Effects of modal truncation errors on transmitted dynamic power estimates in discretely joined component assemblies

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(Received 29 September 1993; revised 10 February 1994; accepted 24 June 1996)

In an earlier paper [J. Acoust. Soc. Am. **97**, 2855–2865 (1995)], the authors proposed a new formulation based on modal synthesis methods for computing the mechanical power transmitted among components in vibrating assemblies. The formulation is exact if all component modes are included in the analysis, but errors may be introduced if incomplete sets of component modes are used. The effects of such modal truncation errors are examined in this paper. A new approach for determining truncation errors in assembly modal properties obtained from modal synthesis formulations is developed and applied, which reveals that the authors' original formulation is susceptible to truncation errors. A modified formulation is developed which retains the advantages of the original, but yields more accurate results over a particular frequency range when small subsets of component modes are used. An example involving the flexural vibration of a beam is examined using both formulations. The example verifies the improved accuracy of the modified formulation, and also illustrates several important issues associated with estimating transmitted dynamic power.

PACS numbers: 43.40.At, 43.20.Ks, 43.40.Cw [PJR]

INTRODUCTION

In a previous paper,[1] the authors proposed a new method for computing structure-borne power transmitted among components in vibrating machine assemblies. The method is based on a component modal synthesis approach, and is exact if complete sets of component modes are used. Unfortunately, most machine elements of practical interest have many more modes than may be practically included, and incomplete sets of component modes often must be used. Consequently, estimates for assembly natural frequencies and modes obtained from modal synthesis may contain errors due to those component modes excluded. Since the mechanical power transmitted among components is of interest, the motions and forces at joint locations must be accurately predicted, which requires that the assembly modes obtained from the synthesis formulation be accurate at the joint locations.

Several modal synthesis formulations have been proposed, with the principal differences among them being the interfacial boundary conditions imposed on the individual components. These include formulations in which components have fixed conditions imposed at the joint locations,[2–5] formulations which use component modes corresponding to free joint locations,[5–7] and hybrid formulations which use both fixed and free component modes.[8–12] Of these methods, a formulation similar to that used by Min *et al.*,[9,10] which is an extension of a formulation originally proposed by Dowell,[11,12] was used in the authors' earlier work.[1] An advantage of this method for determining transmitted mechanical power is that the motion constraints at the joints are enforced using Lagrange multipliers, which are numerically equal to the interfacial forces and moments. These are readily obtained from the forced response of the assembled structure, and permit efficient calculation of transmitted power.

A common feature of all modal synthesis formulations[1–12] is that they produce an eigenvalue problem which must be solved to determine the natural frequencies of the assembly and the transformation relating the assembly modes to those of its components. Most aspects of the algebraic eigenvalue problem have been investigated extensively.[13] Several researchers have dealt with problems involving the relationships between the eigenvalues and eigenvectors of large systems and those of smaller approximate systems. For example, Guyan,[14] Irons,[15] and Leung[16] each proposed a formulation for accurately determining the first few eigenvalues and eigenvectors of large systems by condensation methods in which so-called "slave" degrees of freedom are eliminated from the analysis while "master" degrees of freedom are retained. Wright and Miles[17] proposed an alternative method for efficiently determining the first few eigenvalues of large problems. Each of these methods[14–17] makes use of all terms in the matrices of the exact eigenvalue problems to produce an approximate problem of smaller dimension. In the case of the eigenvalue problems produced by modal synthesis methods, the matrix elements involve inner products between various component modes. Consequently, when the excluded component modes are not known, the large matrices associated with the exact eigenvalue problem are also unknown, and these methods cannot be applied. Other investigators considered the effects of modal truncation on predicted changes of assembly natural frequencies and modes caused by structural modification.[18–23] The scope of these studies was limited to modifications which change local stiffness or mass properties, so that the number assembly modes did not change. This is unlike the eigenvalue problems associated with modal syn-

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thesis formulations, where the number of modes predicted for the assembled structures increases whenever additional component modes are included, and the results of these investigations may not be readily applied either.

Literature on the effects of modal truncation on the results obtained from modal synthesis procedures is sparse. Leung[24] proposed a method for reducing truncation errors in reduced substructure eigenvalue problems. Like the methods of other investigations,[14–16] his approach requires knowledge of the submatrices associated with "slave" degrees of freedom. Meirovitch and Kwak[25,26] proposed the use of so-called quasicomparison functions in a Rayleigh–Ritz method for component modal synthesis. They found that while the Rayleigh–Ritz variational formulation requires only that admissible functions satisfying geometric boundary conditions be used as approximate component modes, the use of quasicomparison functions which more easily satisfy the interfacial force and moment conditions as well gave much more accurate results and increased the rate of convergence as the number of approximate component modes was increased. Rook and Singh[27] recently extended the formulations discussed in this paper to examine effects of incomplete bases on power transmission estimates using component mode and Lanczos vector approaches.

The effects of the use of incomplete sets of component modes on estimates for assembly modal properties and transmitted vibration obtained from the authors' original formulation are examined in this paper, which is intended as a companion to the authors' previous article.[1] It is shown that the original formulation has an inherent flaw which makes it susceptible to modal truncation errors, and an alternative formulation is developed which retains the advantages of the original, but is more robust. To make the paper complete, a summary of the authors' original formulation is included. Examples involving both the original and modified formulations are considered.

I. ORIGINAL FORMULATION: MODAL SYNTHESIS AND DYNAMIC POWER TRANSMISSION IN JOINED COMPONENT ASSEMBLIES

The development of this section is brief, and the reader is referred to the authors' previous paper[1] for details. Consider a vibrating assembly of $R+1$ components. Let one of the components be called the main system, and let the other components be called subsystems. The subsystems may be connected to the main system through joints at any number of discrete points, but may not be connected to each other. Let $\mathscr{H}^0$ and $\mathscr{H}^r$ be the Hilbert spaces for motions of the main system and subsystem r, respectively. The assembly is assumed to be a linear system with real-valued modes. Subsystem r is connected to the main system at the J^r points $[\mathbf{x}_1^r, \mathbf{x}_2^r, \ldots, \mathbf{x}_{J^r}^r]$. The vector $\mathbf{d}_j^r$ is the displacement of the joint at $\mathbf{x}_j^r$ caused by its connection to the main system.

The natural frequencies and modes of the assembled structure are obtained from those of its individual components. In particular, main system natural frequencies ξ_i and modes $\boldsymbol{\phi}_i$ corresponding to free conditions at the joints are required, while natural frequencies ν_k^r and modes $\boldsymbol{\psi}_k^r$ for subsystem r corresponding to fixed conditions at the joints are used.

The steady harmonic motion of the assembled structure may be expressed in terms of the modes of its components. The motion of the main system $\mathbf{q}^0$ is

$$\mathbf{q}^0 = \sum_{i=1}^{N^0} a_i \boldsymbol{\phi}_i e^{-j\Omega t}, \tag{1}$$

where N^0 is the number of main system modes included in the series. Since the component modes $\{\boldsymbol{\psi}_k^r\}$ span the deformation of subsystem r relative to fixed conditions at the joint locations, the motion of subsystem r after assembly may be obtained by superimposing upon these deformations due solely to displacements of the connection points, called static influence functions $\mathbf{g}_j^r$. The internal displacement of subsystem r due to motions imposed at its joints is $\mathbf{q}^r = \mathbf{G}^r \mathbf{d}^r$, where $\mathbf{d}^r = [\mathbf{d}_1^{rT}, \mathbf{d}_2^{rT}, \ldots, \mathbf{d}_{J^r}^{rT}]^T$ is the vector of all displacements constrained by joints to the main system and the columns of $\mathbf{G}^r$ are static influence functions. Hence,

$$\mathbf{q}^r = \left[\sum_{k=1}^{N^r} b_k^r \boldsymbol{\psi}_k^r + \mathbf{G}^r \mathbf{d}^r \right] e^{-j\Omega t}, \tag{2}$$

where N^r is the number of modes of subsystem r included. For the remainder of this development, the ubiquitous term $e^{-j\Omega t}$ will be omitted.

The kinetic and potential energies of the assembly may be computed[1] using the kinematic relations (1) and (2). The displacements of the joints connecting the main system and subsystems occur in the expressions for both $\mathbf{q}^0$ and $\mathbf{q}^r$, so constraints on the joint motions $\mathbf{d}^r$ are necessary. The motions of the main system and subsystems are constrained to be identical at the joint locations so that

$$\mathbf{f}^r = \sum_{i=1}^{N^0} a_i \mathbf{L}^r \boldsymbol{\phi}_i - \mathbf{d}^r = \mathbf{0}, \tag{3}$$

where the selection operator $\mathbf{L}^r$ evaluates the main system modes at the points where subsystem r is joined, so that $\mathbf{L}^r \boldsymbol{\phi} = [\boldsymbol{\phi}(\mathbf{x}_1^r)^T, \ldots, \boldsymbol{\phi}(\mathbf{x}_{J^r}^r)^T]^T$.

Equations of motion for free vibration of the assembled structure are obtained from Lagrange's equation with Lagrange multipliers used to enforce the constraints of equation (3). The vector of generalized coordinates is $\mathbf{y} = [a_1, \ldots, a_{N^0}, b_1^1, \ldots, b_{N^R}^R, d_{11}^1, \ldots, d_{J^R}^R]^T$. If the equations resulting from application Lagrange's equation and the equations of constraint are manipulated to eliminate all joint motions and Lagrange multipliers, then the assembly eigenvalue problem

$$\mathbf{Ax} = \omega^2 \mathbf{Bx} \tag{4}$$

is obtained, where $\mathbf{x} = [a_1, \ldots, b_{N^R}^R]^T$ is the vector of unknown component modal expansion coefficients and $\mathbf{A}$ and $\mathbf{B}$ are self-adjoint matrices with nonzero coefficients

$$A_{ij} = \xi_i^2 \delta_{ij} + \sum_{r=1}^{R} (\mathbf{L}^r \boldsymbol{\phi}_j)^T \mathbf{S}^r \mathbf{L}^r \boldsymbol{\phi}_i, \quad i,j \in \mathcal{N}[1,\ N^0], \tag{5a}$$

$$A_{ij} = (\nu_i^r)^2 \delta_{ij}, \quad i,j \in \mathcal{N}[N^0+1,\ N^0 + \Sigma N^r], \tag{5b}$$

$$B_{ij}=\delta_{ij}+\sum_{r=1}^{R}\langle \mathbf{G}^r\mathbf{L}^r\boldsymbol{\phi}_j,\ \mathbf{M}^r\mathbf{G}^r\mathbf{L}^r\boldsymbol{\phi}_i\rangle,$$

$$i,j\in\mathcal{I}[1,\ N^0], \tag{5c}$$

$$B_{ij}=\delta_{ij},\ \ i,j\in\mathcal{I}[N^0+1,\ N^0+\Sigma N^r] \tag{5d}$$

$$B_{ij}=\langle \boldsymbol{\psi}_k^r,\ \mathbf{M}^r\mathbf{G}^r\mathbf{L}^r\boldsymbol{\phi}^i\rangle,\ \ i\in\mathcal{I}[1,N^0],$$

$$j\in\mathcal{I}[N^0+1,\ N^0+\Sigma N^r],\ \ k=j-\sum_{n=0}^{N^{r-1}}N^n. \tag{5e}$$

The matrix $\mathbf{S}^r$ in Eq. 5(a) is the external stiffness matrix for subsystem r relating forces and motions at the joint locations. Equation (4) yields $(N^0+\Sigma N^r)$ eigenvalues ω_i^2 and eigenvectors $\mathbf{x}_i$. The frequencies ω_i are the undamped natural frequencies of the assembled structure, and the concomitant modes are obtained from the eigenvectors $\mathbf{x}_i$. The matrix $\mathbf{U}$ is defined such that $\mathbf{U}=\text{diag}[[\boldsymbol{\Phi}]\ \ [\boldsymbol{\Psi}^1]\ \ \cdots\ \ [\boldsymbol{\Psi}^R]]$, where $\boldsymbol{\Phi}=[\boldsymbol{\phi}_1,\dots,\boldsymbol{\phi}_{N^0}]$ and $\boldsymbol{\Psi}^r=[\boldsymbol{\psi}_1^r,\dots,\boldsymbol{\psi}_{N^r}^r]$ are matrices of component modes. Let the assembly displacement vector be $\mathbf{q}=[(\mathbf{q}^0)^T,(\mathbf{q}^1)^T,\dots,(\mathbf{q}^R)^T]^T$. Then, the assembly mode functions $\mathbf{v}$ corresponding to $\mathbf{q}$ are

$$\mathbf{v}_i=[\mathbf{I}+\overline{\mathbf{GL}}]\mathbf{U}\mathbf{x}_i, \tag{6}$$

where $\mathbf{I}$ is an identity operator and

$$\overline{\mathbf{G}}=\text{diag}[\mathbf{0}]\ \ [\mathbf{G}^1]\ \ \cdots\ \ [\mathbf{G}^R]], \tag{7a}$$

$$\overline{\mathbf{L}}=\begin{bmatrix}[\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}]\\ [\mathbf{L}^1] & [\mathbf{0}] & \cdots & [\mathbf{0}]\\ \vdots & \vdots & \ddots & [\mathbf{0}]\\ [\mathbf{L}^R] & [\mathbf{0}] & \cdots & [\mathbf{0}]\end{bmatrix}. \tag{7b}$$

The domains of the main system and subsystems intersect only at their boundaries. Hence, the components share no common mass, and the structure of $\mathbf{q}$ ensures that the assembly mass operator is $\mathbf{M}=\text{diag}[[\mathbf{M}^0]\ [\mathbf{M}^1]\ \cdots\ [\mathbf{M}^R]]$. Consequently, since the assembly stiffness operator is self-adjoint, the assembly eigenvectors $\mathbf{v}_i$ may be scaled to be orthonormal with respect to $\mathbf{M}$ so that $\langle\mathbf{v}_i,\ \mathbf{M}\mathbf{v}_j\rangle=\delta_{ij}$.

Because the set $\{\mathbf{v}_i\}$ is an orthonormal basis for the motion of the assembly, the normal mode method may be used to compute its forced response, which may then be used to compute the mechanical power transmitted through joints. The present formulation is well suited to this task, since the forces transmitted at the joints are equal to the Lagrange multipliers used to enforce the constraint conditions. In terms of $\mathbf{q}(\Omega)=\mathbf{V}\eta(\Omega)$, where $\eta(\Omega)$ are modal participation factors,

$$\boldsymbol{\lambda}^r(\Omega)=\Omega^2(\mathbf{G}^r)^T\mathbf{M}^r\mathbf{q}^r-\mathbf{S}^r\mathbf{L}^r\mathbf{q}^0, \tag{8}$$

where $\boldsymbol{\lambda}^r$ is the vector of Lagrange multipliers associated with subsystem r. The power transmitted among the components is computed using the corresponding joint velocities. The total power W_{in} injected into the structure is

$$W_{\text{in}}(\Omega)=\tfrac{1}{2}\,\text{Re}(j\Omega\mathbf{q}^T\mathbf{f}^*), \tag{9}$$

where $\mathbf{f}$ is the external load vector, and the total power W^r transmitted from the main system to subsystem r is

$$W^r(\Omega)=\tfrac{1}{2}\,\text{Re}[-j\Omega(\mathbf{L}^r\mathbf{q}^0)^T(\boldsymbol{\lambda}^r)^*]. \tag{10}$$

By considering cases when all components of $\boldsymbol{\lambda}^r$ are set to zero, except those associated with a particular joint, the vibration transmitted through a particular structural path may be examined.

II. ANALYSIS OF MODAL TRUNCATION ERRORS: ORIGINAL FORMULATION

Consider the eigenvalue problem of Eq. (4). Let the total number of modes associated with the main system component be M^0 and the total number of modes associated with each subsystem component r be M^r. However, suppose that only N^0 main system modes and N^r modes of subsystem r are included in the analysis. A relationship between the solutions of the truncated modal synthesis eigenvalue problem and those of the exact eigenvalue problem in which all component modes are retained is sought.

The matrices of component modes may be partitioned such that $\boldsymbol{\Phi}=[\boldsymbol{\Phi}_1,\ \boldsymbol{\Phi}_2]$, $\boldsymbol{\Psi}^r=[\boldsymbol{\Psi}_1^r\ \ \boldsymbol{\Psi}_2^r]$, where $\boldsymbol{\Phi}_1=[\hat{\boldsymbol{\phi}}_1\ \ \hat{\boldsymbol{\phi}}_2\ \cdots\ \hat{\boldsymbol{\phi}}_{N^0}]$, $\boldsymbol{\Phi}_2=[\hat{\boldsymbol{\phi}}_{N^0+1}\ \ \hat{\boldsymbol{\phi}}_{N^0+2}\ \cdots\ \hat{\boldsymbol{\phi}}_{M^0}]$, $\boldsymbol{\Psi}_1^r=[\boldsymbol{\psi}_1^r\ \ \boldsymbol{\psi}_2^r\ \cdots\ \boldsymbol{\psi}_{N^r}^r]$, and $\boldsymbol{\Psi}_2^r=[\boldsymbol{\psi}_{N^r+1}^r\ \ \boldsymbol{\psi}_{N^r+2}^r\ \cdots\ \boldsymbol{\psi}_{M^r}^r]$. Similarly, the vectors of component modal expansion coefficients may be partitioned such that $\mathbf{a}=[\mathbf{a}_1^T\ \mathbf{a}_2^T]^T$, $\mathbf{b}^r=[(\mathbf{b}_1^r)^T\ (\mathbf{b}_2^r)^T]^T$, where the coefficients in $\mathbf{a}_1$ correspond to the mode vectors in $\boldsymbol{\Phi}_1$, those in $\mathbf{a}_2$ correspond to $\boldsymbol{\Phi}_2$, and so forth. The exact modal synthesis eigenvalue problem in which all component modes are retained may be expressed as

$$\begin{bmatrix}\widetilde{\mathbf{A}}_{00} & & & \\ & \widetilde{\mathbf{A}}_{11} & & \\ & & \ddots & \\ & & & \widetilde{\mathbf{A}}_{RR}\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}^1\\ \vdots\\ \mathbf{b}^R\end{bmatrix}=\widetilde{\omega}^2\begin{bmatrix}\widetilde{\mathbf{B}}_{00} & \widetilde{\mathbf{B}}_{01} & \cdots & \widetilde{\mathbf{B}}_{0R}\\ \widetilde{\mathbf{B}}_{01}^T & \widetilde{\mathbf{B}}_{11} & & \\ \vdots & & \ddots & \\ \widetilde{\mathbf{B}}_{0R}^T & & & \widetilde{\mathbf{B}}_{RR}\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}^1\\ \vdots\\ \mathbf{b}^R\end{bmatrix}, \tag{11}$$

where

$$\widetilde{\mathbf{A}}_{00}=\begin{bmatrix}\mathbf{A}_{00} & \mathbf{A}_{\mathbf{a}_1\mathbf{a}_2}\\ \mathbf{A}_{\mathbf{a}_1\mathbf{a}_2}^T & \mathbf{A}_{\mathbf{a}_2\mathbf{a}_2}\end{bmatrix}, \tag{12a}$$

$$\widetilde{\mathbf{A}}_{rr}=\begin{bmatrix}\mathbf{A}_{rr} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_{\mathbf{b}_2^r\mathbf{b}_2^r}\end{bmatrix},\quad r>0, \tag{12b}$$

$$\widetilde{\mathbf{B}}_{00}=\begin{bmatrix}\mathbf{B}_{00} & \mathbf{B}_{\mathbf{a}_1\mathbf{a}_2}\\ \mathbf{B}_{\mathbf{a}_1\mathbf{a}_2}^T & \mathbf{B}_{\mathbf{a}_2\mathbf{a}_2}\end{bmatrix}, \tag{12c}$$

$$\widetilde{\mathbf{B}}_{rr}=\begin{bmatrix}\mathbf{B}_{rr} & 0\\ 0 & \mathbf{B}_{\mathbf{b}_2^r\mathbf{b}_2^r}\end{bmatrix}, \tag{12d}$$

$$\widetilde{\mathbf{B}}_{0r}=\begin{bmatrix}\mathbf{B}_{0r} & \mathbf{B}_{\mathbf{a}_1\mathbf{b}_2^r}\\ \mathbf{B}_{\mathbf{a}_2\mathbf{b}_1^r} & \mathbf{B}_{\mathbf{a}_2\mathbf{b}_2^r}\end{bmatrix},\quad r>0, \tag{12e}$$

and nonzero matrix coefficients are

$$(\widetilde{\mathbf{A}}_{00})_{ij}=\xi_i^2\delta_{ij}+\sum_{r=1}^{R}(\mathbf{L}^r\boldsymbol{\phi}_j)^T\mathbf{S}^r\mathbf{L}^r\boldsymbol{\phi}_i,\quad i,j\in\mathcal{N}[1,M^0], \tag{13a}$$

$$(\widetilde{\mathbf{A}}_{rr})_{ij}=(\nu_i^r)^2\delta_{ij},\ \ i,j\in\mathcal{N}[1,M^r], \tag{13b}$$

$$(\widetilde{\mathbf{B}}_{00})_{ij}=\delta_{ij}+\sum_{r=1}^{R}\langle\mathbf{G}^r\mathbf{L}^r\boldsymbol{\phi}_j,\mathbf{M}^r\mathbf{G}^r\mathbf{L}^r\boldsymbol{\phi}_i\rangle,$$
$$i,j\in\mathcal{N}[1,M^0], \tag{13c}$$

$$(\widetilde{\mathbf{B}}_{0r})_{ij}=\langle\boldsymbol{\psi}_j^r,\mathbf{M}^r\mathbf{G}^r\mathbf{L}^r\boldsymbol{\phi}_i\rangle,\quad i\in\mathcal{N}[1,M^0],$$
$$j\in\mathcal{N}[1,M^r], \tag{13d}$$

$$(\widetilde{\mathbf{B}}_{rr})_{ij}=\delta_{ij},\quad i,j\in\mathcal{N}[1,M^r]. \tag{13e}$$

The eigenvalue problem of Eq. (11) may be rearranged to separate the coefficients associated with component modes included in the truncated analysis from those associated with excluded modes, so that the eigenvector of component modal expansion coefficients of the exact problem is partitioned as $\widetilde{\mathbf{x}}=[\mathbf{x}_1^T\ \ \mathbf{x}_2^T]^T$. This gives

$$\begin{bmatrix}\mathbf{A} & \mathbf{A}_{12}\\ \mathbf{A}_{12}^T & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}=\widetilde{\omega}^2\begin{bmatrix}\mathbf{B} & \mathbf{B}_{12}\\ \mathbf{B}_{12}^T & \mathbf{B}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}, \tag{14}$$

where

$$\mathbf{x}_1=[a_1\ a_2\ \cdots\ a_{N^0}\ b_1^1\ b_2^1\ \cdots\ b_{N^R}^R]^T,$$

$$\mathbf{x}_2=[a_{N^0+1}\ a_{N^0+2}\ \cdots\ a_{M^0}\ b_{N^1+1}^1\ b_{N^1+2}^1\ \cdots\ b_{M^R}^R]^T.$$

The submatrices $\mathbf{A}$ and $\mathbf{B}$ in Eq. (14) are those associated with the approximate truncated eigenvalue problem and are defined by Eq. (5). The other submatrices are

$$\mathbf{A}_{12}=\begin{bmatrix}\mathbf{A}_{\mathbf{a}_1\mathbf{a}_2} & 0 & \cdots & 0\\ 0 & 0 & & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}, \tag{15a}$$

$$\mathbf{A}_{22}=\begin{bmatrix}\mathbf{A}_{\mathbf{a}_2\mathbf{a}_2} & & & \\ & \mathbf{A}_{\mathbf{b}_2^1\mathbf{b}_2^1} & & \\ & & \ddots & \\ & & & \mathbf{A}_{\mathbf{b}_2^R\mathbf{b}_2^R}\end{bmatrix}, \tag{15b}$$

$$\mathbf{B}_{12}=\begin{bmatrix}\mathbf{B}_{\mathbf{a}_1\mathbf{a}_2} & \mathbf{B}_{\mathbf{a}^1\mathbf{b}_2^1} & \cdots & \mathbf{B}_{\mathbf{a}_1\mathbf{b}_2^R}\\ \mathbf{B}_{\mathbf{a}_2\mathbf{b}_1^1}^T & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \\ \mathbf{B}_{\mathbf{a}_2\mathbf{b}_1^R}^T & \mathbf{0} & & \mathbf{0}\end{bmatrix}, \tag{15c}$$

$$\mathbf{B}_{22}=\begin{bmatrix}\mathbf{B}_{\mathbf{a}_2\mathbf{a}_2} & \mathbf{B}_{\mathbf{a}_2\mathbf{b}_2^1} & \cdots & \mathbf{B}_{\mathbf{a}_2\mathbf{b}_2^R}\\ \mathbf{B}_{\mathbf{a}_2\mathbf{b}_2^1}^T & \mathbf{B}_{\mathbf{b}_2^1\mathbf{b}_2^2} & & \\ \vdots & & \ddots & \\ \mathbf{B}_{\mathbf{a}_2\mathbf{b}_2^R}^T & & & \mathbf{B}_{\mathbf{b}_2^R\mathbf{b}_2^R}\end{bmatrix}. \tag{15d}$$

If Eq. (14) is manipulated to eliminate $\mathbf{x}_2$, then

$$[(\mathbf{A}-\widetilde{\omega}^2\mathbf{B})-(\mathbf{A}_{12}-\widetilde{\omega}^2\mathbf{B}_{12})(\mathbf{A}_{22}-\widetilde{\omega}\,\mathbf{B}_{22})^{-1}$$
$$\times(\mathbf{A}_{12}-\widetilde{\omega}^2\mathbf{B}_{12})^T]\mathbf{x}_1=\mathbf{0}. \tag{16}$$

Clearly, if the quantity $(\mathbf{A}_{12}-\widetilde{\omega}^2\mathbf{B}_{12})(\mathbf{A}_{22}-\widetilde{\omega}^2\mathbf{B}_{22})^{-1}(\mathbf{A}_{12}-\widetilde{\omega}^2\mathbf{B}_{12})^T$ is small compared to $(\mathbf{A}-\widetilde{\omega}^2\mathbf{B})$, then the eigenvalues and eigenvectors of the truncated eigenvalue problem, ω_i^2 and $\mathbf{x}_i$, will be close to those of the exact eigenvalue problem, $\widetilde{\omega}_i^2$ and $\widetilde{\mathbf{x}}$. The matrices $\mathbf{A}_{12}$ and $\mathbf{B}_{12}$ are responsible for the coupling between the component modes included in the analysis and those excluded. Consequently, if the norms of these matrices are small compared to those of $\mathbf{A}$ and $\mathbf{B}$, the excluded modes will be weakly coupled and accurate results may be expected from the truncated eigenvalue problem. However, note that the excluded component modes $\boldsymbol{\Phi}_2$ and $\boldsymbol{\Psi}_2^r$ which are needed to compute the elements of the matrices $\mathbf{A}_{12}$, $\mathbf{A}_{22}$, etc., are usually not known. Consequently, the matrix $(\mathbf{A}_{12}-\widetilde{\omega}^2\mathbf{B}_{12})(\mathbf{A}_{22}-\widetilde{\omega}^2\mathbf{B}_{22})^{-1}(\mathbf{A}_{12}-\widetilde{\omega}^2\mathbf{B}_{12})^T$ is also unknown, and bounds on the errors associated with the truncated eigenvalue problem may not be readily obtained. However, because all modal synthesis formulations are essentially variations of the Rayleigh–Ritz method,[25,26] it is possible to state an inclusion principle relating the eigenvalues ω_i^2 of the truncated eigenvalue problem to those of the exact problem, $\widetilde{\omega}_i^2$. This relation is

$$\widetilde{\omega}_1^2\leqslant\omega_1^2\leqslant\omega^2_{(N^0+\Sigma N^r)}\leqslant\widetilde{\omega}^2_{(M^0+\Sigma M^r)}. \tag{17}$$

III. EXAMPLE: FLEXURAL VIBRATION OF A JOINED BEAM SYSTEM

Consider the fixed/fixed beam illustrated in Fig. 1. The beam is treated as a two component assembly, and the formulation of Sec. I is applied. The left portion of the beam of length l^0 is considered the main system, with free conditions imposed at the joint location as indicated in Fig. 1(b). The right portion of the beam of length l^1 shown in Fig. 1(c) is considered the subsystem. Fixed conditions are imposed at the joint location. The natural frequencies and modes of the beam components are

$$\xi_n,\ \nu_n^1=\left[\frac{(\beta_n^i l^i)^4EI}{\rho A(l^i)^4}\right]^{1/2}, \tag{18a}$$

$$\boldsymbol{\phi}_n(x),\ \boldsymbol{\psi}_n^1(x)=\cosh\beta_n^i x-\cos\beta_n^i x$$
$$-\left(\frac{\cosh\beta_n^i l^i-\cos\beta_n^i l^i}{\sinh\beta_n^i l^i-\sin\beta_n^i l^i}\right)$$
$$\times(\sinh\beta_n^i x-\sin\beta_n^i x), \tag{18b}$$

where the beam eigenvalues $\beta_n^i l^i$ are those associated with fixed/free beams for the main system ($i=0$), and those associated with fixed/fixed beams for the subsystem ($i=1$), as

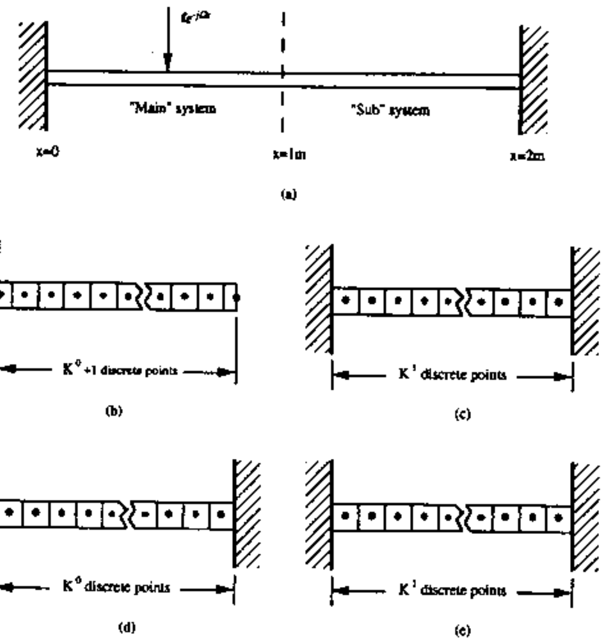


FIG. 1. Joined beam/bar system considered as example case. (a) Assembled structure. (b) Main system component for original formulation. (c) Subsystem component for original formulation. (d) Main system component for modified formulation. (e) Subsystem component for modified formulation.

given by Gorman[28]. The single joint connecting the components has two degrees of freedom: transverse displacement and rotation. Hence, the joint motion is $\mathbf{d}^1=[q(l^0) \quad \theta(l^0)]^T$ and the corresponding subsystem static influence function and external stiffness matrices are

$$\mathbf{G}^1(x)=[\mathbf{g}_1^1(x) \quad \mathbf{g}_2^1(x)], \tag{19a}$$

where

$$\mathbf{g}_1^1(x)=2(x/l^1)^3-3(x/l^1)^2+1, \tag{19b}$$

$$\mathbf{g}_2^1(x)=l^1[(x/l^1)^3-2(x/l^1)^2+(x/l^1)], \tag{19c}$$

and

$$\mathbf{S}^1=\left(\frac{EI}{(l^1)^3}\right)\begin{bmatrix} 12 & 6l^1 \\ 6l^1 & 4(l^1)^2 \end{bmatrix}. \tag{19d}$$

Although continuous component modes and static influence functions could be used directly, implementation is simplified if the functions are evaluated at discrete points, and the resulting vectors of discrete values are used instead, as illustrated in Fig. 1. The main system modes are evaluated at K^0+1 points, so that one point is at the joint. Hence, the discretized main system mass matrix is $\mathbf{M}^0=(\rho A l^0/K^0)\times\mathrm{diag}[\frac{1}{2} \;\; 1 \;\; 1 \;\; \cdots \;\; 1 \;\; 1 \;\; \frac{1}{2}]$. Since the subsystem modes and static influence functions need not be evaluated at the joint, they are evaluated at K^1 discrete points, and the discretized subsystem mass matrix is $\mathbf{M}^1=(\rho A l^1/K^1)\mathrm{diag}[1 \;\; 1 \;\; \cdots \;\; 1]$. Because the joint constrains a rotation degree of freedom, the continuous main system joint selection operator $\mathbf{L}^1$ must evaluate derivatives of the main modes at the joint location. For the discretized main system modes, derivatives are evaluated numerically. Because the joint is at the end of the beam, a five point single-sided finite difference approximation is used.[29] Hence, the discretized $2\times(K^0+1)$ main system selection operator is

$$\mathbf{L}^1=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1/4h & -4/3h & 3/h & -4/h & 25/12h \end{bmatrix}, \tag{20}$$

where $h=l^0/K^0$ is the length of one discretized main system element as shown in Fig. 1.

TABLE I. Natural frequency estimates for fixed/fixed beam from original formulation.

Modal index n	Natural frequency f_n (Hz) $N^0=N^1=5$	$N^0=N^1=10$	$N^0=N^1=15$	$N^0=N^1=20$	Exact value
1	0.2653	0.2610	0.2603	0.2603	0.2570
2	0.7090	0.7084	0.7084	0.7084	0.7084
3	1.4546	1.4194	1.4140	1.4140	1.3888
4	2.3028	2.2965	2.2962	2.2962	2.2955
5	3.6033	3.5065	3.4924	3.4924	3.4292
6	4.8243	4.7936	4.7920	4.7920	4.7893
7	6.7421	6.5252	6.4972	6.4972	6.3765
8	8.3169	8.2016	8.1967	8.1967	8.1906
9	11.0330	10.4875	10.4298	10.4297	10.2307
10	16.0409	12.5261	12.5137	12.5136	12.4982

For the specific case considered, the joint was located at the center of the assembly, and component lengths $l^0=l^1=1.0$ m were used. Physical constants were $\rho=1.0$ kg/m^3, $E=10\,000$ N/m^2, $I=8.333\times10^{-9}$ m^4, and $A=0.01$ m^2. The number of discrete points at which the component modes were evaluated were $K^0=K^1=100$. Estimates for the first ten natural frequencies of the assembly obtained by including different numbers of component modes appear in Table I. The estimates are better for even numbered modes than for odd numbered modes. Note that the rate at which the natural frequencies converge to the corresponding exact values is slow, as little difference is observed between the natural frequency estimates for $N^0=N^1=10$ and those for $N^0=N^1=20$. Results obtained for the asymmetric modes ($n=2,4,\ldots$) were virtually indistinguishable from the corresponding exact results over the entire domain up to mode ten. Results for the first three symmetric modes ($n=1,3,\ldots$) appear in Fig. 2 with the corresponding exact fixed/fixed beam modes. The symmetric mode estimates exhibit errors in both displacement and slope, which are most severe at the joint location, $x=1$ m. The modes of Fig. 2 were calculated for the case $N^0=N^1=10$, but including additional component modes had little effect on the local errors in the mode shapes. Although the modes obtained from the synthesis formulation seem reasonable in a global sense, local errors at the joint are cause for concern, since they affect estimates for vibration transmitted through the joints.

The forced response of the assembly was computed for the case of a concentrated force of 1 N magnitude and uniform harmonic content applied at the point $x=0.5$ m, as indicated in Fig. 1(a). The first ten assembly modes were used in the normal mode expansion, synthesized modes obtained from 20 component modes ($N^0=N^1=10$) were used,

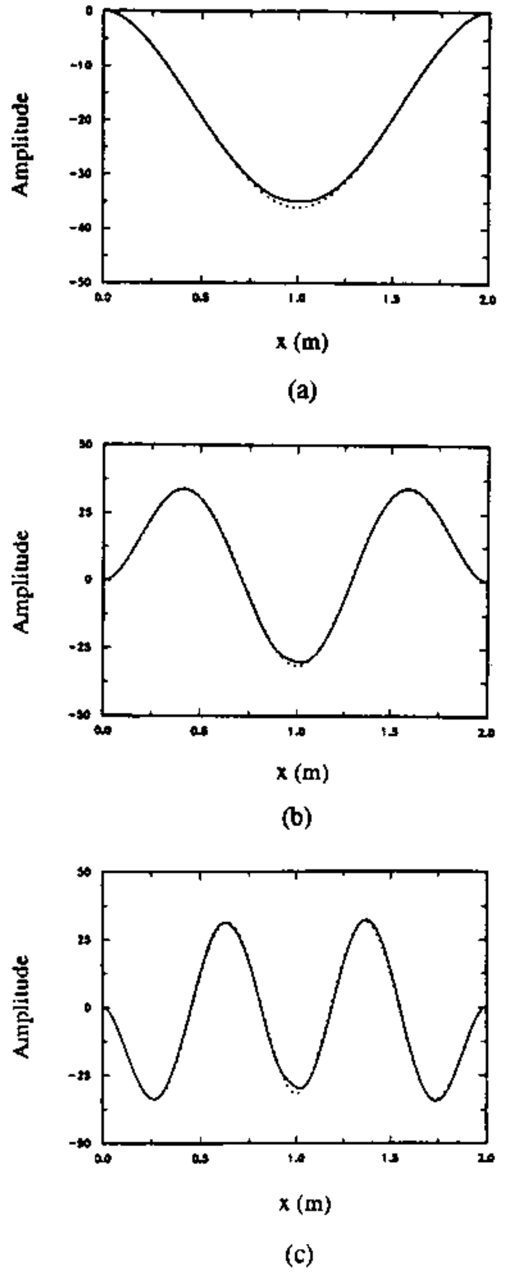


FIG. 2. Odd-numbered assembly modes for fixed/fixed beam obtained from formulation of Sec. I. (a) First mode. (b) Third mode. (c) Fifth mode. Solid lines are synthesized modes; dashed lines are exact results.

and damping ratios $\zeta=0.01$ were applied to all modes. Corresponding results using the same number of exact modes for the assembled beam were also obtained. The magnitudes of the transverse displacement and rotation at the joint location are shown in Fig. 3. While the predicted transverse displacement estimate exhibits modest errors near some of the assembly natural frequencies, the joint rotation exhibits significant errors near natural frequencies corresponding to symmetric assembly modes. These are due to the slopes of the estimated symmetric assembly modes at the joint, which are non zero due to modal truncation. The transverse force and bending moment at the joint appear in Fig. 4. These estimates are also generally reasonable, with errors of a few decibels at particular frequencies.

Estimates for power transmitted from the main to the subsystem through the transverse force/displacement and bending moment/rotation paths appear in Figs. 5 and 6, respectively. The estimated force/displacement power magnitude and phase agree with exact results at most frequencies, with magnitude errors of 0–4 dB observed near some of the natural frequencies. However, the power transmitted through the moment/rotation path exhibits much greater errors, with magnitude differences of 10 dB occurring at those frequencies where estimates for joint rotation and bending moment are relatively poor. Finally, the real part of the power transmitted through both paths, which is of primary interest, is shown in Fig. 7. Virtually all of the transmitted power is concentrated in narrow bands near the assembly natural frequencies. Estimates for the real power transmitted through the force/displacement path are rather poor at several frequencies, and the estimated power transmitted through the moment/rotation path is even worse, with errors exceeding 50% at some frequencies. Particularly disturbing are the negative values appearing in both power spectra. These indicate power flow from the subsystem to the main system, which is not possible for this example since only the main system is excited directly. The phase spectra of Figs. 5 and 6 near the frequencies where negative values are predicted reveal that the phase changes abruptly from 90° to −90° over an extremely narrow range. Consequently, modest errors in phase estimates in these bands are the reason for negative power predictions.

This example illustrates an inherent flaw of the formulation of Sec. I. While accurate estimates of assembly natural frequencies are necessary for accurate estimates of the power transmitted among components, assembly mode shape estimates with little error at the joint locations are also required, since estimates for forces and motions at the joints are proportional to these values. The reason for relatively large local errors in the estimated assembly modes is the choice of interfacial boundary conditions imposed on the main system component. Because free interfacial conditions are imposed, each of the main system modes $\boldsymbol{\phi}_i$ has zero-valued forces and moments at the joint locations. However, when the structure is assembled, non zero forces and moments act at the joints, and the exact assembly modes have non zero forces and moments at these locations as well. This paradox does not pose a problem if the set of component modes is complete. However, when a subset of the component modes is used, it is not possible to satisfy the nonzero force and moment conditions accurately unless a very large number of free interface main system modes is included. Similar observations were reported by Meirovitch and Kwak[24] for the selection of admissible functions to be used in a Rayleigh–Ritz formulation.

IV. MODIFIED MODAL SYNTHESIS FORMULATION

Because of the limitations of the formulation of Sec. I an alternative is needed. The problem with the original formulation was the type of interfacial conditions used to determine the main system component modes. In contrast, the fixed interface subsystem modes easily satisfy the nonzero interfacial force and moment conditions when only a few

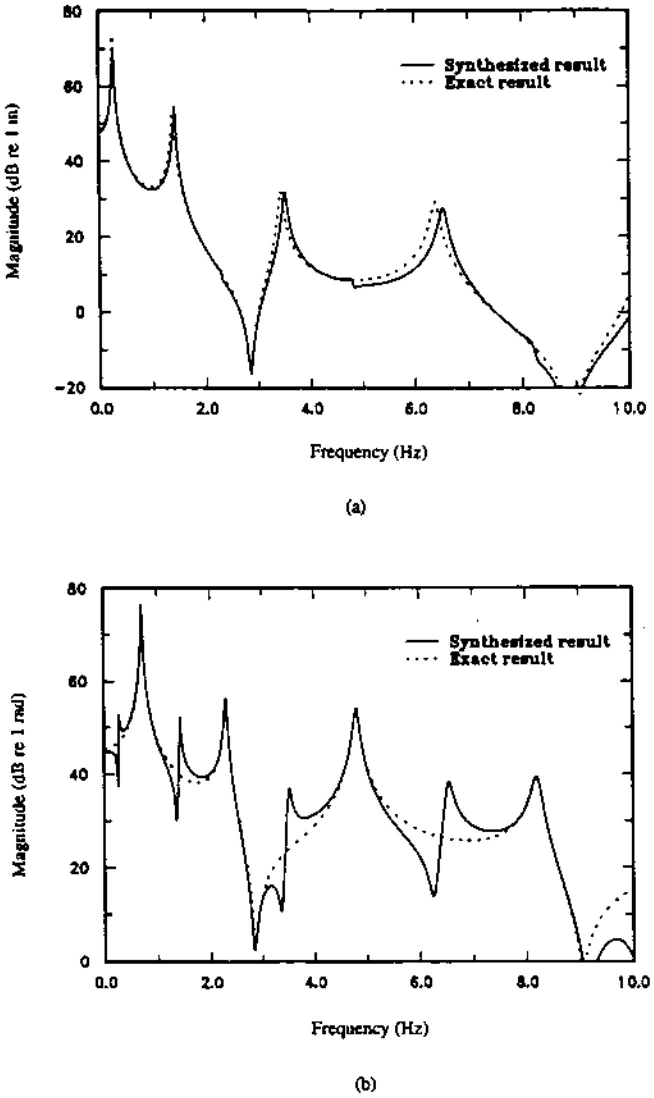


FIG. 3. Motions at joint location for fixed/fixed beam obtained from original formulation of Sec. I. (a) Transverse displacement. (b) Rotary displacement.

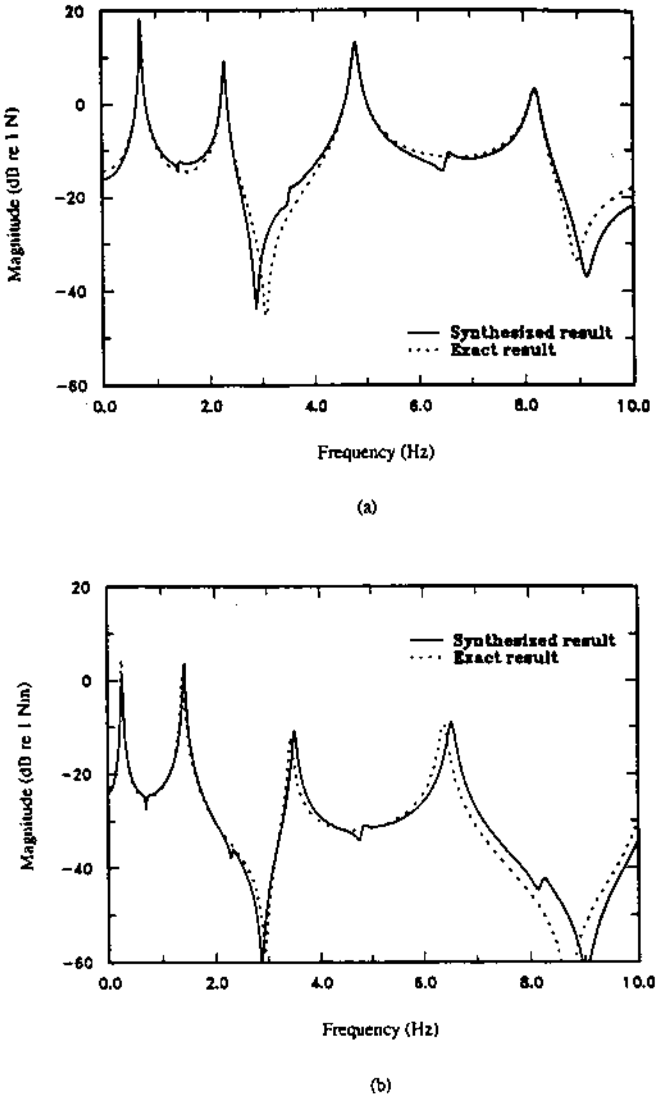


FIG. 4. Forces, at joint location for fixed/fixed beam obtained from original formulation of Sec. I. (a) Transverse force. (b) Bending moment.

modes are included, since each mode has nonzero force and moment conditions at the joint locations. Consequently, the portion of the assembly motion Hilbert space spanned by the subsystem modes exhibits less error than the portion spanned by the main system modes. Evidence of this can be seen in Fig. 2. These observations suggest the following approach for obtaining a more robust formulation: Let all components in the assembly be treated as "subsystem" components with zero-motion conditions imposed at all joint locations. This is similar to the approach used in modal synthesis formulations originally proposed by Hurty[2] and by Craig and Bampton[3].

Consider again a vibrating assembly of $R+1$ components. As before, the subsystem components modes ψ_k^r and eigenvalues $(\nu_k^r)^2$ correspond to zero displacement conditions imposed at the joint locations. In contrast, the main system component modes ϕ_i and eigenvalues ξ_i^2 also correspond to fixed interfacial conditions. All modes are scaled to be orthonormal with respect to the appropriate mass operators. Because the mode shapes of all components have zero values at the joint locations, internal displacements due to motions at the joints must be superimposed upon these. Hence, the kinematic relations for the main system component and for subsystem component r are, respectively,

$$\mathbf{q}^0=\left[\sum_{i=1}^{N^0} a_i\boldsymbol{\phi}_i+\mathbf{G}^0\mathbf{d}^0\right]e^{-j\Omega t} \tag{21a}$$

$$\mathbf{q}^r=\left[\sum_{k=1}^{N^r} b_k^r\boldsymbol{\psi}_k^r+\mathbf{G}^r\mathbf{d}^r\right]e^{-j\Omega t}, \tag{21b}$$

where $\mathbf{d}^0$ is the vector of all main system displacements constrained by joints, and $\mathbf{G}^0$ is the corresponding matrix of main system static influence functions. In the following, the ubiquitous term $e^{-j\Omega}$ will be omitted. The kinetic energies of the main system and subsystem components are, respectively,

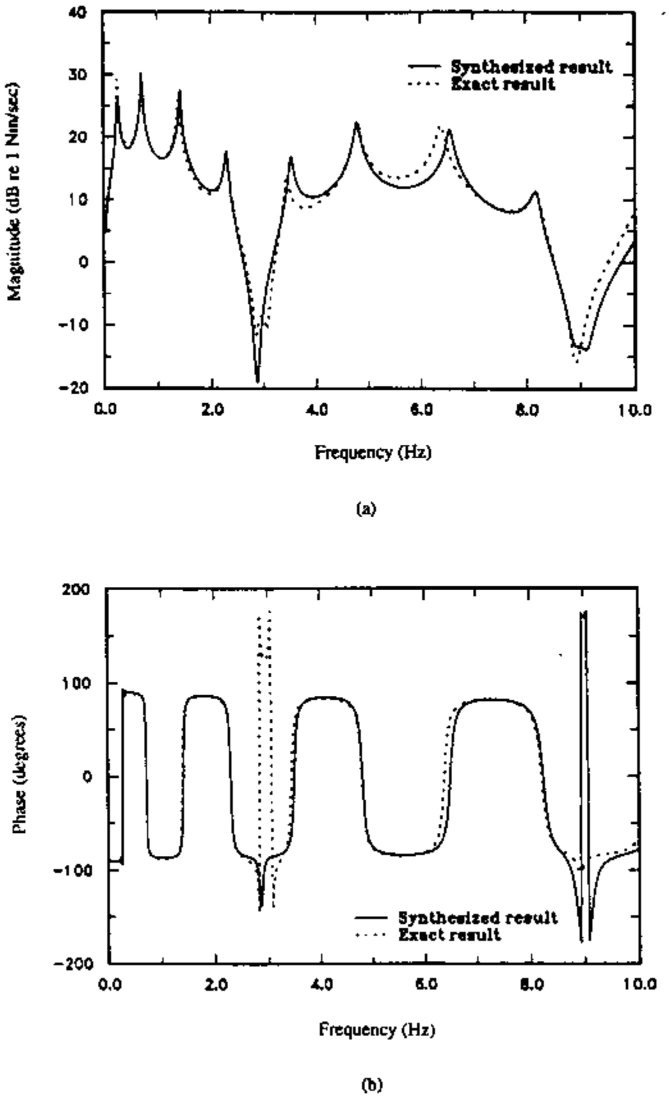


FIG. 5. Power transmitted by transverse force/transverse displacement path for fixed/fixed beam obtained from original formulation of Sec. I. (a) Magnitude. (b) Phase.

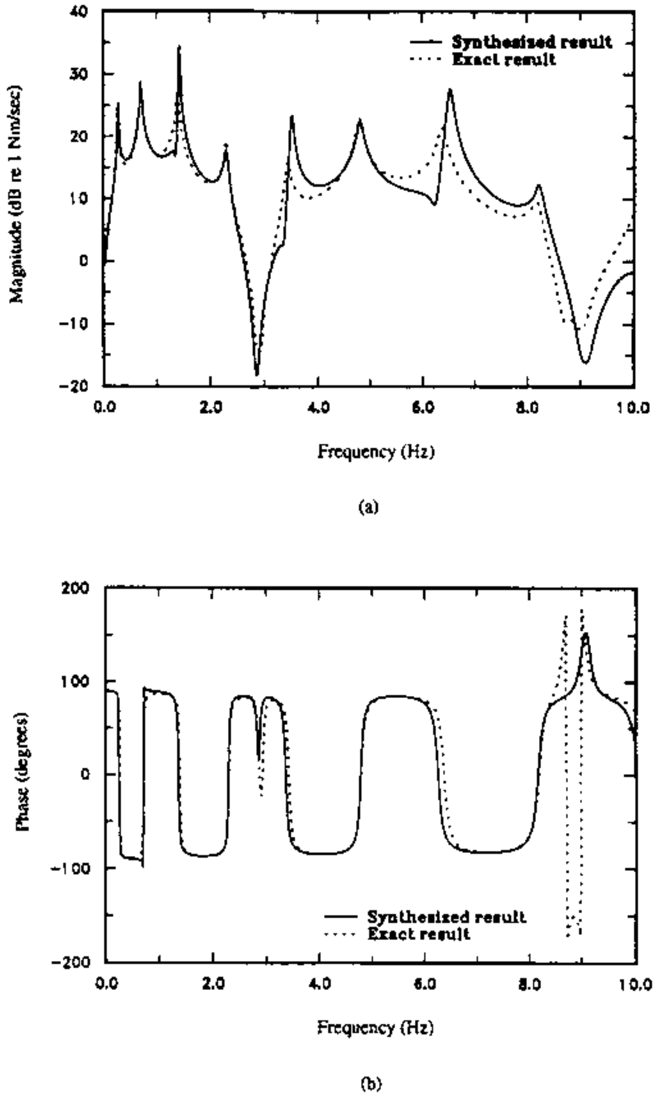


FIG. 6. Power transmitted by moment/rotation path for fixed/fixed beam obtained from original formulation of Sec. I. (a) Magnitude. (b) Phase.

$$T^0=\frac{1}{2}\,\Omega^2\left(\sum_{i=1}^{N^0} a_i^2+2\sum_{i=1}^{N^0} a_i\langle \boldsymbol{\phi}_i,\, \mathbf{M}^0\mathbf{G}^0\mathbf{d}^0\rangle + \langle \mathbf{G}^0\mathbf{d}^0,\, \mathbf{M}^0\mathbf{G}^0\mathbf{d}^0\rangle\right), \tag{22a}$$

$$T^r=\frac{1}{2}\,\Omega^2\left(\sum_{k=1}^{N^r} (b_k^r)^2+2\sum_{k=1}^{N^r} b_k^k\langle \boldsymbol{\psi}_i,\, \mathbf{M}^r\mathbf{G}^r\mathbf{d}^r\rangle + \langle \mathbf{G}^r\mathbf{d}^r,\, \mathbf{M}^r\mathbf{G}^r\mathbf{d}^r\rangle\right). \tag{22b}$$

The component potential energies are

$$U^0=\frac{1}{2}\left[\sum_{i=1}^{N^0} a_i^2\xi_i^2+(\mathbf{d}^0)^H\mathbf{S}^0\mathbf{d}^0\right], \tag{23a}$$

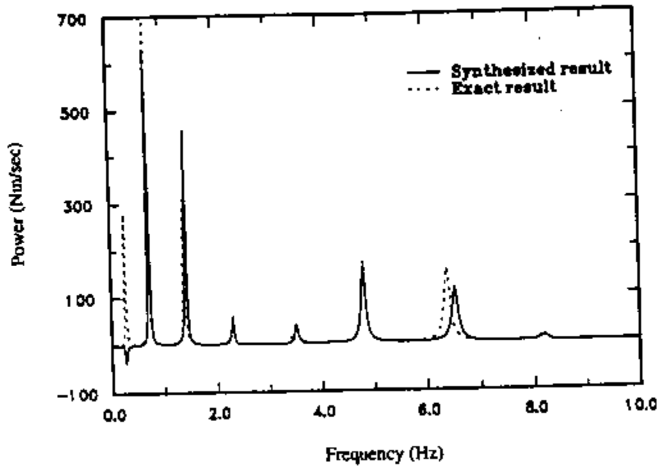
$$U^r=\frac{1}{2}\left[\sum_{k=1}^{N^r} (b_k^r)^2(\nu_k^r)^2+(\mathbf{d}^r)^H\mathbf{S}^r\mathbf{d}^r\right], \tag{23b}$$

where $\mathbf{S}^0$ is the main system external stiffness matrix relating forces and motions at the joint locations.

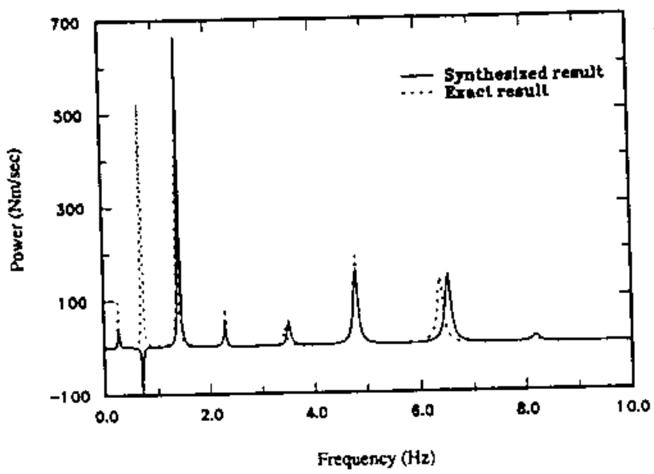
Since joint motions occur in motion expressions for both the main system and the subsystems, it is again necessary to impose constraints that the motions of the components at the joint locations be identical. This may be expressed for each subsystem component as

$$\mathbf{f}^r=\hat{\mathbf{L}}^r\mathbf{d}^0-\mathbf{d}^r=\mathbf{0}, \tag{24}$$

where $\hat{\mathbf{L}}^r$ is an operator which evaluates the main system joint displacement vector $\mathbf{d}^0$ at the points where the main system is connected to subsystem r, and is different from the operator $\mathbf{L}^r$ used in the original formulation which evaluates the main system modes $\boldsymbol{\phi}$ at the joint locations. The equations of motion for the assembly again are obtained from Lagrange's equation,



(a)



(b)

FIG. 7. Real power transmitted through joint for fixed/fixed beam obtained from original formulation of Sec. I. (a) Transverse force/transverse displacement path. (b) Moment/rotation path.

$$\frac{\partial L}{\partial y_i}+\sum_{r=1}^{R}(\boldsymbol{\lambda}^r)^T\frac{\partial \mathbf{f}^r}{\partial y_i}=0, \tag{25}$$

where λ^r is the vector of Lagrange multipliers associated with the constraints imposed on subsystem r. The vector of generalized coordinates is $\mathbf{y}=[a_1,\ \ldots,\ a_{N^0},\ b_1^1,\ \ldots,\ b_{N^R}^R,\ (\mathbf{d}^0)^T,\ \ldots,(\mathbf{d}^R)^T]^T$.

If the equations resulting from application equations (24) and (25) are manipulated to eliminate all Lagrange multipliers and joint displacement vectors except for $\mathbf{d}^0$, then the eigenvalue problem

$$\mathbf{A}\mathbf{x}=\omega^2\mathbf{B}\mathbf{x} \tag{26}$$

is obtained, where $\mathbf{x}=[a_1,\ \ldots,\ a_{N^0},\ b_1^1,\ \ldots,\ b_{N^R}^R,\ (\mathbf{d}^0)^T]^T$ and the self-adjoint matrices $\mathbf{A}$ and $\mathbf{B}$ have nonzero entries

$$A_{ij}=\xi_i^2\delta_{ij},\ \ i,\ j\in\mathcal{N}[1,\ N^0], \tag{27a}$$

$$A_{ij}=(\nu_k^r)^2\delta_{ij},\ \ i,\ j\in\mathcal{N}\left[N^0+1,\ \sum_{n=0}^{R}N^n\right],\quad k=i-\sum_{n=1}^{r-1}N^n, \tag{27b}$$

$$\mathbf{A}=\mathbf{S}^0+\sum_{r=1}^{R}(\hat{\mathbf{L}}^r)^T\mathbf{S}^r\hat{\mathbf{L}}^r,\quad i,\ j\in\mathcal{N}\left[\sum_{n=0}^{R}N^n+1,\ \sum_{n=1}^{R}N^n+d\right], \tag{27c}$$

$$B_{ij}=\delta_{ij},\ \ i,\ j\in N\left[1,\ \sum_{n=0}^{R}N^n\right], \tag{27d}$$

$$B_{ij}=\langle\boldsymbol{\phi}_i,\ \mathbf{M}^0\mathbf{G}^0\rangle_k,\ \ i\in\mathcal{N}[1,N^0]\quad j\in\mathcal{N}\left[\sum_{n=0}^{R}N^n+1,\ \sum_{n=0}^{R}N^n+d\right],\quad k=j-\sum_{n=0}^{R}N^n, \tag{27e}$$

$$B_{ij}=\langle\boldsymbol{\psi}_l^r,\ \mathbf{M}^r\mathbf{G}^r\hat{\mathbf{L}}^r\rangle_k,\ \ i\in\mathcal{N}\left[N^0+1\ \sum_{n=0}^{R}N^n\right],\quad j\in N\left[\sum_{n=0}^{R}N^n+1,\ \sum_{n=0}^{R}N^n+d\right],\quad k=j-\sum_{n=0}^{R}N^n,\ \ l=i-\sum_{n=0}^{r-1}N^n, \tag{27f}$$

$$\mathbf{B}=(\mathbf{G}^0)^T\mathbf{M}^0\mathbf{G}^0+\sum_{r=1}^{R}(\hat{\mathbf{L}}^r)^T(\mathbf{G}^r)^T\mathbf{M}^r\mathbf{G}^r\hat{\mathbf{L}}^r,\quad i,\ j\in\mathcal{N}\left[\sum_{n=0}^{R}N^n+1,\ \sum_{n=0}^{R}N^n+d\right], \tag{27g}$$

where d is the dimension of $\mathbf{d}^0$. The eigenvalues ω_i^2 of Eq. (26) are the squares of the natural frequencies of the assembly, and the corresponding assembly modes are obtained from the concomitant eigenvectors, $\mathbf{x}_i$. Defining the matrix $\mathbf{G}$ as

$$\mathbf{G}=\begin{bmatrix}\mathbf{G}^0\\ \mathbf{G}^1\hat{\mathbf{L}}^1\\ \vdots\\ \mathbf{G}^R\hat{\mathbf{L}}^R\end{bmatrix}, \tag{28}$$

the assembly modes $\mathbf{v}_i$ are related to the corresponding expansion coefficient eigenvectors $\mathbf{x}_i$ by $\mathbf{v}_i=[\mathbf{U}\quad\mathbf{G}]\mathbf{x}_i$, where $\mathbf{U}$ is the matrix of component modes used in Eq. (6). After computing natural frequencies and modes for the assembled structure, the normal mode method is used to compute its forced response. The interfacial forces (Lagrange multipliers), input power, and power transmitted among components are obtained from the forced response as they were in the original formulation.

V. ANALYSIS OF MODAL TRUNCATION ERRORS: MODIFIED FORMULATION

An expression is sought for the modified formulation relating the approximate eigenvalues and eigenvectors associated with the truncated modal synthesis eigenvalue problem to those of the corresponding exact eigenvalue problem in which all modes are included. Again consider the main system component to have a total of M^0 modes, of which N^0 are included in the analysis, and subsystem r to have M^r modes, of which N^r are included. Let the matrices of component modes $\mathbf{\Phi}$ and $\mathbf{\Psi}^r$ be partitioned as they were in Sec. II to separate included and excluded component modes. Similarly, the vectors of component modal expansion coefficients are partitioned so that $\mathbf{a}=[\mathbf{a}_1^T \;\; \mathbf{a}_2^T]^T$, where $\mathbf{a}_1$ is the vector of modal expansion coefficients corresponding to main system modes included in the analysis, $\mathbf{a}_2$ is the vector of coefficients corresponding to main system modes excluded, and so forth. The exact modal synthesis eigenvalue problem in which all modes of all components are included is

$$\begin{bmatrix} \tilde{\mathbf{A}}_{00} & & & & \\ & \tilde{\mathbf{A}}_{11} & & & \\ & & \ddots & & \\ & & & \tilde{\mathbf{A}}_{RR} & \\ & & & & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b}^1 \\ \vdots \\ \mathbf{b}^R \\ \mathbf{d}^0 \end{bmatrix} = \tilde{\omega}^2 \begin{bmatrix} \tilde{\mathbf{B}}_{00} & \mathbf{0} & \cdots & \mathbf{0} & \tilde{\mathbf{B}}_{0\mathbf{d}} \\ \mathbf{0}^T & \tilde{\mathbf{B}}_{11} & & & \tilde{\mathbf{B}}_{1\mathbf{d}} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0}^T & & & \tilde{\mathbf{B}}_{RR} & \tilde{\mathbf{B}}_{R\mathbf{d}} \\ \tilde{\mathbf{B}}_{0\mathbf{d}}^T & \tilde{\mathbf{B}}_{1\mathbf{d}}^T & \cdots & \tilde{\mathbf{B}}_{R\mathbf{d}}^T & \mathbf{G}^T\mathbf{M}\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b}^1 \\ \vdots \\ \mathbf{b}^R \\ \mathbf{d}^0 \end{bmatrix}, \tag{29}$$

where

$$\tilde{\mathbf{A}}_{00} = \begin{bmatrix} \mathbf{A}_{00} & \\ & \tilde{\mathbf{A}}_{\mathbf{a}_2\mathbf{a}_2} \end{bmatrix}, \tag{30a}$$

$$\tilde{\mathbf{A}}_{rr} = \begin{bmatrix} \mathbf{A}_{rr} & \\ & \tilde{\mathbf{A}}_{\mathbf{b}_2^r\mathbf{b}_2^r} \end{bmatrix}, \tag{30b}$$

$$\tilde{\mathbf{B}}_{00} = \begin{bmatrix} \mathbf{B}_{00} & \\ & \tilde{\mathbf{B}}_{\mathbf{a}_2\mathbf{a}_2} \end{bmatrix}, \tag{30c}$$

$$\tilde{\mathbf{B}}_{rr} = \begin{bmatrix} \mathbf{B}_{rr} & \\ & \tilde{\mathbf{B}}_{\mathbf{b}_2^r\mathbf{b}_2^r} \end{bmatrix}, \tag{30d}$$

$$\tilde{\mathbf{B}}_{o\mathbf{d}} = [\mathbf{B}_{0\mathbf{d}} \;\; \tilde{\mathbf{B}}_{\mathbf{a}_2\mathbf{d}}], \tag{30e}$$

$$\tilde{\mathbf{B}}_{r\mathbf{d}} = [\mathbf{B}_{r\mathbf{d}} \;\; \tilde{\mathbf{B}}_{\mathbf{b}_2^r\mathbf{d}}], \tag{30f}$$

with $r>0$ and nonzero elements of the matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are

$$(\tilde{\mathbf{A}}_{00})_{ij} = \xi_i^2\delta_{ij}, \quad i,\, j \in \mathcal{N}[1,\, M^0], \tag{31a}$$

$$(\tilde{\mathbf{A}}_{rr})_{ij} = (\nu_i^r)^2\delta_{ij}, \quad i,\, j \in \mathcal{N}[1,\, M^r], \tag{31b}$$

$$\mathbf{S} = \mathbf{S}^0 + \sum_{r=1}^{R} (\hat{\mathbf{L}}^r)^T\mathbf{S}^r\hat{\mathbf{L}}^r, \tag{31c}$$

$$(\tilde{\mathbf{B}}_{00})_{ij} = \delta_{ij}, \quad i,\, j \in \mathcal{N}[1,\, M^0], \tag{31d}$$

$$(\tilde{\mathbf{B}}_{rr})_{ij} = \delta_{ij}, \quad i,\, j \in \mathcal{N}[1,\, M^r], \tag{31e}$$

$$(\tilde{\mathbf{B}}_{0\mathbf{d}})_{ij} = \langle \boldsymbol{\phi}_i,\, \mathbf{M}^0\mathbf{G}^0\rangle_j, \quad i \in \mathcal{N}[1,\, M^0], \; j \in \mathcal{N}[1,\, d], \tag{31f}$$

$$(\tilde{\mathbf{B}}_{r\mathbf{d}})_{ij} = \langle \boldsymbol{\psi}_i^r,\, \mathbf{M}^r\mathbf{G}^r\hat{\mathbf{L}}^r\rangle_j, \quad i \in \mathcal{N}[1,\, M^r], \; j \in \mathcal{N}[1,\, d], \tag{31g}$$

$$\mathbf{G}^T\mathbf{M}\mathbf{G} = (\mathbf{G}^0)^T\mathbf{M}^0\mathbf{G}^0 + \sum_{r=1}^{R} (\hat{\mathbf{L}}^r)^T(\mathbf{G}^r)^T\mathbf{M}^r\mathbf{G}^r\hat{\mathbf{L}}^r. \tag{31h}$$

If the exact eigenvalue problem of Eq. (29) is rearranged so that the coefficients associated with the component modes included in the analysis are separated from those associated with the excluded modes, it becomes

$$\begin{bmatrix} \mathbf{A} & A_{12} \\ \mathbf{A}_{12}^T & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \tilde{\omega}^2 \begin{bmatrix} \mathbf{B} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^T & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}, \tag{32}$$

where

$$\mathbf{x}_1 = [a_1 \;\; a_2 \;\cdots\; a_{N^0} \;\; b_1^1 \;\; b_2^1 \;\cdots\; b_{N^R}^R \;\; (\mathbf{d}^0)^T]^T, \tag{33a}$$

$$\mathbf{x}_2 = [a_{N^0+1} \;\; a_{N^0+2} \;\cdots\; a_{M^0} \;\; b_{N^1+1}^1 \;\; b_{N^1+2}^1 \;\cdots\; b_{M^R}^R]^T. \tag{33b}$$

are, respectively, the vectors of expansion coefficients associated with those modes included in the analysis and those excluded. The submatrices $\mathbf{A}$ and $\mathbf{B}$ in Eq. (32) are the matrices associated with the truncated modal synthesis eigenvalue problem and are defined by Eq. (27). The other submatrices are

$$\mathbf{A}_{12} = \mathbf{0}, \tag{34a}$$

$$\mathbf{A}_{22} = \begin{bmatrix} \mathbf{A}_{\mathbf{a}_2\mathbf{a}_2} & & & \\ & \mathbf{A}_{\mathbf{b}_2^1\mathbf{b}_2^1} & & \\ & & \ddots & \\ & & & \mathbf{A}_{\mathbf{b}_2^R\mathbf{b}_2^R} \end{bmatrix}, \tag{34b}$$

$$\mathbf{B}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & \vdots \\ \mathbf{0} & & & \mathbf{0} \\ \mathbf{B}_{\mathbf{a}_2\mathbf{d}}^T & \mathbf{B}_{\mathbf{b}_2^1\mathbf{d}}^T & \cdots & \mathbf{B}_{\mathbf{b}_2^R\mathbf{d}}^T \end{bmatrix}, \tag{34c}$$

$$\mathbf{B}_{22} = \begin{bmatrix} \mathbf{B}_{\mathbf{a}_2\mathbf{a}_2} & & & \\ & \mathbf{B}_{\mathbf{b}_2^1\mathbf{b}_2^1} & & \\ & & \ddots & \\ & & & \mathbf{B}_{\mathbf{b}_2^R\mathbf{b}_2^R} \end{bmatrix}. \tag{34d}$$

If the coefficient vector $\mathbf{x}_2$ is eliminated from Eq. (32) and the fact that $\mathbf{A}_{12}=\mathbf{0}$ is applied, then the relation

$$[(\mathbf{A} - \tilde{\omega}^2\mathbf{B}) - \tilde{\omega}^4\mathbf{B}_{12}(\mathbf{A}_{22} - \tilde{\omega}^2\mathbf{B}_{22})^{-1}\mathbf{B}_{12}^T]\mathbf{x}_1 = \mathbf{0} \tag{35}$$

is obtained. Hence, if the matrix $\bar{\omega}^4 \mathbf{B}_{12}(\mathbf{A}_{22}-\bar{\omega}^2\mathbf{B}_{22})^{-1}\mathbf{B}_{12}^T$ is small compared to $(\mathbf{A}-\bar{\omega}^2\mathbf{B})$, the approximate eigenvalues and eigenvectors of the truncated modal synthesis problem, ω_i^2 and $\mathbf{x}_i$, will be close to those of the exact problem, $\bar{\omega}_i^2$ and $\bar{\mathbf{x}}_i$. Because the excluded component modes are usually not known, the coefficients of matrices other than **A** and **B** in Eq. (35) are also unknown, and error bounds may not be readily determined. However, observe, that coupling between the included modes and excluded modes in the modified formulation is only due to the matrix $\mathbf{B}_{12}$, while in the original formulation they were also coupled by a nonzero matrix $\mathbf{A}_{12}$. In addition, the matrix $\mathbf{B}_{12}$ of the modified formulation is more sparsely populated than the corresponding matrix of the original formulation. This suggests that the modified formulation may be inherently less susceptible to modal truncation than the original. The inclusion principle of Eq. (17) is also valid for the modified formulation.

VI. EXAMPLE: MODIFIED FORMULATION APPLIED TO A JOINED BEAM SYSTEM

The fixed/fixed beam system was again analyzed using the modified formulation of Sec. IV. The assembly is shown in Fig. 1(a), with main system and subsystem components as indicated in Fig. 1(d) and (e), respectively. Fixed interface conditions are imposed on both components. The component modal properties are obtained from Eq. (18), with fixed/fixed beam eigenvalues[28] used for both components. The subsystem static influence function matrix is again obtained from Eq. (19), and that of the main system is

$$\mathbf{G}^0(x)=[\mathbf{g}_1^0(x) \ \ \mathbf{g}_2^0(x)], \tag{36a}$$

where

$$\mathbf{g}_1^0(x)=3(x/l^0)^2-2(x/l^0)^3, \tag{36b}$$

$$\mathbf{g}_2^0(x)=l^0[(x/l^0)^3-(x/l^0)^2]. \tag{36c}$$

The external stiffness matrix for each component is obtained from Eq. (19d), with the length, cross section, and material properties appropriate for each component applied. The main system and subsystem modes and static influence function matrices are evaluated at K^0 and K^1 discrete points, respectively. Because only two components are involved, the constraint operator $\hat{\mathbf{L}}^1$ is an identity operator.

Natural frequencies and modes of the assembly were obtained from the modified formulation. The joint was again located at the center of the beam, and the beam properties used were the same as those of Sec. III. Component modes and static influence function matrices were evaluated at $K^0=K^1=100$ points. Natural frequency estimates for the first ten modes appear in Table II. The estimates converge to the corresponding exact values very rapidly as the number of component modes is increased. In fact, the estimates for the natural frequencies for symmetric modes $(n=1,3,\ldots)$ obtained from the modified formulation using only ten component modes $(N^0=N^1=5)$ are more accurate than those obtained from the original formulation using 40 component modes $(N^0=N^1=20)$. Errors in estimates for the first natural frequency obtained from both formulations with varying numbers of component modes are shown in Fig. 8. The slope

TABLE II. Natural frequency estimates for fixed/fixed beam from modified formulation.

Modal index n	Natural frequency f_n (Hz) $N^0=N^1=5$	$N^0=N^1=10$	$N^0=N^1=15$	$N^0=N^1=20$	Exact value
1	0.2570	0.2570	0.2570	0.2570	0.2570
2	0.7084	0.7084	0.7084	0.7084	0.7084
3	1.3888	1.3887	1.3887	1.3887	1.3888
4	2.2956	2.2955	2.2955	2.2955	2.2955
5	3.4312	3.4293	3.4292	3.4292	3.4292
6	4.7908	4.7896	4.7896	4.7896	4.7893
7	6.3910	6.3775	6.3771	6.3770	6.3765
8	8.2015	8.1906	8.1904	8.1903	8.1906
9	10.3026	10.2333	10.2314	10.2312	10.2307
10	12.5763	12.4991	12.4982	12.4975	12.4982

of the error for the original formulation becomes nearly horizontal as the number of component modes increases, indicating poor convergence. In contrast, the modified formulation exhibits excellent convergence characteristics. Figure 8 indi-

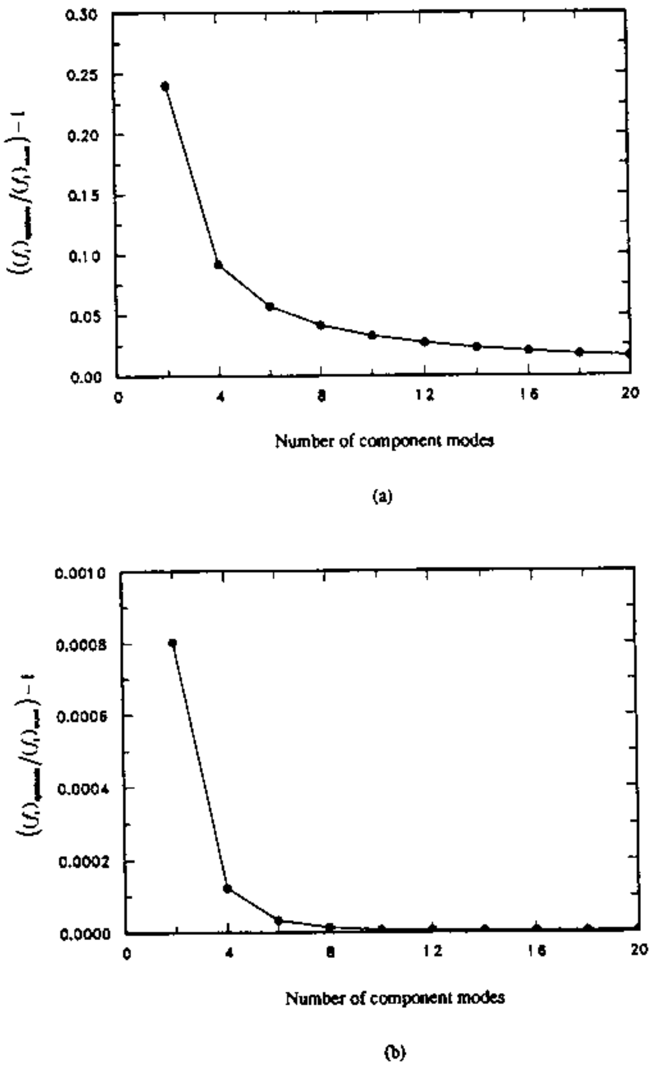


FIG. 8. Convergence of estimates for first natural frequency of fixed/fixed beam assembly. (a) Results from original formulation of Sec. I. (b) Results from modified formulation of Sec. IV.

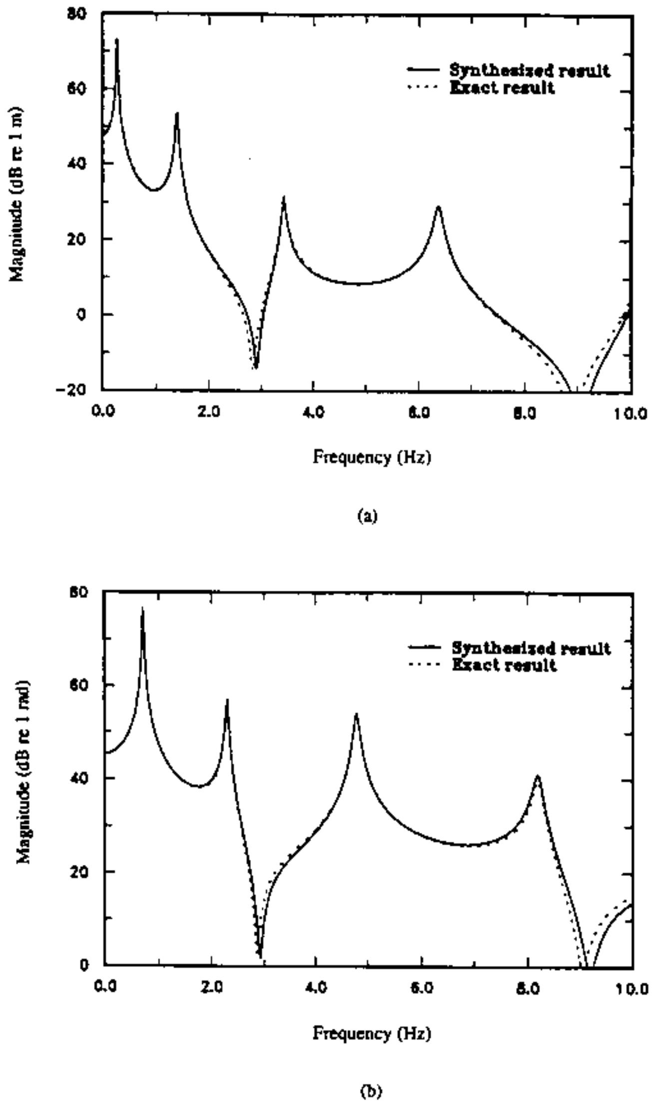


FIG. 9. Motions at joint location for fixed/fixed beam obtained from modified formulation of Sec. IV. (a) Transverse displacement. (b) Rotary displacement.

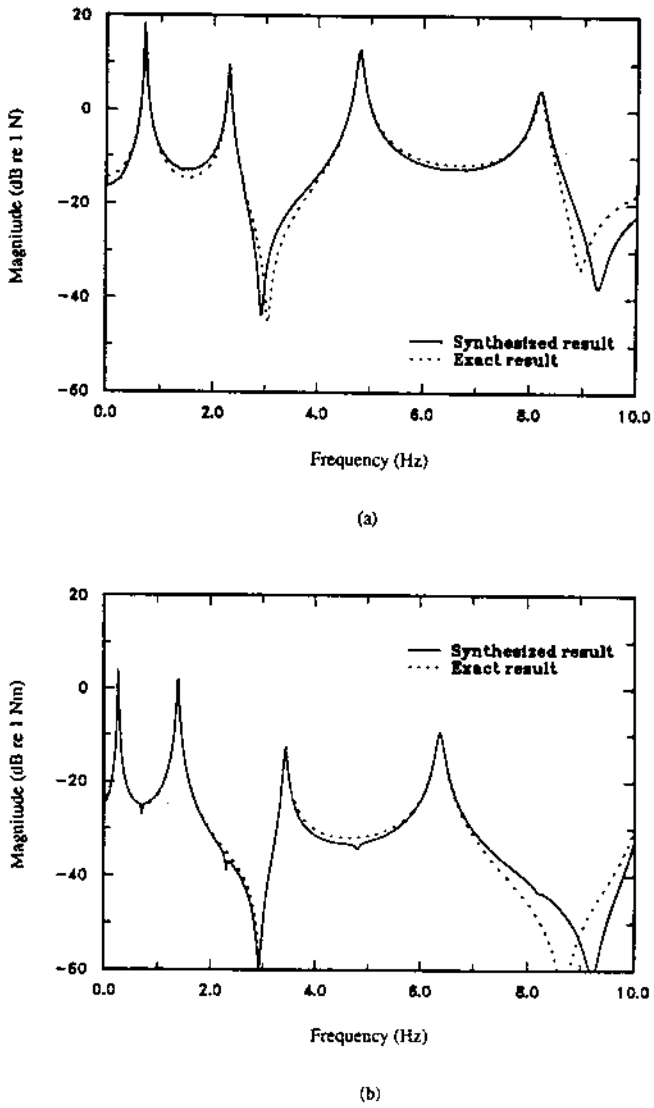


FIG. 10. Forces at joint location for fixed/fixed beam obtained from modified formulation of Sec. IV. (a) Transverse force. (b) Bending moment.

cates that errors in the first natural frequency estimates obtained from the modified formulation are two orders of magnitude less than those in the estimates obtained from the original formulation. The improved performance of the modified formulation is due to greatly improved estimates for assembly mode shape deflection and shape at the joint. Assembly modes obtained from the modified formulation using the first ten modes of each component were found to be visually indistinguishable from the corresponding exact results over the entire domain.

The degree of coupling between included and excluded component modes was investigated numerically for both formulations. The matrices **A** and **B** corresponding to both the original and modified formulation eigenvalue problems of Eqs. (4) and (26), respectively, were computed using 100 component modes ($N^0 = N^1 = 50$). Since both components in this example have an infinite number of modes, these were assumed to be reasonable approximations to the matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ of the corresponding exact eigenvalue problems of Eqs. (11) and (29). Truncated analyses in which only 20 component modes were included ($N^0 = N^1 = 10$) were considered for both formulations, and measures of the coupling between the 20 modes included and the 80 excluded were sought, as defined by Eqs. (16) and (35), respectively, for the original and modified formulations. The degree of coupling was estimated by computing the ratios of the norms of the coupling matrices $\mathbf{A}_{12}$ and $\mathbf{B}_{12}$ defined by Eqs. (15) and (34) to those of the corresponding truncated modal synthesis matrices **A** and **B**. Results for the original formulation are $\|\mathbf{A}_{12}\|/\|\mathbf{A}\| = 0.0425$ and $\|\mathbf{B}_{12}\|/\|\mathbf{B}\| = 0.7703$, and results for the modified formulation are $\|\mathbf{A}_{12}\|/\|\mathbf{A}\| = 0$ and $\|\mathbf{B}_{12}\|/\|\mathbf{B}\| = 0.0062$. Hence, the coupling between included and excluded modes in this example is much stronger for the original formulation than it is for the modified formulation, as the analyses of Sec. II and Sec. V suggest.

The forced response of the assembly was computed for the same excitation as the example of Sec. III. The motions at the joint location are shown in Fig. 9. Both components of joint motion are in excellent agreement with the exact results, and the joint rotation of Fig. 9(b) is significantly more

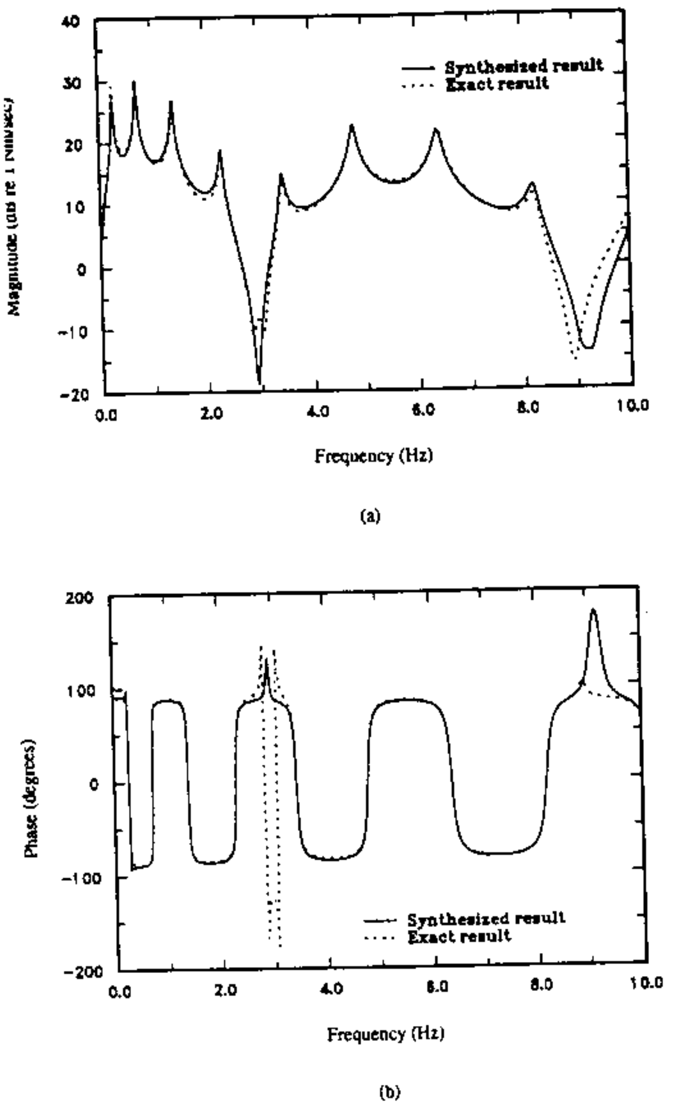


FIG. 11. Power transmitted by transverse force/transverse displacement for fixed/fixed beam obtained from modified formulation of Sec. IV. (a) Magnitude. (b) Phase.

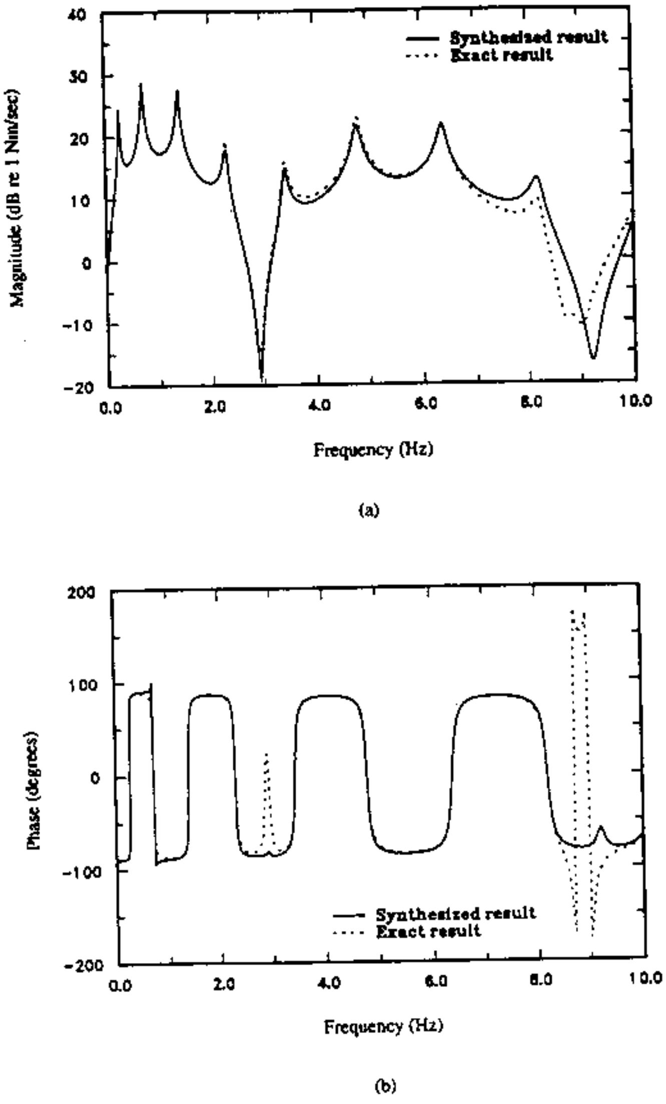


FIG. 12. Power transmitted by moment/rotation for fixed/fixed beam obtained from modified formulation of Sec. IV. (a) Magnitude. (b) Phase.

accurate than that of Fig. 3(b), obtained from the original formulation. Transverse force and bending moment magnitudes at the joint appear in Fig. 10, and both agree closely with corresponding exact results. The complex power transmitted through the transverse force/displacement path and moment/rotation path are shown in Figs. 11 and 12, respectively. Magnitude and phase agree closely with the exact solutions for both cases. Both complex power results are more accurate than those of Figs. 5 and 6 obtained from the original formulation, but the power transmitted through the moment/rotation path exhibits particular improvement. The real parts of the power transmitted through each path appear in Fig. 13. The transmitted power is concentrated in narrow bands near the assembly natural frequencies, as expected, and the agreement between estimated and exact results is good at most frequencies. However, there are two exceptions. The estimates for real power transmitted through the force/displacement path at the first natural frequency and through the moment/rotation path at the second natural frequency are both negative, which is not possible as the subsystem is not directly excited. The magnitudes of the complex power estimates at both of these frequencies match the corresponding exact results very well. The reason for the errors is the behavior of the phase of the complex power estimates in the neighborhoods of these frequencies. In both cases, the phase abruptly changes from +90° to −90° over narrow frequency bands due to low damping. Consequently, although the phase estimates are generally quite good, significant phase errors occur in these bands. This phase sensitivity is an inherent difficulty associated with estimating the real power transmitted among components. Since it is unlikely that the abrupt changes in the complex power phase in lightly damped structures can be estimated with sufficient accuracy from a reasonable number of component modes, it may be better to examine the complex power magnitude and phase directly and note those frequency bands with significant phase sensitivity than to examine the real transmitted power estimates directly.

Errors in the transmitted real power estimates obtained from both formulations using increasing numbers of compo-

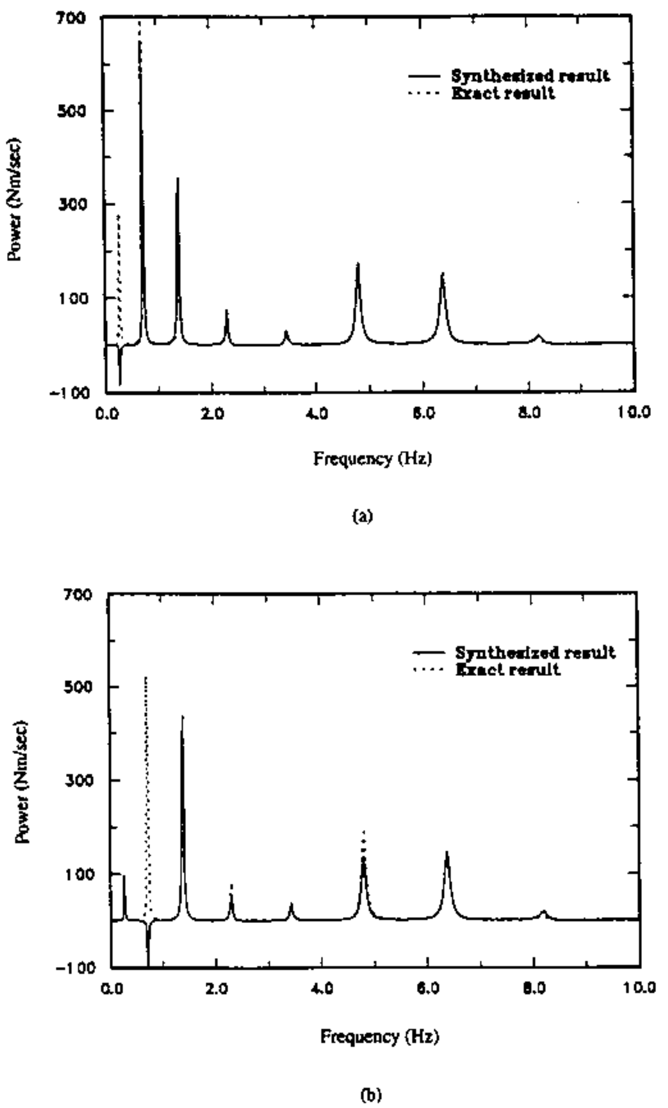


FIG. 13. Real power transmitted through joint for fixed/fixed beam obtained from modified formulation of Sec. IV. (a) Transverse force/transverse displacement path. (b) Moment/rotation path.

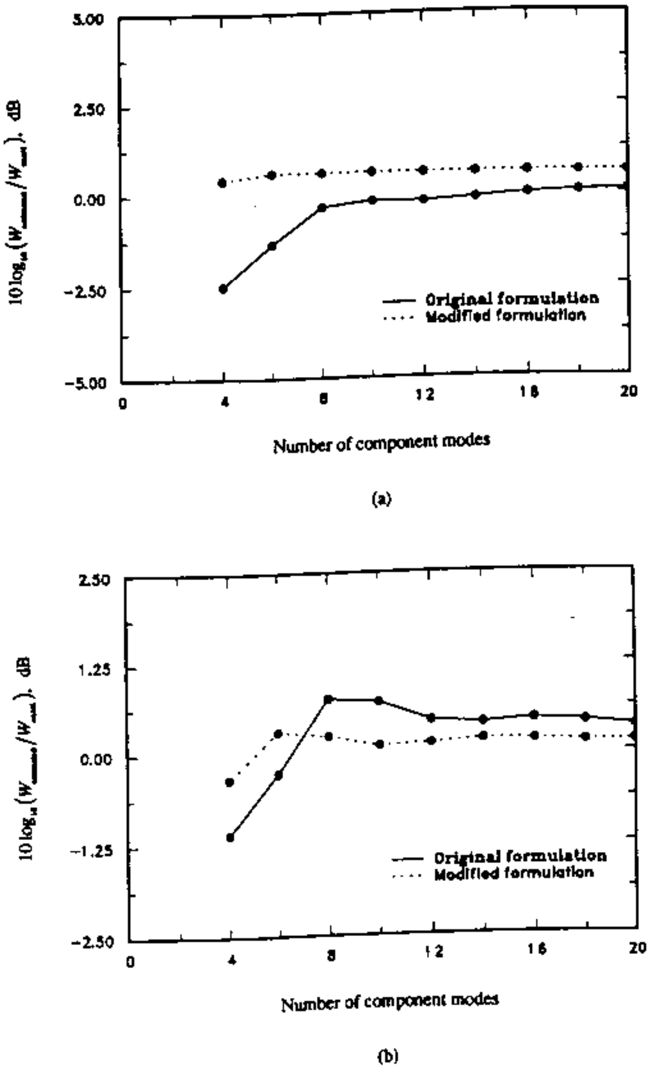


FIG. 14. Errors in real transmitted power estimates at third assembly natural frequency for fixed/fixed beam. (a) Errors for transverse force/transverse displacement path. (b) Errors for bending moment/rotation path.

nent modes appear in Fig. 14. Estimates for power transmitted at the third assembly natural frequency through both transmission paths obtained from the modified formulation converge to accurate values with many fewer modes included than corresponding estimates obtained from the original formulation. Figure 14(b) indicates that including only six component modes in the modified formulation produced results more accurate than those obtained from the original formulation using 20 modes. Figure 14(a) indicates that for larger numbers of modes, the original formulation produced a power estimate for the transverse force/displacement path which is more accurate than that of the modified formulation at the third natural frequency. However, the errors from the modified formulation in Fig. 14(a) converge to their asymptotic values with fewer modes than those of the original formulation.

VII. CONCLUSIONS

The effects of using incomplete sets of component modes on estimates for assembly modal properties and vibration transmitted among components have been examined. Two significant developments emerge from this study. First, an approach for determining errors in assembly modal property estimates obtained from modal synthesis formulations due to component modal truncation has been presented and applied. Second, a formulation has been developed which permits transmitted vibration in machine assemblies to be accurately predicted in terms of the modal properties of its individual components, even when only a small subset of component modes is used.

It has been demonstrated that accurate estimates of transmitted mechanical power require assembly modes which are locally accurate at the joint locations, in addition to accurate natural frequency estimates. It has also been shown that the fixed/free modal synthesis formulation used in the authors' earlier paper[1] is susceptible to modal truncation errors, and a more robust alternative formulation been developed. The modified formulation retains the advantages of the original, including the unique physical insight which the method affords, since the vibration transmitted through various paths may be expressed in terms of the modal properties of individual components. The examples considered indicate that the modified formulation works well, but care must be used when interpreting results due to abrupt changes in the phase of the complex power near resonance frequencies. Clearly, all assembly modes with significant participation factors in the frequency range of interest must be included to ensure accuracy, and sufficient component modes must be included to ensure that all assembly modes needed will be predicted accurately. The examples considered suggest that the number of component modes included should exceed the number of desired assembly modes by a factor of 1.2–1.5 for accurate results, but these numbers may be specific to the problem considered. The effects of component modal truncation on estimated assembly natural frequencies and modes have been quantified mathematically for fixed/free and fixed/fixed interface modal synthesis formulations. This fills a void in the literature, as few previous investigations have addressed this issue.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the financial support of this work by Copeland Corporation, Edison Welding Institute, and The American Society of Heating, Refrigeration, and Air Conditioning Engineers.

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