe that the input power and the power transmitted to the subsystem are nearly equal at the higher frequencies, indicating that most of the input power is transmitted to the subsystem through the top joint and dissipated within the subsystem. Power transmission ratio results for the same case are illustrated in Fig. 9. Figure 9 is significantly different from the analogous proportionally damped system result of Fig. 5. The top joint power ratio is again greater than one in the neighborhood of $\Omega = 1.8$ rad/s, but the thermodynamic requirement that the net power ratio not exceed unity is clearly satisfied.

The effect of localized joint damping on the input and transmitted dynamic power was also investigated. Magnitudes of the input power, the power transmitted through the top joint, and the power transmitted through the bottom joint are shown in Fig. 10, Fig. 11, and Fig. 12, respectively, for joint damping parameter values of $c_j = 0.125$ N/(m/s), $c_j = 0.500$ N/(m/s), and $c_j = 1.000$ N/(m/s). The effects of joint damping are similar for the input power and the power transmitted through the top joint. Increasing joint damping tends to increase power away from the resonant peaks and reduce it near them, and has greater effect at the higher frequencies. The effect of joint damping on the power transmitted through the bottom joint is also greater at the higher frequencies, but is much more pronounced than it is for the input power or the power transmitted through the top joint. The top joint power magnitudes of Fig. 11 are quite different from those of Fig. 6 with uniform modal damping ratios. Finally, observe that the results of Figs. 10, 11, and 12 for $c_j = 1.000$ N/(m/s) are quite different from those for $c_j = 0.125$ N/(m/s), which is the only joint damping case that could be addressed without the extended formulation of Sec. III.

V. CONCLUSION

A new analysis framework for computing the mechanical power transmitted through joints in vibrating machine assemblies has been developed. This approach is unique because it is based on component modal synthesis ideas. The use of component modes and static influence functions as a basis for the motion of a vibrating assembly offers informa-

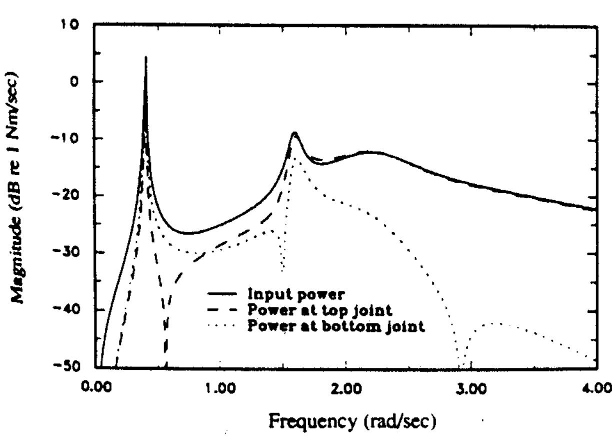

FIG. 8. Input and transmitted dynamic power for nonproportionally damped four dof system example with $c_j = 1.00$.

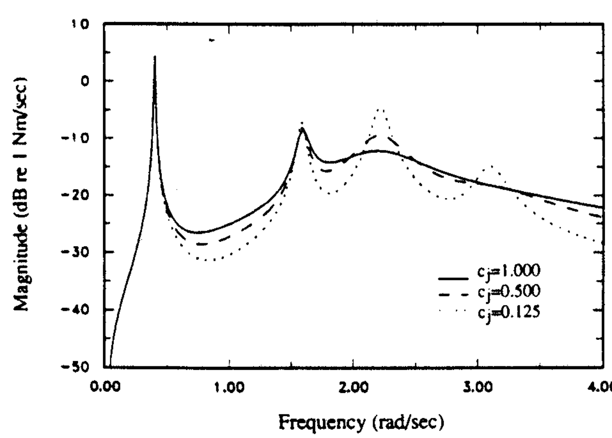

FIG. 10. Effect of joint damping parameter c_j on input power for nonproportionally damped four dof system example.

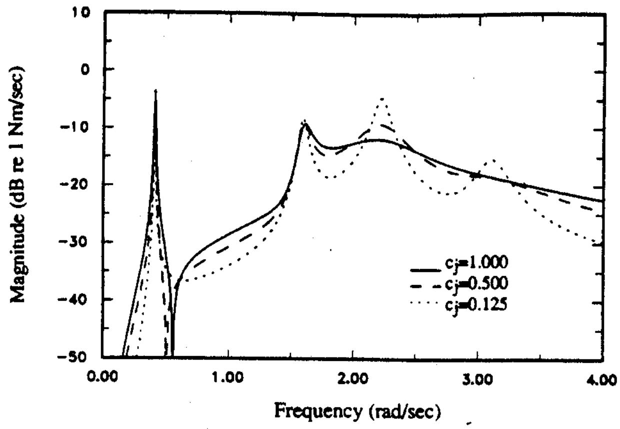

FIG. 11. Effect of joint damping parameter c_j on power transmitted through the top joint for nonproportionally damped four dof system example.

tion about which transmission paths and component modal properties most affect vibration transmission at a particular frequency. Particularly convenient in the formulation is the ease with which interfacial forces and moments may be evaluated. Both forces and motions at joints may be expressed in terms of component modal participation factors, which afford unique insight into vibration transmission phenomena. The use of a component modal basis also provides a convenient approach for the analysis of assemblies with irregularly shaped components that do not fit any orthogonal coordinate system. Although examples of simple lumped parameter systems were considered in this paper, the analysis method is not limited to components with discrete-valued mode functions. Components with continuous mode functions may be included as well, if inner products appropriate for the Hilbert spaces associated with the spatially continuous component motions are used. Such components frequently have large numbers of modes, however, and it is not practical or desirable to include too many of them in analyses of vibration transmission in a particular frequency range. When some modes are excluded, errors may result from the contributions of those modes excluded from the analysis.

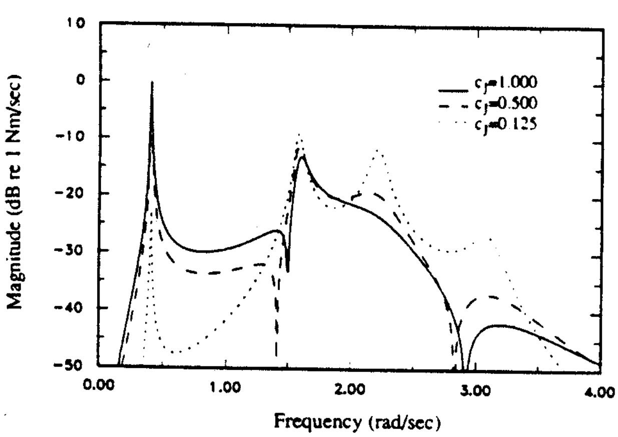

FIG. 12. Effect of joint damping parameter c_j on power transmitted through the bottom joint for nonproportionally damped four dof system example.

This modal truncation issue has been addressed by the authors and will be the topic of a future publication.

Particularly significant in the new formulation is the capability to analyze assemblies with arbitrary viscous damping. The extended modal synthesis formulation for damped component assemblies is itself a new development, and the formulation permits study of the effects of relatively heavy localized damping at joints, which often occurs in machine assemblies, on transmitted vibration. The example cases considered suggest that these effects often may be significant. It should be noted, however, that the additional damping associated with the joints must be included in the subsystem component models. Currently, very little data for appropriate viscous damping coefficients for practical joints exist. Further work to experimentally determine these data would certainly enhance the utility of the new formulation.

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