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A NEW GEAR MESH INTERFACE DYNAMIC MODEL TO PREDICT MULTI-DIMENSIONAL FORCE COUPLING AND EXCITATION

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Abstract—A new gear mesh interface dynamic model is developed which presents a true, three-dimensional representation of the forces and moments generated within and transmitted via a gear mesh interface. Force coupling effects due to vibratory changes in the instantaneous plane-of-action are included directly. The new formulation facilitates an examination of the many simplifying assumptions that have been made either explicitly or implicitly by prior investigators, thus allowing existing models to be compared on a common mathematical basis.

1. INTRODUCTION

The study of gear dynamics is essential to the design of quiet and reliable power transmission products exhibiting high torque-to-weight ratios and acceptable levels of gear noise. Critical to every gear dynamic analysis is the expression used to quantify the vibratory source strength and force coupling associated with the gear mesh interface. Virtually all gear dynamic models consider a scalar expression for the force generated within the gear mesh and consider mostly kinematic transmission error and variation in mesh stiffness as the primary sources of noise and vibration [1, 2]. Such models are not adequate to describe coupled motions excited by certain gear tooth flank errors, system misalignment, mass unbalance and vibratory changes in the instantaneous plane-of-action. The analysis of helical gear systems further requires that axial and rocking (rotation about an axis parallel to the line-of-action) motions be included [3–9]. Typically, these additional degrees-of-freedom (DOF) are not properly included in the corresponding gear mesh interface model which must consider a multi-dimensional vector of forces and moments generated within and transmitted via the gear mesh; such a comprehensive gear mesh interface model does not exist in the literature. Another research issue in gear dynamics involves the numerous assumptions that are commonly employed in the development of governing differential equations associated with any gear train model and the tractability of the resulting problem formulation. Specifically, a mathematical formulation is needed so that any tacit assumptions can be identified and their consequences clearly understood. This paper attempts to clarify and resolve these issues.

In this paper a new six DOF gear mesh interface model that may be employed in dynamic analyses of internal and external spur and helical gears, is formulated by extending the current, well accepted theory [1, 2]. This model is then reduced to linear time-varying (LTV) and linear time-invariant (LTI) forms and compared with selected helical gear pair models available in the literature. Gear bodies are assumed to be rigid except for the elastic compliance of meshing gear

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teeth and any tooth separations are neglected; although the theory can be readily extended to include gear backlash phenomenon.

2. PROBLEM FORMULATION

2.1. Coordinate systems and vector notation

The cylindrical element shown in Fig. 1 represents a typical external gear body. Two Cartesian coordinate systems are shown. The non-rotating *geometric* reference frame (X^i_G, Y^i_G, Z^i_G) is used to describe the position and orientation of gear body i with the intended rotational motion $\theta^i(t)$ occurring about the Z^i_G axis. The mean location and orientation of the geometric frame with respect to the *inertial* frame (X, Y, Z) are described by time-invariant global position vector $\bar{\mathbf{R}}^i_G$ and angular misalignment vector $\bar{\boldsymbol{\Theta}}^i_G$ which respectively represent the translation of the origin O^i_G and the rotational orientation of the (X^i_G, Y^i_G, Z^i_G) frame with respect to the inertial frame. The vibratory translational displacement of gear i from its mean location $\bar{\mathbf{R}}^i_G$ is described by the time-varying displacement vector $\mathbf{R}^i_{Gm}(t)$. Thus, the instantaneous translational position of gear i is given by the vector sum $\mathbf{R}^i_G(t) = \bar{\mathbf{R}}^i_G + \mathbf{R}^i_{Gm}(t)$. Similarly, the vibratory angular displacement of gear i from its ideal angular position defined by $\theta^i(t)$ and $\bar{\boldsymbol{\Theta}}^i_G$ is described by the time-varying angular displacement vector $\boldsymbol{\Theta}^i_{Gm}(t)$. Thus, the instantaneous angular orientation of gear i with respect to the inertial frame is given by the vector sum $\boldsymbol{\Theta}^i_G(t) = \bar{\boldsymbol{\Theta}}^i_G + \boldsymbol{\Theta}^i_{Gm}(t)$. Here $\boldsymbol{\Theta}^i_G(t)$ is a three dimensional rotation vector whose elements are assumed to be small. The displacement vector $\mathbf{q}^i(t) = \{\mathbf{R}^{iT}(t) \quad \boldsymbol{\Theta}^{iT}(t)\}^T$ denotes the generalized coordinates of gear body i. Hence, the complete description of the instantaneous position and orientation of gear i with respect to the inertial frame requires that both $\mathbf{q}^i(t)$ and $\theta^i(t)$ be specified. The *nominal rotation angle* $\theta^i(t)$ is the only large angle considered. Coordinate vector $\mathbf{q}^i(t) = \bar{\mathbf{q}}^i + \mathbf{q}^i_m(t)$ can be written as the vector sum of a time-invariant coordinate vector $\bar{\mathbf{q}}^i = \{\bar{\mathbf{R}}^{iT} \quad \bar{\boldsymbol{\Theta}}^{iT}\}^T$, which describes the mean position of gear i and the orientation of its axes, and a time-varying vibratory coordinate vector $\mathbf{q}^i_m(t) = \{\mathbf{R}^{iT}_m(t) \quad \boldsymbol{\Theta}^{iT}_m(t)\}^T$, which describes the deviation of gear body i from its ideal position defined by $\bar{\mathbf{q}}^i$ and $\theta^i(t)$.

A complete description of the functional gear geometry is assumed to be known relative to the (X^i_G, Y^i_G, Z^i_G) frame. In practice, various forms of gear manufacturing errors, system misalignment and mass unbalance can exist making it necessary to define several additional stationary and rotating reference frames in order to properly describe these conditions in terms of common gear design, manufacturing and inspection parameters [10].

2.2. Dynamic mesh force concept

The translational forces acting on gear i as a result of its meshing with gear j are denoted by mesh force vector $\mathbf{F}^{ij}(t)$ having dimension three. Any moments acting on gear i are denoted by mesh moment vector $\mathbf{T}^{ij}(t)$ also of dimension three. The vectors $\mathbf{F}^{ij}(t)$ and $\mathbf{T}^{ij}(t)$ are combined to form a generalized force vector $\mathbf{Q}^{ij}(t) = \{\mathbf{F}^{ijT}(t) \quad \mathbf{T}^{ijT}(t)\}^T$ of dimension six. The transmitted gear mesh force $\mathbf{Q}^{ij}(t) = \bar{\mathbf{Q}}^{ij} + \mathbf{Q}^{ij}_m(t)$ is decomposed into a static component $\bar{\mathbf{Q}}^{ij}$ due to the mean transmitted load and a vibratory component $\mathbf{Q}^{ij}_m(t)$ which arises due to the meshing action. Further, $\mathbf{Q}^{ij}_m(t) = \mathbf{Q}^{ij}_{me}(t) + \mathbf{Q}^{ij}_{md}(t)$ is written as the vector sum of conservative or elastic forces $\mathbf{Q}^{ij}_{me}(t)$ and dissipative forces $\mathbf{Q}^{ij}_{md}(t)$.

The dissipative force vector $\mathbf{Q}^{ij}_{md}$ describes non-conservative gear mesh forces and moments arising from sliding friction, internal hysteresis, fluid film lubrication effects, viscous damping and the like. These mechanisms are as yet poorly understood, especially in the context of gear dynamics. To this end, an energy-equivalent viscous damping model is assumed to account for all dissipative effects occurring within the gear mesh interface. The accuracy of model predictions will only be affected in resonance regimes since the steady state forced response is dictated mostly by $\mathbf{Q}^{ij}_{me}$ in off resonance regimes.

Vibratory elastic deformations of mating gear teeth give rise to $\mathbf{Q}^{ij}_{me}(t)$. A Hookean expression for $\mathbf{Q}^{ij}_{me}$ is assumed having the form

$$\mathbf{Q}^{ij}_{me}(t) = -\mathbf{K}^{ij}(t)[\boldsymbol{\delta}^{ij}_m(t)]^n \tag{1}$$

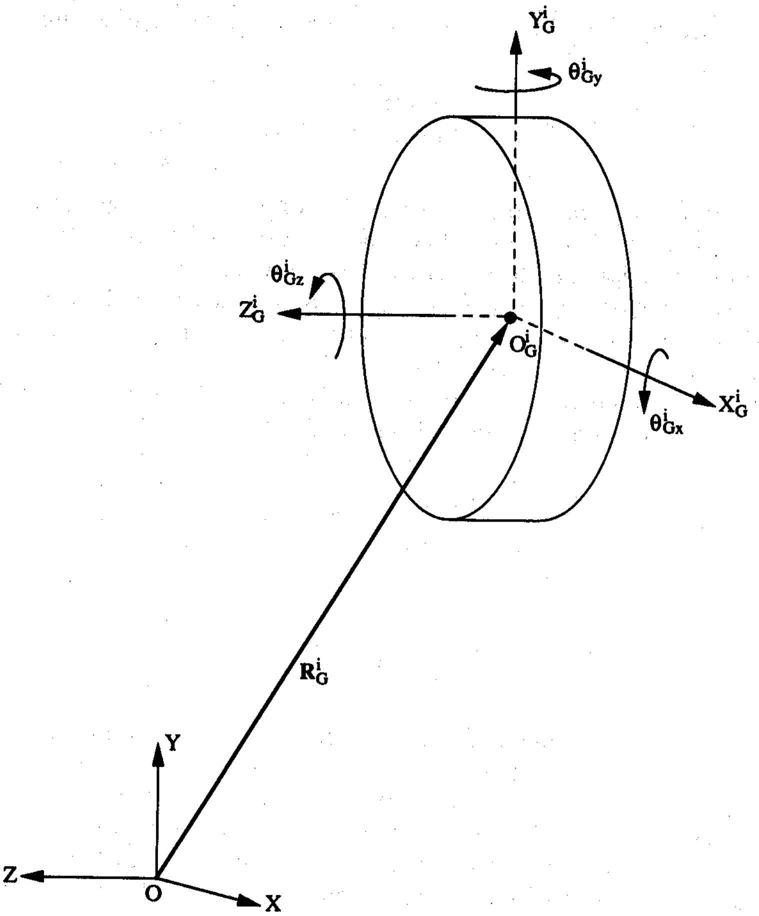

Fig. 1. Definition of gear body reference frames and coordinate vectors.

where $\mathbf{K}^{ij}(t)$ is a mesh stiffness matrix, $\boldsymbol{\delta}_{\mathrm{m}}^{ij}(t)$ is a vector of equivalent displacements and n is an exponent. Only $n = 1$ is considered in this study. The negative sign in equation (1) implies that the elements of $\mathbf{Q}_{\mathrm{me}}^{ij}$ are acting on gear body i in the direction of positive $\boldsymbol{\delta}_{\mathrm{m}}^{ij}$. The vibratory displacement vector $\boldsymbol{\delta}_{\mathrm{m}}^{ij}(t)$ is separated into two distinct components: $\boldsymbol{\delta}_{q}^{ij}(t)$ due to gear body displacements $\mathbf{q}^{i}(t)$ and $\mathbf{q}^{j}(t)$, and $\boldsymbol{\delta}_{\epsilon}^{ij}(t)$ due to deviations of gear tooth profiles from perfect conjugate form over some finite region(s) of contact; such tooth errors or deviations may be described by a generic error function, say $\epsilon^{ij}(t)$. Accordingly,

$$\boldsymbol{\delta}_{\mathrm{m}}^{ij}(t) = \boldsymbol{\delta}_{q}^{ij}(t) - \boldsymbol{\delta}_{\epsilon}^{ij}(t) \tag{2}$$

and the expression for $\mathbf{Q}_{\mathrm{me}}^{ij}$ becomes

$$\mathbf{Q}_{\mathrm{me}}^{ij}(t) = -\mathbf{K}^{ij}(t)\boldsymbol{\delta}_{q}^{ij}(t) + \mathbf{K}^{ij}(t)\boldsymbol{\delta}_{\epsilon}^{ij}(t) \tag{3}$$

The term $-\mathbf{K}^{ij}\boldsymbol{\delta}_{q}^{ij}$ may be viewed as a source of parametric excitation and force coupling in the dynamic equation of motion and $\mathbf{K}^{ij}\boldsymbol{\delta}_{\epsilon}^{ij}$ as an external forcing excitation. In general, $\mathbf{K}^{ij}(t) = \mathbf{K}^{ij}[\mathbf{Q}^{ij}(t), \mathbf{q}^{i}(t), \mathbf{q}^{j}(t), \epsilon^{ij}(t)]$ and $\epsilon^{ij}(t) = \epsilon^{ij}[\mathbf{Q}^{ij}(t), \mathbf{K}^{ij}(t), \mathbf{q}^{i}(t), \mathbf{q}^{j}(t)]$. Hence explicit computation of $\mathbf{K}^{ij}(t)$ and $\epsilon^{ij}(t)$ independent of a complete dynamic analysis is impossible.

2.3. Motion analysis

In an ideal geared system where only pure rotational motion exists, the position of each body i is described by a single rotational coordinate $\theta^{i}(t)$. The relationship between any two bodies in direct contact may be written as $\theta^{i} = \lambda^{ij}\theta^{j}$ where the *motion transmission ratio* λ^{ij} corresponds to a one-dimensional, time-invariant kinematic constraint. For a gear pair $\lambda^{ij} = -a^{j}/a^{i}$ is the ratio of the base radii and corresponds to the kinematic condition commonly referred to as *conjugacy*. Hence, the position of each gear may be defined in terms of a single system rotational parameter, say

$$\theta^*(t) = \int_0^t \Omega^*(\tau)\,d\tau \tag{4}$$

where $\Omega^*(t) = \Omega^i/n^i$ is a system velocity parameter and n^i is an integer given by $n^i = N^i/\mathrm{GCF}\,(N^i, N^j)$. Here GCF (N^i, N^j) is the greatest common factor of gear tooth numbers N^i and N^j and the initial value of θ^* is assumed to be zero without any loss in generality.

Displacement vector $\mathbf{q}^i_{\mathrm{m}}(t) = \mathbf{q}^i_{\mathrm{m}}(t;\theta^*) = \mathbf{q}^i_{\mathrm{mo}}(\theta^*) + \mathbf{q}^i_{\mathrm{mg}}(t)$ may be further expanded in terms of a spatially-varying *nominal* component $\mathbf{q}^i_{\mathrm{mo}}(\theta^*)$ and a time-varying *dynamic* component $\mathbf{q}^i_{\mathrm{mg}}(t)$. The motion described by $\mathbf{q}^i_{\mathrm{mo}}(\theta^*)$ includes system misalignment effects and deflections due to static loading or mean transmitted torque as the system is rotated by angle θ^*. Under a quasi-static condition as Ω^* approaches zero, denoted here by subscript o,

$$\lim_{\Omega^*\to 0} \mathbf{q}^i_{\mathrm{m}}(t) = \mathbf{q}^i_{\mathrm{mo}}(\theta^*_{\mathrm{o}})$$

where

$$\lim_{\Omega^*\to 0} \theta^* = \theta^*_{\mathrm{o}}.$$

Hence, $\mathbf{q}^i_{\mathrm{mo}}(\theta^*)$ may be determined from a quasi-static or low frequency analysis, or even measured experimentally under quasi-static loaded conditions. Under operating conditions when $\Omega^* > 0$, $\mathbf{q}^i_{\mathrm{mo}}(\theta^*)$ and other spatially-varying parameters, such as gear mesh stiffness and kinematic transmission error, give rise to dynamic forces and moments which result in dynamic displacements $\mathbf{q}^i_{\mathrm{mg}}(t)$ about the instantaneous nominal position $\mathbf{q}^i_{\mathrm{mo}}(\theta^*)$. The transformation from the spatial domain to the time domain given non-zero Ω^* is straightforward by using equation (4).

3. HELICAL GEAR PAIR MODEL

3.1. *Formulation*

An external helical involute gear pair denoted by ij is shown in Fig. 2. Each gear i is represented by a base cylinder of radius a^i and width $2b^i$. The origins of two non-rotating geometric reference frames $(X^i_{\mathrm{G}}, Y^i_{\mathrm{G}}, Z^i_{\mathrm{G}})$ and $(X^j_{\mathrm{G}}, Y^j_{\mathrm{G}}, Z^j_{\mathrm{G}})$ are located on the respective gear bodies i and j as shown such that the orientation of either frame remains parallel to the inertial frame (X, Y, Z). Intended rotation is assumed to be about the Z^i_{G} axis. A third mesh reference frame $(\chi^{ij}, \gamma^{ij}, \varphi^{ij})$ is defined having its origin located at the instantaneous pitch point defined by position vector $\mathbf{R}^{ij}_{\mathrm{p}}(t)$ which is given by

$$\mathbf{R}^{ij}_{\mathrm{p}}(t) = \left(1 - \frac{a^i}{a^i + a^j}\right)\mathbf{R}^i_{\mathrm{G}}(t) + \left(\frac{a^i}{a^i + a^j}\right)\mathbf{R}^j_{\mathrm{G}}(t) \tag{5}$$

For the case shown with gear i acting as the driven gear and $\Omega^i \geqslant 0$ or gear i acting as the driving gear and $\Omega^i \leqslant 0$, the directions of the χ^{ij}, γ^{ij} and φ^{ij} axes are defined by unit vectors $\mathbf{u}^{ij}(t)$, $\mathbf{v}^{ij}(t)$ and $\mathbf{w}^{ij}(t)$, respectively which are given by

$$\mathbf{u}^{ij}(t) = \mathbf{A}^{ij}\mathbf{U}^{ij}(\mathbf{R}^j_{\mathrm{G}} - \mathbf{R}^i_{\mathrm{G}}) \tag{6}$$

$$\mathbf{v}^{ij}(t) = \mathbf{A}^{ij}\mathbf{V}^{ij}(\mathbf{R}^j_{\mathrm{G}} - \mathbf{R}^i_{\mathrm{G}}) \tag{7}$$

$$\mathbf{w}^{ij}(t) = \mathbf{A}^{ij}\{0 \quad 0 \quad 1\}^{\mathrm{T}} \tag{8}$$

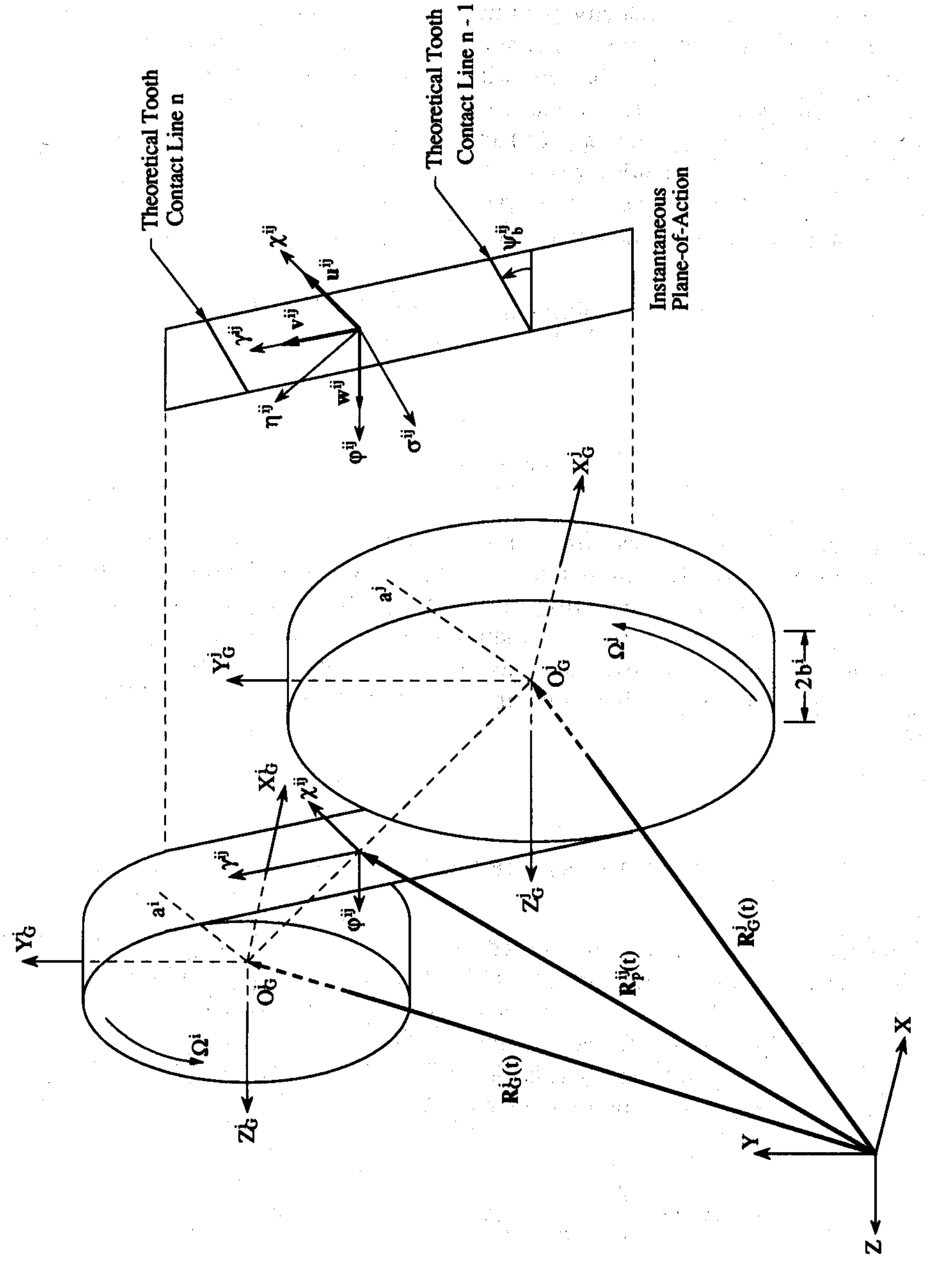

Fig. 2. Gear mesh interface model for a helical gear pair showing the instantaneous plane-of-action.

where

$$\mathbf{U}^{ij}(t) = \frac{a^i + a^j}{\| \mathbf{R}^j_{\mathrm{G}} - \mathbf{R}^i_{\mathrm{G}} \|^2} \begin{bmatrix} \dfrac{1}{\sqrt{\beta^{ij2}-1}} & -\sqrt{\beta^{ij2}-1} & 0 \\ \sqrt{\beta^{ij2}-1} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

$$\mathbf{V}^{ij}(t) = \frac{a^i + a^j}{\| \mathbf{R}^j_{\mathrm{G}} - \mathbf{R}^i_{\mathrm{G}} \|^2} \begin{bmatrix} -\sqrt{\beta^{ij2}-1} & -1 & 0 \\ 1 & -\sqrt{\beta^{ij2}-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$\mathbf{R}^i_{\mathrm{G}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{R}^i_{\mathrm{G}}(t) \tag{11}$$

$$\beta^{ij}(t) = \frac{\| \mathbf{R}^j_{\mathrm{G}} - \mathbf{R}^i_{\mathrm{G}} \|}{a^i + a^j} \geqslant 1 \tag{12}$$

Here ($_\vee$) denotes a two-dimensional vector defined in the instantaneous plane-of-rotation and $\mathbf{A}^{ij}(t)$ is a time-varying mean rotation matrix given approximately by

$$\mathbf{A}^{ij}(t) \cong \begin{bmatrix} 1 & 0 & \theta^{ij}_y(t) \\ 0 & 1 & -\theta^{ij}_x(t) \\ -\theta^{ij}_y(t) & \theta^{ij}_x(t) & 1 \end{bmatrix} \tag{13}$$

Here $\theta^{ij}_x(t) = (a^i\theta^i_{\mathrm{G}x}(t) + a^j\theta^j_{\mathrm{G}x}(t))/(a^i + a^j)$ and $\theta^{ij}_y(t) = (a^i\theta^i_{\mathrm{G}y}(t) + a^j\theta^j_{\mathrm{G}y}(t))/(a^i + a^j)$ are the instantaneous average rotation angles of gear pair ij. Both θ^{ij}_x and θ^{ij}_y are assumed to be small such that $\cos\theta^{ij}_i \cong 1$, $\sin\theta^{ij}_i \cong \theta^{ij}_i$ and $\sin\theta^{ij}_i \sin\theta^{ij}_j \cong 0$. Consequently, only changes in the direction of $(\chi^{ij}, \gamma^{ij}, \varphi^{ij})$ due to translational motions in the plane-of-rotation and rotations about the X^i_{G} and Y^i_{G} axes are considered. Axial and torsional motions along and about the Z^i_{G} axis do not affect the direction of the instantaneous line-of-action. The above formulation can be readily extended to other cases. For example, $\mathbf{U}^{ij}(t)$ and $\mathbf{V}^{ij}(t)$ are replaced by their respective transposes in equations (6) and (7) when gear i acts as the driven gear with $\Omega^i \leqslant 0$ or when gear i acts as the driving gear with $\Omega^i \geqslant 0$. Similar results can be obtained for an internal–external helical gear mesh or an epicyclic gear arrangement.

3.2. Instantaneous load distribution

The instantaneous zone of contact and a hypothetical, yet reasonable, load distribution normal to tooth contact lines is shown in Fig. 3(a) corresponding to an arbitrary position θ^* in the meshing cycle. The actual zone of contact and load distribution are problem specific and other forms are also possible. The $(\chi^{ij}, \eta^{ij}, \sigma^{ij})$ frame is rotated by an angle ψ^{ij}_b about the χ^{ij} axis from the $(\chi^{ij}, \gamma^{ij}, \varphi^{ij})$ frame such that the η^{ij} axis is coincident with the instantaneous tooth normal. The angle ψ^{ij}_b corresponds to the base helix angle and is assumed to be constant. Only the resultant normal force $F^{ij}_\eta(t)$ acting along the η^{ij} axis and moment $T^{ij}_\chi(t)$ acting about the χ^{ij} axis are assumed to exist as shown in Fig. 3(b). Any resultant force F^{ij}_σ acting along the σ^{ij} axis is assumed to be zero. If sliding friction were to be considered, $F^{ij}_\sigma(t)$ would be non-zero. The instantaneous load distribution $P^{ij}(\chi, \sigma, t)$ normal to the contacting tooth surfaces may be computed as follows over a given instantaneous contact region $S^{ij}(t)$

$$\int_{S^{ij}(t)} P^{ij}(\chi, \sigma, t)\,\mathrm{d}\chi\,\mathrm{d}\sigma = F^{ij}_\eta(t) \tag{14a}$$

$$\int_{S^{ij}(t)} P^{ij}(\chi, \sigma, t)\sigma\,\mathrm{d}\chi\,\mathrm{d}\sigma = T^{ij}_\chi(t) \tag{14b}$$

$$\int_{S^{ij}(t)} G^{ij}[\chi, \sigma; x, y, P^{ij}(x, y, t), \epsilon^{ij}(x, y)]P^{ij}(x, y, t)\,\mathrm{d}x\,\mathrm{d}y = \delta^{ij}(\chi, \sigma, t) \tag{14c}$$

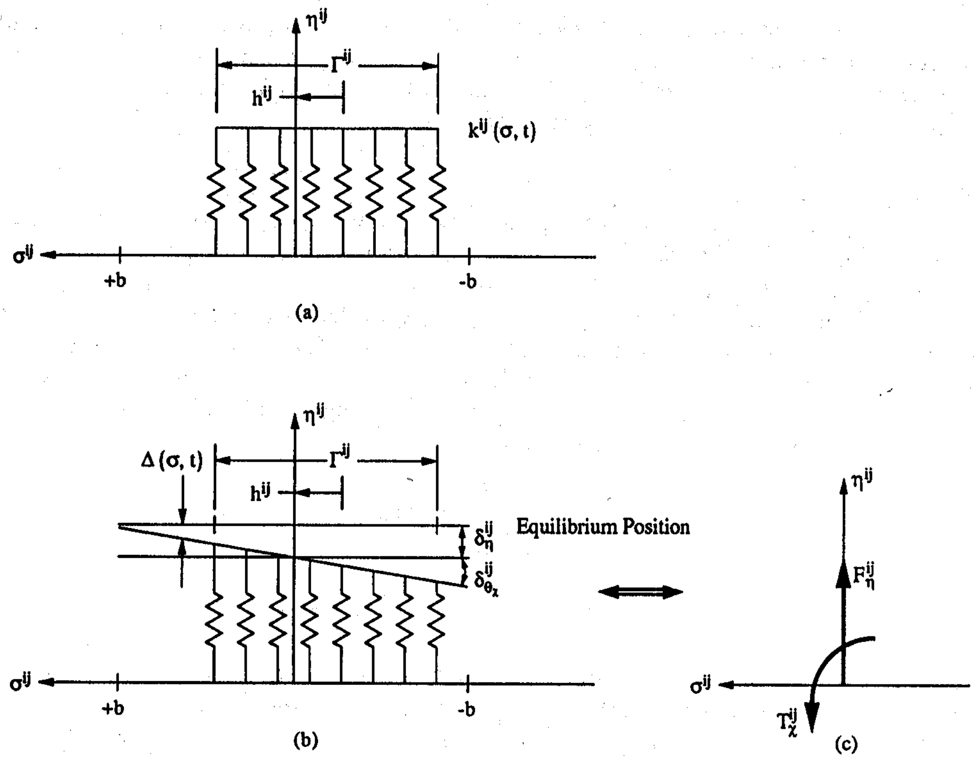

Fig. 3. Load distribution within a helical gear mesh interface: (a) instantaneous load distribution over the zones of contact; (b) resultant normal force associated with each contact zone; (c) resultant normal force and moment.

Here G^{ij} is a two-dimensional dynamic Green's function with units of compliance per area that depends on both P^{ij} and ϵ^{ij}. Displacement function $\delta^{ij}(\chi, \sigma, t)$ represents equivalent displacement errors in the instantaneous contact regime due to motions of gear bodies, tooth errors and elastic deflections. The unknowns are $\delta^{ij}(t)$ and $P^{ij}(t)$. The domain of integration $S^{ij}(t)$ is the instantaneous region(s) of contact which depend upon G^{ij}, $\delta^{ij}(t)$, and $P^{ij}(t)$. The constraint forces $F_\eta^{ij}(t)$ and $T_\chi^{ij}(t)$ cannot be determined independent of a complete dynamic analysis and furthermore, their evaluation is difficult. Typically, this problem has been reduced to a quasi-static analysis in only one dimension in order to obtain approximate solutions for $P^{ij}(t)$ and $\delta^{ij}(t)$. Accordingly, the quasi-static load distribution $P_{\text{no}}^{ij}(\sigma, \theta_\text{o}^*)$ corresponding to the n^{th} tooth pair may be obtained by evaluating integrals of the form

$$\int_{L_{\text{no}}^{ij}(\theta_\text{o}^*)} G_{\text{no}}^{ij}[\sigma; y, P_{\text{no}}^{ij}(y, \theta_\text{o}^*), \epsilon_{\text{no}}^{ij}(y, \theta_\text{o}^*)] P_{\text{no}}^{ij}(y, \theta_\text{o}^*)\,\mathrm{d}y = \delta_{\text{no}}^{ij}(\sigma, \theta_\text{o}^*) \tag{15a}$$

subject to the constraint

$$\sum_{n=1}^{N(\theta_\text{o}^*)} \int_{L_{\text{no}}^{ij}(\theta_\text{o}^*)} P_{\text{no}}^{ij}(\sigma, \theta_\text{o}^*)\,\mathrm{d}\sigma = F_{\eta\text{o}}^{ij} \tag{15b}$$

where G_{no}^{ij} is a static, scalar Green's function with units of compliance per length, $\delta_{\text{no}}^{ij}(\sigma, \theta_\text{o}^*)$ represents displacement errors occurring along the n^{th} contact line and $N(\theta_\text{o}^*)$ corresponds to the number of teeth in contact at meshing position θ_o^*. The actual domain of integration $L_{\text{no}}^{ij}(\theta_\text{o}^*)$ depends upon the quasi-static motions of gear bodies, misalignment, shaft and bearing deflections, tooth errors and tooth deformations. Typically $L_{\text{no}}^{ij}(\theta_\text{o}^*)$ is assumed to be the theoretical line-of-contact determined from a purely kinematic analysis. Systems of equations similar to (15) have been solved by using a variety of techniques ranging in complexity from simplex type algorithms [11–13] to sophisticated finite element codes with specialized gear contact elements [14]. The end results are invariably scalar expressions for equivalent mesh stiffness and the so-called *loaded transmission error* computed normal to the tooth surface and acting along the theoretical line-of-action.

3.3. Functional mesh force vector formulation

Gear contact analyses of the type discussed in the previous section are beyond the scope of this work. Instead, functional forms for mesh stiffness and transmission error are defined which have meaningful physical interpretations and are assumed to be available from an independent quasi-static analysis or measurement. Specifically, an equivalent scalar stiffness function $k^{ij}(\sigma, t)$ having units of stiffness per length is assumed. This stiffness function $k^{ij}(\sigma, t) = k^{ij}(t)$ is further assumed to be uniform along the instantaneous σ^{ij} axis over contact length $\Gamma^{ij}(t)$ and centered about axial position $\sigma^{ij} = h^{ij}(t)$ as shown in Fig. 4(a). Accordingly, the resultant normal force $F_\eta^{ij}(t)$ and moment $T_\chi^{ij}(t)$ shown in Fig. 4(b) are given by

$$F_\eta^{ij}(t) = -\int_{h^{ij}(t)-\Gamma^{ij}(t)/2}^{h^{ij}(t)+\Gamma^{ij}(t)/2} k^{ij}(t)\Delta^{ij}(\sigma, t)\,\mathrm{d}\sigma \tag{16}$$

$$T_\chi^{ij}(t) = \int_{h^{ij}(t)-\Gamma^{ij}(t)/2}^{h^{ij}(t)+\Gamma^{ij}(t)/2} k^{ij}(t)\sigma\Delta^{ij}(\sigma, t)\,\mathrm{d}\sigma \tag{17}$$

where $\Delta^{ij}(\sigma, t) = \delta_\eta^{ij}(t) + \sigma\delta_{\theta_\chi}^{ij}(t)$ is an equivalent displacement function along the σ^{ij} axis as shown in Fig. 4(c). The $\delta_\eta^{ij}(t)$ component represents the instantaneous displacement normal to the σ^{ij} axis and the $\delta_{\theta_\chi}^{ij}(t)$ component represents the instantaneous rotation about the χ^{ij} axis. Here $\delta_\eta^{ij}(t)$ and $\delta_{\theta_\chi}^{ij}(t)$ are stiffness-weighted averages for all contacting teeth at any instant in time. Carrying out the above integrations yields

$$F_\eta^{ij}(t) = -K^{ij}(t)[\delta_\eta^{ij}(t) + h^{ij}(t)\delta_{\theta_\chi}^{ij}(t)] \tag{18}$$

$$T_\chi^{ij}(t) = K^{ij}(t)\{h^{ij}(t)\delta_\eta^{ij}(t) + [h^{ij2}(t) + \tfrac{1}{12}\Gamma^{ij2}(t)]\delta_{\theta_\chi}^{ij}(t)\} \tag{19}$$

where $K^{ij}(t) = k_{(t)}^{ij}\Gamma^{ij}(t)$ is the instantaneous mesh stiffness. The error functions $\delta_\eta^{ij}(t) = \delta_{\eta q}^{ij}(t) - \epsilon_\eta^{ij}(t)$ and $\delta_{\theta_\chi}^{ij}(t) = \delta_{\theta_\chi q}^{ij}(t) - \epsilon_{\theta_\chi}^{ij}(t)$ are further decomposed into components $\delta_{\eta q}^{ij}(t)$ and $\delta_{\theta_\chi q}^{ij}(t)$ due to gear motions described by $\mathbf{q}^i(t)$ and $\mathbf{q}^j(t)$ and components $\epsilon_\eta^{ij}(t)$ and $\epsilon_{\theta_\chi}^{ij}(t)$ due to kinematic errors. The kinematic error function $\epsilon_\eta^{ij}(t)$ may be written in terms of kinematic error functions $\epsilon_\gamma^{ij}(t)$ and $\epsilon_\varphi^{ij}(t)$ defined along the γ^{ij} and φ^{ij} axes respectively which are related by

$$\epsilon_\eta^{ij}(t) = \epsilon_\gamma^{ij}(t)\cos\psi_b^{ij} + \epsilon_\varphi^{ij}(t)\sin\psi_b^{ij} \tag{20}$$

Here $\epsilon_\gamma^{ij}(t)$ is the so-called *manufacturing transmission error* acting along the line-of-action. Writing $\delta_{\eta q}^{ij}(t)$ and $\delta_{\theta_\chi q}^{ij}(t)$ in terms of $\mathbf{q}^i(t)$ and $\mathbf{q}^j(t)$ and transforming $F_\eta^{ij}(t)$ to the (X_G^i, Y_G^i, Z_G^i) frame, the three-dimensional vector of vibratory elastic forces $\mathbf{F}_{Gme}^{ij}(t) = \{F_{Gmex}^{ij}(t)F_{Gmey}^{ij}(t)F_{Gmez}^{ij}(t)\}^T$ acting on gear body i is given by

$$\mathbf{F}_{Gme}^{ij}(t) = -\mathbf{W}^{ij}(t)\mathbf{K}^{ij}(t)\boldsymbol{\delta}_m^{ij}(t) \tag{21a}$$

where $\mathbf{W}^{ij}(t) = [\mathbf{u}^{ij}(t) \quad \mathbf{v}^{ij}(t) \quad \mathbf{w}^{ij}(t)]$ is a time-varying directional transformation matrix (3×3) whose columns are unit vectors $\mathbf{u}^{ij}(t)$, $\mathbf{v}^{ij}(t)$ and $\mathbf{w}^{ij}(t)$. The mesh stiffness matrix $\mathbf{K}^{ij}(t)$ (3×6) is given by

$$\mathbf{K}^{ij}(t) = K^{ij}(t)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & cs & c\kappa(t) & -cs & c^2 \\ 0 & cs & s^2 & s\kappa(t) & -s^2 & cs \end{bmatrix}^{ij} \tag{21b}$$

$$c^{ij} = \cos\psi_b^{ij}; \quad s^{ij} = \sin\psi_b^{ij}; \quad \kappa^{ij}(t) = h^{ij}(t)/b^i \tag{21c–e}$$

where $\kappa^{ij}(t)$ is a dimensionless time-varying parameter which describes translational–rotational

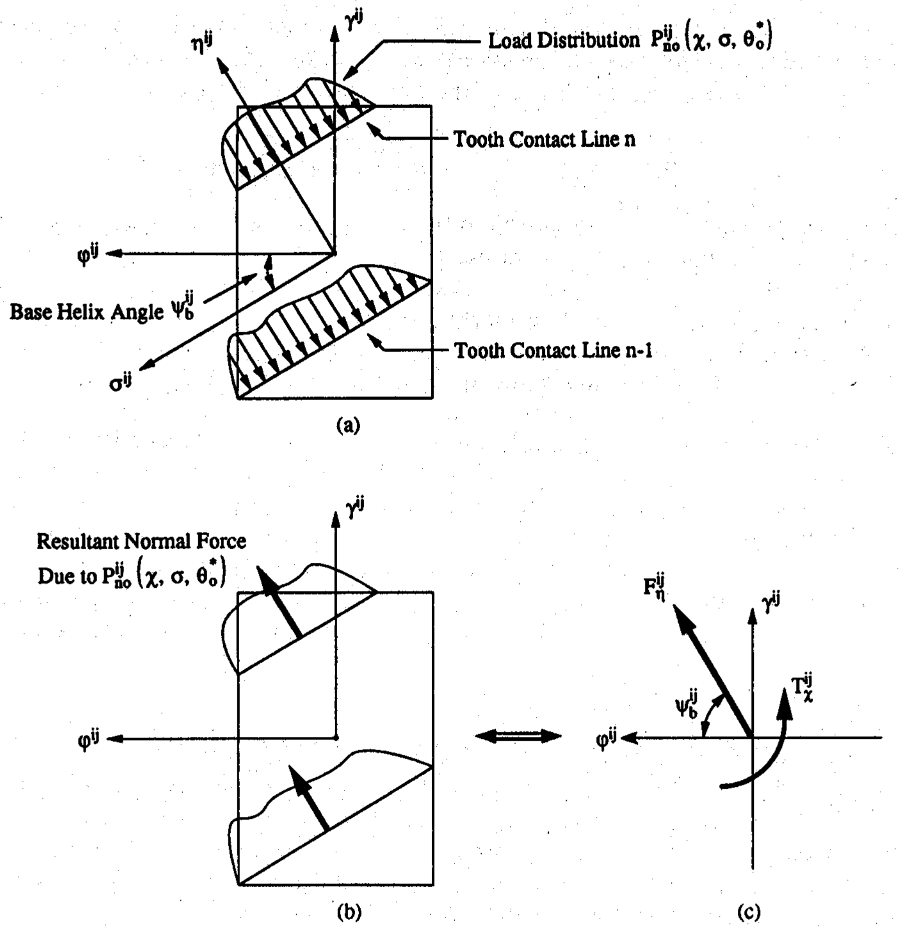

Fig. 4. Functional gear mesh interface model formulation: (a) functional gear mesh stiffness parameters; (b) relative mesh displacement functions; (c) resultant normal force and moment.

force coupling. The components of the six dimensional relative displacement vector $\boldsymbol{\delta}^{ij}_{m}(t) = \boldsymbol{\delta}^{ij}_{q}(t) - \boldsymbol{\delta}^{ij}_{\epsilon}(t)$ are given by

$$\boldsymbol{\delta}^{ij}_{q}(t) = \begin{bmatrix} \mathbf{W}^{ijT} & 0 \\ 0 & \mathbf{W}^{ijT} \end{bmatrix} [\mathbf{D}^i \mathbf{q}^i_m \oplus \mathbf{D}^j \mathbf{q}^j_m] \tag{22}$$

$$\boldsymbol{\delta}^{ij}_{\epsilon}(t) = \{0 \quad \epsilon^{ij}_{\gamma}(t) \quad \epsilon^{ij}_{\varphi}(t) \quad b^i \epsilon^{ij}_{\theta_\chi}(t) \quad 0 \quad 0\}^{\mathrm{T}} \tag{23}$$

where $\oplus = \{- \quad - \quad - \quad - \quad + \quad +\}$ is a vector operator defined in accordance with the sign convention adopted here. The diagonal transformation matrix $\mathbf{D}^i$ (6×6) and moment arm $a^i_{\mathrm{p}}(t)$ are given respectively by

$$\mathbf{D}^i(t) = \mathrm{diag}[1 \quad 1 \quad 1 \quad b^i \quad a^i_{\mathrm{p}}(t) \quad a^i] \tag{24}$$

$$a^i_{\mathrm{p}}(t) = [\mathbf{R}^{ij}_{\mathrm{p}}(t) - \mathbf{R}^i_{\mathrm{G}}(t)] \cdot \mathbf{u}^{ij}(t) \tag{25}$$

Similarly, the scalar vibratory elastic moment $T^{ij}_{\chi\mathrm{me}}(t)$ is given by

$$T^{ij}_{\chi\mathrm{me}}(t) = K^{ij}(t) b^i [0 \quad \kappa(t)c \quad \kappa_{\mathrm{m}}(t)s \quad \mu_{\mathrm{m}}(t) \quad -\kappa(t)s \quad \kappa(t)c]^{ij} \boldsymbol{\delta}^{ij}_{\mathrm{m}}(t) \tag{26a}$$

$$\mu^{ij}(t) = \kappa^{ij2}(t) + \frac{\Gamma^{ij2}(t)}{12 b^{i2}} \tag{26b}$$

where $\mu^{ij}(t)$ is a time-varying dimensionless parameter which describes rotational–rotational force coupling. Accordingly, the three-dimensional vector of vibrational elastic moments $\mathbf{T}^{ij}_{\mathrm{Gme}}(t) = \{T^{ij}_{\mathrm{Gmex}}(t)\ T^{ij}_{\mathrm{Gmey}}(t)\ T^{ij}_{\mathrm{Gmez}}(t)\}^{\mathrm{T}}$ acting on gear body i about axes $(X^i_{\mathrm{G}}, Y^i_{\mathrm{G}}, Z^i_{\mathrm{G}})$ is given by

$$\mathbf{T}^{ij}_{\mathrm{Gme}}(t) = [\mathbf{R}^{ij}_{\mathrm{p}}(t) - \mathbf{R}^{i}_{\mathrm{G}}(t)] \times \mathbf{F}^{ij}_{\mathrm{Gme}}(t) + T^{ij}_{\chi\mathrm{me}}(t)\mathbf{u}^{ij}(t) \tag{27}$$

Following the sign convention adopted here, the vector of vibratory elastic forces and moments acting on gear body j are given respectively by

$$\mathbf{F}^{ji}_{\mathrm{G}}(t) = -\mathbf{F}^{ij}_{\mathrm{G}}(t) \tag{28}$$

$$\mathbf{T}^{ji}_{\mathrm{Gme}}(t) = [\mathbf{R}^{ij}_{\mathrm{p}}(t) - \mathbf{R}^{j}_{\mathrm{G}}(t)] \times \mathbf{F}^{ji}_{\mathrm{Gme}}(t) + T^{ji}_{\chi\mathrm{me}}(t)\mathbf{u}^{ij}(t) \tag{29}$$

The generalized vector of forces and moments associated with reference frame $(X^i_{\mathrm{G}}, Y^i_{\mathrm{G}}, Z^i_{\mathrm{G}})$ acting on gear body i is defined as $\mathbf{Q}^{ij}_{\mathrm{G}}(t) = \{\mathbf{F}^{ij\mathrm{T}}_{\mathrm{G}}(t)\quad \mathbf{T}^{ij\mathrm{T}}_{\mathrm{G}}(t)\}^{\mathrm{T}}$. The mean or static component $\bar{\mathbf{Q}}^{ij}_{\mathrm{G}}$ of $\mathbf{Q}^{ij}_{\mathrm{G}}(t)$ may be combined with $\mathbf{Q}^{ij}_{\mathrm{Gme}}(t)$ to form a single mesh force expression by including static deflections $\bar{\boldsymbol{\delta}}^{ij}$ of the system in the relative displacement function $\boldsymbol{\delta}^{ij}(t) = \bar{\boldsymbol{\delta}}^{ij} + \boldsymbol{\delta}^{ij}_{\mathrm{m}}(t)$. This expression may then be substituted into equations (21a) and (26a) to yield the generalized force vector $\bar{\mathbf{Q}}^{ij}_{\mathrm{G}} + \mathbf{Q}^{ij}_{\mathrm{Gme}}(t)$. On the other hand, $\bar{\mathbf{Q}}^{ij}_{\mathrm{G}}$ and $\mathbf{Q}^{ij}_{\mathrm{Gme}}(t)$ may be considered separately as functions of $\bar{\boldsymbol{\delta}}^{ij}$ and $\boldsymbol{\delta}^{ij}_{\mathrm{m}}(t)$, i.e. $\bar{\mathbf{Q}}^{ij}_{\mathrm{G}} = \bar{\mathbf{Q}}^{ij\mathrm{T}}_{\mathrm{G}}(\bar{\boldsymbol{\delta}}^{ij})$ and $\mathbf{Q}^{ij}_{\mathrm{Gme}}(t) = \mathbf{Q}^{ij\mathrm{T}}_{\mathrm{Gme}}[\boldsymbol{\delta}^{ij}_{\mathrm{m}}(t)]$. The proper choice depends on the specific analysis.

4. REDUCED LTV GEAR MESH INTERFACE MODEL

The mesh force and moment equations presented in the previous section are non-linear and furthermore, evaluation of instantaneous meshing parameters $k^{ij}(t)$, $h^{ij}(t)$ and $\Gamma^{ij}(t)$ in the context of a complete dynamic analysis is virtually impossible at present. Therefore the problem is reduced to a tractable form by employing a few analytical simplifications while still retaining as many unique features of the proposed mesh force expression as possible.

4.1. *Quasi-static excitation analysis technique*

Gear mesh force vector $\mathbf{Q}^{ij}(t) = \bar{\mathbf{Q}}^{ij} + \mathbf{Q}^{ij}_{\mathrm{mo}}(\theta^*) + \mathbf{Q}^{ij}_{\mathrm{mg}}(t)$, mesh stiffness matrix $\mathbf{K}^{ij}(t) = \bar{\mathbf{K}}^{ij} + \mathbf{K}^{ij}_{\mathrm{mo}}(\theta^*) + \mathbf{K}^{ij}_{\mathrm{mg}}(t)$, kinematic error function $\epsilon^{ij}(t) = \epsilon^{ij}_{\mathrm{mo}}(\theta^*) + \epsilon^{ij}_{\mathrm{mg}}(t)$ and equivalent displacement vector $\boldsymbol{\delta}^{ij}(t) = \bar{\boldsymbol{\delta}}^{ij} + \boldsymbol{\delta}^{ij}_{\mathrm{mo}}(\theta^*) + \boldsymbol{\delta}^{ij}_{\mathrm{mg}}(t)$ may be expressed as the sum of mean, nominal and dynamic components. A fundamental assumption implicit to many gear dynamic analyses is to neglect the dynamic stiffness and error terms $\mathbf{K}^{ij}_{\mathrm{mg}}(t)$ and $\epsilon^{ij}_{\mathrm{mg}}(t)$, respectively. Hence $\boldsymbol{\delta}^{ij}_{\mathrm{mo}}(\theta^*) = \boldsymbol{\delta}^{ij}_{q\mathrm{o}}(\theta^*) + \boldsymbol{\delta}^{ij}_{\epsilon\mathrm{o}}(\theta^*)$ and $\boldsymbol{\delta}^{ij}_{\mathrm{mg}}(t) = \boldsymbol{\delta}^{ij}_{q\mathrm{g}}(t)$ since $\boldsymbol{\delta}^{ij}_{\epsilon\mathrm{g}}(t) = 0$ by the assumption that $\epsilon^{ij}_{\mathrm{mg}}(t) = 0$. The vibratory elastic gear mesh force vector $\mathbf{Q}^{ij}_{\mathrm{me}}(t) \cong \mathbf{K}^{ij}_{\mathrm{o}}(\theta^*)[\boldsymbol{\delta}^{ij}_{\mathrm{mo}}(\theta^*) + \boldsymbol{\delta}^{ij}_{q\mathrm{g}}(t)]$ is then computed *approximately* from a dynamic system analysis in terms of spatially-varying parameters $\mathbf{K}^{ij}_{\mathrm{o}}(\theta^*) = \bar{\mathbf{K}}^{ij} + \mathbf{K}^{ij}_{\mathrm{mo}}(\theta^*)$ and $\boldsymbol{\delta}^{ij}_{\mathrm{mo}}(\theta^*)$, which are assumed to be known prior to the dynamic analysis, and the time-varying relative displacement vector $\boldsymbol{\delta}^{ij}_{\mathrm{qg}}(t)$, which is usually the unknown.

To this end, approximate expressions for the functional mesh stiffness parameters $k^{ij}(t) \cong k^{ij}_{\mathrm{o}}(\theta^*)$, $h^{ij}(t) \cong h^{ij}_{\mathrm{o}}(\theta^*)$ and $\Gamma^{ij}(t) \cong \Gamma^{ij}_{\mathrm{o}}(\theta^*)$ or equivalently $K^{ij}(t) \cong K^{ij}_{\mathrm{o}}(\theta^*)$, $\kappa^{ij}(t) \cong \kappa^{ij}_{\mathrm{o}}(\theta^*)$ and $\mu^{ij}(t) \cong \mu^{ij}_{\mathrm{o}}(\theta^*)$ are assumed. Similarly, approximate expressions for the transmission error excitation functions $\epsilon^{ij}_{\eta}(t) \cong \epsilon^{ij}_{\eta\mathrm{o}}(\theta^*)$ and $\epsilon^{ij}_{\theta_\chi}(t) \cong \epsilon^{ij}_{\theta_\chi\mathrm{o}}(\theta^*)$ or equivalently $\epsilon^{ij}_{\gamma}(t) \cong \epsilon^{ij}_{\gamma\mathrm{o}}(\theta^*)$, $\epsilon^{ij}_{\varphi}(t) \cong \epsilon^{ij}_{\varphi\mathrm{o}}(\theta^*)$ and $\epsilon^{ij}_{\theta_\chi}(t) \cong \epsilon^{ij}_{\theta_\chi\mathrm{o}}(\theta^*)$ are assumed. Further, $\mathbf{q}^{i}_{\mathrm{o}}(\theta^*_{\mathrm{o}})$ and $\mathbf{q}^{j}_{\mathrm{o}}(\theta^*_{\mathrm{o}})$ are also assumed to be known from a separate analysis.

4.2. *Analytical simplifications*

By expanding the terms of equation (21) as the products of nominal and dynamic components, the expression for $\mathbf{F}^{ij}_{\mathrm{G}}(t)$ may be written as follows

$$\mathbf{F}^{ij}_{\mathrm{G}}(t) = -[\mathbf{W}^{ij}_{\mathrm{o}} + \mathbf{W}^{ij}_{\mathrm{mg}}][\mathbf{K}^{ij}_{\mathrm{o}} + \mathbf{K}^{ij}_{\mathrm{mg}}]\{\boldsymbol{\delta}^{ij}_{\mathrm{o}} + \boldsymbol{\delta}^{ij}_{\mathrm{mg}}\} \tag{30a}$$

Neglecting products of dynamic components, which are assumed to be small, the nominal and dynamic components of $\mathbf{F}_{G}^{ij}(t)=\mathbf{F}_{Go}^{ij}(\theta^*)+\mathbf{F}_{Gmg}^{ij}(t)$ may be approximated as

$$\mathbf{F}_{Go}^{ij}(\theta^*)=-\mathbf{W}_{o}^{ij}\mathbf{K}_{o}^{ij}\boldsymbol{\delta}_{o}^{ij} \tag{30b}$$

$$\mathbf{F}_{Gmg}^{ij}(t)\cong-\mathbf{W}_{o}^{ij}\mathbf{K}_{o}^{ij}\boldsymbol{\delta}_{mg}^{ij}-\mathbf{W}_{mg}^{ij}\mathbf{K}_{o}^{ij}\boldsymbol{\delta}_{o}^{ij} \tag{30c}$$

Here the dynamic mesh stiffness matrix $\mathbf{K}_{mg}^{ij}(t)$ has been neglected. The nominal force vector $\mathbf{F}_{Go}^{ij}$ of equation (30b) represents the mean and vibratory gear mesh forces arising directly from the quasi-static loaded transmission error vector $\boldsymbol{\delta}_{o}^{ij}(\bar{\mathbf{Q}}^{ij},\theta_{o}^{*})$. The first term in equation (30c) represents the dynamic mesh force vector acting in the nominal plane-of-action due to dynamic gear displacements $\mathbf{q}_{mg}^{i}$ and $\mathbf{q}_{mg}^{j}$. The second term in equation (30c) describes out-of-plane forces due to dynamic changes in the instantaneous plane-of-action and represents a more generalized formulation of the "whirling-feedback" term originally proposed by Daws [15].

The displacement excitation vector $\boldsymbol{\delta}^{ij}(t)=\bar{\boldsymbol{\delta}}^{ij}+\boldsymbol{\delta}_{q}^{ij}(t)+\boldsymbol{\delta}_{c}^{ij}(t)$ may be approximated as $\boldsymbol{\delta}^{ij}(t)\cong\boldsymbol{\delta}_{o}^{ij}(\theta^*)+\boldsymbol{\delta}_{qg}^{ij}(t)$ by neglecting the dynamic error term $\boldsymbol{\delta}_{cg}^{ij}(t)$. Consequently, the spatially-varying component $\boldsymbol{\delta}_{mo}^{ij}(\theta^*)$ of the transmission error vector $\boldsymbol{\delta}_{o}^{ij}(\theta^*)$ is given by

$$\boldsymbol{\delta}_{mo}^{ij}(\theta^*)=\boldsymbol{\delta}_{qo}^{ij}(\theta^*)+\boldsymbol{\delta}_{co}^{ij}(\theta^*) \tag{31a}$$

$$\boldsymbol{\delta}_{qo}^{ij}(\theta^*)=\begin{bmatrix}\mathbf{W}_{o}^{ijT} & 0\\ 0 & \mathbf{W}_{o}^{ijT}\end{bmatrix}[\mathbf{D}_{o}^{i}\mathbf{q}_{mo}^{i}\oplus\mathbf{D}_{o}^{j}\mathbf{q}_{mo}^{j}] \tag{31b}$$

$$\boldsymbol{\delta}_{co}^{ij}(\theta^*)=\{0\quad \epsilon_{\gamma o}^{ij}\quad \epsilon_{\varphi o}^{ij}\quad b^{i}\epsilon_{\theta_{\chi}o}^{ij}\quad 0\quad 0\}^{T} \tag{31c}$$

or equivalently

$$\boldsymbol{\delta}_{co}^{ij}(\theta^*)=\{0\quad \epsilon_{\eta o}^{ij}\quad 0\quad b^{i}\epsilon_{\theta_{\chi}o}^{ij}\quad 0\quad 0\}^{T} \tag{31d}$$

Equations (31c) and (31d) are equivalent expressions for $\boldsymbol{\delta}_{co}^{ij}$ in accordance with the constraint equation (20). The dynamic transmission error vector $\boldsymbol{\delta}_{mg}^{ij}(t)=\boldsymbol{\delta}_{qg}^{ij}(t)$ is given approximately by

$$\boldsymbol{\delta}_{mg}^{ij}(t)\cong\begin{bmatrix}\mathbf{W}_{o}^{ijT} & 0\\ 0 & \mathbf{W}_{o}^{ijT}\end{bmatrix}[\mathbf{D}_{o}^{i}\mathbf{q}_{mg}^{i}\oplus\mathbf{D}_{o}^{j}\mathbf{q}_{mg}^{j}]+\begin{bmatrix}\mathbf{W}_{mg}^{ijT} & 0\\ 0 & \mathbf{W}_{mg}^{ijT}\end{bmatrix}\boldsymbol{\delta}_{o}^{ij}+\begin{bmatrix}\mathbf{W}_{o}^{ijT} & 0\\ 0 & \mathbf{W}_{o}^{ijT}\end{bmatrix}[\mathbf{D}_{mg}^{i}\mathbf{q}_{mo}^{i}\oplus\mathbf{D}_{mg}^{j}\mathbf{q}_{mo}^{j}] \tag{32}$$

Products of dynamic terms have been neglected in equation (32). The second term in (32) is neglected since both $\|\mathbf{W}_{mg}^{ij}\|$ and $\|\boldsymbol{\delta}_{o}^{ij}\|$ are relatively small and the third term is neglected since the $\|\mathbf{D}_{mg}^{i}\|$ are normally insignificant; these assumptions are examined further in Section 4.3. Hence the expression for $\delta_{mg}^{ij}(t)$ is reduced to

$$\boldsymbol{\delta}_{mg}^{ij}(t)\cong\begin{bmatrix}\mathbf{W}_{o}^{ijT} & 0\\ 0 & \mathbf{W}_{o}^{ijT}\end{bmatrix}[\mathbf{D}_{o}^{i}\mathbf{q}_{mg}^{i}\oplus\mathbf{D}_{o}^{j}\mathbf{q}_{mg}^{j}] \tag{33}$$

The expression for moment $T_{\chi}^{ij}(t)=T_{\chi o}^{ij}(\theta^*)+T_{\chi mg}^{ij}(t)$ is similarly reduced by employing assumptions from the development above. The expressions for the nominal $T_{\chi o}^{ij}(\theta^*)$ and dynamic $T_{\chi mg}^{ij}(t)$ components are given below

$$T_{\chi o}^{ij}(\theta^*)=K_{o}^{ij}b^{i}[0\quad \kappa_{o}c\quad \kappa_{o}s\quad \mu_{o}\quad -\kappa_{o}s\quad \kappa_{o}c]^{ij}\boldsymbol{\delta}_{o}^{ij} \tag{34a}$$

$$T_{\chi mg}^{ij}(t)=K_{o}^{ij}b^{i}[0\quad \kappa_{o}c\quad \kappa_{o}s\quad \mu_{o}\quad -\kappa_{o}s\quad \kappa_{o}c]^{ij}\boldsymbol{\delta}_{mg}^{ij} \tag{34b}$$

Accordingly, the moment vector $\mathbf{T}_{G}^{ij}(t)=\mathbf{T}_{Go}^{ij}(\theta^*)+\mathbf{T}_{Gmg}^{ij}(t)$ resulting from $\mathbf{F}_{G}^{ij}$ and T_{χ}^{ij} is given approximately by

$$\mathbf{T}_{G}^{ij}(t)\cong[\mathbf{R}_{po}^{ij}-\mathbf{R}_{Go}^{i}]\times\mathbf{F}_{G}^{ij}+[\mathbf{R}_{pg}^{ij}-\mathbf{R}_{Gg}^{i}]\times\mathbf{F}_{Go}^{ij}+T_{\chi}^{ij}\mathbf{u}_{o}^{ij}+T_{\chi o}^{ij}\mathbf{u}_{mg}^{ij} \tag{35}$$

Nominal and dynamic components are combined in the above moment expression. The significance of the terms $T^{ij}_{\chi o}$, $\mathbf{u}^{ij}_{mg}$ and $[\mathbf{R}^{ij}_{pg} - \mathbf{R}^{i}_{Gg}] \times \mathbf{F}^{ij}_{Go}$ depend upon $\|T^{ij}_{\chi o}\|$, $\|\mathbf{F}^{ij}_{Go}\|$, $\|\mathbf{q}^{i}_{mg}\|$ and $\|\mathbf{q}^{j}_{mg}\|$. Typically all of these terms are negligible in precision geared systems. Accordingly, the following simplified expression for $\mathbf{T}^{ij}_{G}(t)$ may be used in practice

$$\mathbf{T}^{ij}_{G}(t) \cong [\mathbf{R}^{ij}_{po} - \mathbf{R}^{i}_{Go}] \times \mathbf{F}^{ij}_{G} + T^{ij}_{\chi}\mathbf{u}^{ij}_{o} \tag{36}$$

Any dynamic component of $\mathbf{T}^{ij}_{G}(t)$ due to dynamic changes in the instantaneous plane-of-action are considered only through the corresponding dynamic changes in $\mathbf{F}^{ij}_{G}(t)$ given by the term $-\mathbf{W}^{ij}_{mg}\mathbf{K}^{ij}_{o}\boldsymbol{\delta}^{ij}_{o}$ in equation (30c). Similarly, the force and moment vectors acting on gear body j are given by

$$\mathbf{F}^{ji}_{G}(t) = -\mathbf{F}^{ij}_{G}(t) \tag{37}$$

$$\mathbf{T}^{ji}_{G}(t) \cong [\mathbf{R}^{ij}_{po} - \mathbf{R}^{j}_{Go}] \times \mathbf{F}^{ji}_{G} + T^{ji}_{\chi}\mathbf{u}^{ij}_{o} \tag{38}$$

respectively, in accordance with the sign convention adopted here.

4.3. Significance of out-of-plane mesh force terms

The significance of the out-of-plane mesh force term $-\mathbf{W}^{ij}_{mg}\mathbf{K}^{ij}_{o}\boldsymbol{\delta}^{ij}_{o}$ depends largely upon $\|\mathbf{F}^{ij}_{Go}\|$, $\|\mathbf{q}^{i}_{mg}\|$ and $\|\mathbf{q}^{j}_{mg}\|$. The force transformation matrix $\mathbf{W}^{ij}(t) = \mathbf{W}^{ij}_{o}[\mathbf{q}^{i}_{o}(\theta^*), \mathbf{q}^{j}_{o}(\theta^*)] + \mathbf{W}^{ij}_{mg}[\theta^*; \mathbf{q}^{i}_{mg}(t), \mathbf{q}^{j}_{mg}(t)]$ is a nonlinear function of $\mathbf{q}^{i}(t)$ and $\mathbf{q}^{j}(t)$; but it can be linearized about the nominal operating condition $\mathbf{W}^{ij}_{o}[\mathbf{q}^{i}_{o}(\theta^*), \mathbf{q}^{j}_{o}(\theta^*)] = \mathbf{W}^{ij}_{o}(\theta^*)$. An approximate transformation matrix $\hat{\mathbf{W}}^{ij}_{mg}[\theta^*; \mathbf{q}^{i}_{mg}(t), \mathbf{q}^{j}_{mg}(t)]$ is defined such that it is a linear function of $\mathbf{q}^{i}_{mg}$ and $\mathbf{q}^{j}_{mg}$. By substituting $\hat{\mathbf{W}}^{ij}_{mg}[\theta^*; \mathbf{q}^{i}_{mg}(t), \mathbf{q}^{j}_{mg}(t)]$ for $\mathbf{W}^{ij}_{mg}(t)$ in equation (32) the resulting gear mesh force and moment equations will be LTV in nature. The derivation of $\hat{\mathbf{W}}^{ij}_{mg}[\theta^*; \mathbf{q}^{i}_{mg}(t), \mathbf{q}^{j}_{mg}(t)]$ for the general case is tedious. However, an examination of the coordinate transformation equations (6–13) and mesh force equations (21–25) reveals that variations in $\mathbf{v}^{ij}(t)$ due to translational displacements in the direction of $\mathbf{u}^{ij}(t)$ will have the most significant effect on the gear mesh force in the plane-of-rotation. Assuming for the moment that rotation angles $\theta^{ij}_{x} = \theta^{ij}_{y} = 0$, i.e. $\mathbf{A}^{ij} = \mathbf{I}_3$ the exact equations for $\mathbf{W}^{ij}$ may be formulated in terms of two physical parameters: mean operating pressure angle $\bar{\phi}$ and a dimensionless vector parameter $\Delta\mathbf{r}^{ij}/\|\bar{\mathbf{R}}^{j}_{G} - \bar{\mathbf{R}}^{i}_{G}\|$. Here $\|\bar{\mathbf{R}}^{j}_{G} - \bar{\mathbf{R}}^{i}_{G}\|$ is the mean gear center distance in the plane-of-rotation and the components of vector $\Delta\mathbf{r}^{ij} = \{\Delta r^{ij}_{u} \quad \Delta r^{ij}_{v} \quad \Delta r^{ij}_{w}\}^{T}$ represent the relative displacements of gear pair ij in the direction of $\bar{\mathbf{u}}^{ij}$, $\bar{\mathbf{v}}^{ij}$ and $\bar{\mathbf{w}}^{ij}$, respectively. Both Δr^{ij}_{v} and Δr^{ij}_{w} have no effect on $\mathbf{W}^{ij}$. The variation of $\|\Delta\mathbf{F}^{ij}_{G}\|/\|\bar{\mathbf{F}}^{ij}_{G}\|$ with $|\Delta r^{ij}_{u}|/\|\bar{\mathbf{R}}^{j}_{G} - \bar{\mathbf{R}}^{i}_{G}\|$ is nearly linear over the range $|\Delta r^{ij}_{u}|/\|\bar{\mathbf{R}}^{j}_{G} - \bar{\mathbf{R}}^{i}_{G}\| = 0$ to 0.010 for even large values of $\bar{\phi} \leqslant 45°$. Typically, values of $|\Delta r^{ij}_{u}|$ due to misalignments, static deflections and/or dynamic displacements will normally be less than 0.002 and $|\Delta\bar{\theta}^{ij}_{i}| \leqslant 0.1°$ for precision parallel axis gear systems, regardless of size. However, $\|\Delta\mathbf{F}^{ij}_{G}\|/\|\bar{\mathbf{F}}^{ij}_{G}\|$ is less than two percent for even relatively large values of $|\Delta r^{ij}_{u}| = 0.010$ and $|\Delta\bar{\theta}^{ij}_{i}| \leqslant 0.5°$ [9]. Therefore the term $-\mathbf{W}^{ij}_{mg}\mathbf{K}^{ij}_{o}\boldsymbol{\delta}^{ij}_{o}$ in equation (30c) may be neglected under most conditions provided that $|\Delta r^{ij}_{u}|$ and $|\Delta\bar{\theta}^{ij}_{i}|$ do not exceed the conservative limits given above. Further, the inclusion of the $-\mathbf{W}^{ij}_{mg}\mathbf{K}^{ij}_{o}\boldsymbol{\delta}^{ij}_{o}$ term is inappropriate in analyses which neglect sliding friction since gear mesh frictional forces are of the same order.

4.4. Analytical simplification of directional transformation matrix

In lieu of the above discussion, it is appropriate to replace $\mathbf{W}^{ij}(t)$ with its spatially averaged mean value $\bar{\mathbf{W}}^{ij}$ in analyses which consider stationary gear drive arrangements. This is equivalent to assuming that the $(\chi^{ij}, \gamma^{ij}, \varphi^{ij})$ frame is time-invariant. However, for epicyclic gear drives, $\mathbf{W}^{ij}_{o}(\theta^*)$ will exhibit large variations as a function of θ^*. Thus the notation $\mathbf{W}^{ij}_{o}(\theta^*)$ is retained in the present formulation for generality. In such cases, $\mathbf{W}^{ij}_{o}(\theta^*)$ is typically computed from a purely kinematic analysis. In a similar fashion, the diagonal transformation matrix $\mathbf{D}^{i}(t)$, which contain a single time-varying element $a^i_p(t)$, may be replaced by its mean value $\bar{\mathbf{D}}^i$ in most cases without any appreciable loss in accuracy. However, the spatially-varying moment arm $\mathbf{R}^{ij}_{po}$–$\mathbf{R}^i_{Go}$ should not be replaced by its mean value when misalignment effects are significant; these terms give rise to complex amplitude and angle modulation effects [16].

5. DISCUSSION

5.1. *Energy-equivalent viscous damping*

An energy-equivalent viscous damping force $\mathbf{F}^{ij}_{Gd}$ is assumed having the following form

$$\mathbf{F}^{ij}_{Gd}(t) \approx -\mathbf{W}^{ij}_o \mathbf{C}^{ij}_o \dot{\boldsymbol{\delta}}^{ij}_m \tag{39}$$

Here $(\dot{\ })$ denotes differentiation with respect to time and $\mathbf{C}^{ij}_o$ is a viscous damping coefficient matrix which is assumed to be available from a separate quasi-static analysis. Any moment T^{ij}_χ due to dissipative phenomena is assumed to be negligible. Accordingly, the dissipative moment vector $\mathbf{T}^{ij}_{Gd}(t)$ is solely resultant from damping force $\mathbf{F}^{ij}_{Gd}$ and is given approximately by

$$\mathbf{T}^{ij}_{Gd}(t) \cong [\mathbf{R}^{ij}_{po} - \mathbf{R}^i_{Go}] \times \mathbf{F}^{ij}_{Gd} \tag{40}$$

The dissipative force and moment vectors acting on gear body j are found in a similar fashion by following the sign convention of equations (37) and (38).

5.2. *Limitations imposed by quasi-static excitation assumptions*

The applicability of any gear dynamic model which employs functional mesh parameters determined from an independent quasi-static analysis is limited by the implicit assumptions that dynamic gear displacements and mesh force have negligible influence on the instantaneous mesh stiffness and transmission error, i.e. $\mathbf{K}^{ij}_{mg}(t) = 0$ and $\delta^{ij}_{cg}(t) = 0$. These fundamental assumptions, which are tacit to nearly every gear dynamic analysis in the literature, are violated when the gear pair is operated at or near a mesh resonance where the magnitude of dynamic displacements, mesh forces and moments may be significantly larger than their corresponding static values or when tooth contact is lost due to gear backlash phenomenon.

5.3. *Comparison with the literature*

The proposed elastic gear mesh interface model is compared with selected LTV models available in the literature [5–9] by reducing it to various simplified forms. First, the $(\chi^{ij}, \gamma^{ij}, \varphi^{ij})$ frame is assumed to be time-invariant, i.e. $\mathbf{W}^{ij}(t) = \bar{\mathbf{W}}^{ij}$, and further $\mathbf{R}^{ij}_p(t) - \mathbf{R}^i_G(t) = \bar{\mathbf{R}}^{ij}_p - \bar{\mathbf{R}}^i_G$. Consequently, $\mathbf{D}^i(t) = \bar{\mathbf{D}}^i$. Additionally, if the (X^i_G, Y^i_G, Z^i_G) and (X^j_G, Y^j_G, Z^j_G) frames are chosen such that both are parallel with the $(\chi^{ij}, \gamma^{ij}, \varphi^{ij})$ frame, $\bar{\mathbf{W}}^{ij} = \mathbf{I}_3$ and the six DOF generalized force vector $\mathbf{Q}^{ij}_G(t)$ acting on gear body i is given by

$$\mathbf{Q}^{ij}(t) = -\mathbf{K}^{ij}_{Mo}\boldsymbol{\delta}^{ij}(t) \tag{41a}$$

$$\mathbf{K}^{ij}_{Mo}(\bar{\mathbf{Q}}^{ij}, \theta^*) = K^{ij}_o \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & cs & \kappa^{ij}_o c & -cs & c^2 \\ 0 & cs & s^2 & \kappa^{ij}_o s & -s^2 & cs \\ 0 & \kappa^{ij}_o b^i c & \kappa^{ij}_o b^i s & \mu^{ij}_o b^i & -\kappa^{ij}_o b^i s & \kappa^{ij}_o b^i c \\ 0 & -a^i cs & -a^i s^2 & -\kappa^{ij}_o a^i s & a^i s^2 & -a^i cs \\ 0 & a^i c^2 & a^i cs & \kappa^{ij}_o a^i c & -a^i cs & a^i c^2 \end{bmatrix} \tag{41b}$$

Here $\mathbf{K}^{ij}_{Mo}$ is the approximate mesh stiffness matrix (6 × 6) which is notably different from the (3 × 6) mesh stiffness matrix of equation (21b). The nominal and dynamic components of the six DOF transmission error vector $\boldsymbol{\delta}^{ij}(t) = \boldsymbol{\delta}_{o}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*}) + \boldsymbol{\delta}_{mg}^{ij}(t)$ are given respectively by

$$\boldsymbol{\delta}_{o}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*}) = \bar{\boldsymbol{\delta}}^{ij} + \boldsymbol{\delta}_{qo}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*}) - \boldsymbol{\delta}_{\epsilon o}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*}) = \begin{Bmatrix} x_{o}^{i} - x_{o}^{j} \\ y_{o}^{i} - y_{o}^{j} \\ z_{o}^{i} - z_{o}^{j} \\ b^{i}\theta_{xo}^{i} - b^{j}\theta_{xo}^{j} \\ a^{i}\theta_{yo}^{i} + a^{j}\theta_{yo}^{j} \\ a^{i}\theta_{zo}^{i} + a^{j}\theta_{zo}^{j} \end{Bmatrix} - \begin{Bmatrix} 0 \\ \epsilon_{yo}^{ij} \\ \epsilon_{zo}^{ij} \\ b^{i}\epsilon_{\theta_{x}o}^{ij} \\ 0 \\ 0 \end{Bmatrix} \quad (41c)$$

$$\boldsymbol{\delta}_{mg}^{ij}(t) = \bar{\mathbf{D}}^{i}\mathbf{q}_{mg}^{i}(t) \oplus \bar{\mathbf{D}}^{j}\mathbf{q}_{mg}^{j}(t) = \begin{Bmatrix} x_{g}^{i} - x_{g}^{j} \\ y_{g}^{i} - y_{g}^{j} \\ z_{g}^{i} - z_{g}^{j} \\ b^{i}\theta_{xg}^{i} - b^{j}\theta_{xg}^{j} \\ a^{i}\theta_{yg}^{i} + a^{j}\theta_{yg}^{j} \\ a^{i}\theta_{zg}^{i} + a^{j}\theta_{zg}^{j} \end{Bmatrix} \quad (41d)$$

Equation (41b) shows that only five principal excitation and force coupling DOF exist. Excitation and force coupling orthonormal to the plane-of-action are neglected by the assumptions $\mathbf{W}^{ij}(t) = \bar{\mathbf{W}}^{ij}$ and $F_{\sigma}^{ij} \equiv 0$. The quasi-static transmission error vector $\boldsymbol{\delta}_{o}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*})$ is highly dependent upon system parameters other than gear mesh parameters, such as shaft and bearing stiffness characteristics. For instance, in the unloaded case where misalignment and bearing clearance effects are neglected, all terms in equation (41c) are zero except for the term ϵ_{yo}^{ij} which is the so-called quasi-static manufacturing transmission error discussed earlier. This is because all other gear pair DOF are constrained by shaft and bearing elements. Previous investigators have considered only scalar transmission error functions as excitation [1–9]. In this case, the approximate expression for $\boldsymbol{\delta}_{o}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*})$ becomes

$$\boldsymbol{\delta}_{o}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*}) \cong \begin{Bmatrix} 0 \\ \xi_{o}^{i}(\bar{F}_{y}^{ij}, \theta_{o}^{*}) \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (42)$$

where $\xi_{o}^{i}(\bar{F}_{y}^{ij}, \theta_{o}^{*})$ is the torsional, or line-of-action, quasi-static loaded transmission error which includes quasi-static deflections and kinematic errors normal to the tooth surface in the theoretical plane-of-action due to mean applied load $\bar{F}_{y}^{ij}$. Since the stiffness parameters $\kappa_{o}^{ij}(\theta_{o}^{*})$ and $\mu_{o}^{ij}(\theta_{o}^{*})$ are retained in the expression for $\mathbf{K}_{Mo}^{ij}(\bar{\mathbf{Q}}^{ij}, \theta_{o}^{*})$, the resulting mesh force expression describes only five DOF force coupling. If coupling terms $\kappa_{o}^{ij}(\theta_{o}^{*})$ and $\mu_{o}^{ij}(\theta_{o}^{*})$ are neglected, mesh force coupling is reduced to only four DOF. This is consistent with the formulations of Kiyono [5] and Kücükay [7]. Equation (41) may be reduced to an even more simplified form which results in a system of LTI equations of motion. This is accomplished by assuming that the excitation term $\boldsymbol{\delta}_{o}^{ij}(\theta_{o}^{*})$ is independent of $\mathbf{K}_{Mo}^{ij}(\theta_{o}^{*})$ and replacing $\mathbf{K}_{Mo}^{ij}(\theta_{o}^{*})$ with its spatially-averaged mean value $\bar{\mathbf{K}}_{M}^{ij}$. By neglecting all other spatially-varying and time-varying effects, the resulting equations of motion will indeed be LTI in nature. See for instance the 10 DOF helical gear dynamic model developed by Kahraman [9] which has only four DOF effective mesh force coupling.

6. CLOSURE

The dynamic analysis of geared systems is extremely complex and inherently non-linear. This analysis shows that various simplifying assumptions and approximations must be made in order to obtain analytically tractable formulations which are applicable to specific classes of problems. To this end, a new model is developed which presents a true, three-dimensional representation of the forces and moments generated within and transmitted via the gear mesh interface. This model is written in terms of functional, spatially-varying mesh stiffness and transmission error parameters which are assumed to be available from independent quasi-static elastic analyses or determined from empirical studies which relate these parameters to various gear design specifications, errors and misalignment conditions. The general formulation allows various existing models to be compared on the same mathematical basis. Different LTV and LTI models are obtained depending upon the simplifying assumptions. The vector formulation lends itself to an automated multibody analysis of geared power transmission systems including epicyclic gear drives. Force coupling due to vibratory changes in the instantaneous plane-of-action are included directly. In a follow up paper by the authors [17], numerical examples are presented to illustrate the importance of elastic force transmissibility via the gear mesh interface.

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