Identification and quantification of stick-slip induced brake groan events using experimental and analytical investigations

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To describe the brake creep-groan phenomenon and several types of stick-slip motions, we propose both analytical and experimental investigations for an automatic transmission equipped vehicle. A lumped torsional model is employed to approximate the dynamics of the real mechanical system. This model assigns inertia to the drive, brake rotor, brake caliper and tire/vehicle, and by appropriate algorithms the simulation time histories include the effects of the friction non-linearity coupling brake and rotor. We consider how to force this particular system, from what physical state, and finding appropriate parameters and solutions. Important computational issues, as related to the stick-slip or slip-stick transitions, are addressed with the algorithms. Driving forces are assigned with two functions, one appropriate for comparison with the test and the other to study stick-slip orbits, which have been found to have various types of motion depending on the controlling brake actuation parameters. Since the groan features a discontinuous friction force and finite-repeating motions some comparisons are made in the frequency and time-frequency domains. These demonstrate the expected stick-slip frequency and multiple orders. © 2008 Institute of Noise Control Engineering.

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1 INTRODUCTION

The brake creep groan type noise and vibration problem may occur in vehicles when the driver slowly moves some small distance and stops (e.g. in stop-go traffic, at traffic lights, in garage maneuvers). In automatic transmission (AT) vehicles, the torque converter impeller applies small torque on the stationary turbine and hence the rest of the powertrain. With slow pedal release the brake friction torque will be overcome by the driving torque (impeller) and the brake rotor will start to slip against the brake pad. This initial slip of the rotor excites vibration in the driveline and brake subsystems, allowing a condition where the rotor and pad may stick again. In practice this stick-slip motion may repeat until the rotor and pad reach some steady-sliding state, or stick and even halt the vehicle. The friction-induced motion is highly dependent on pedal actuation. Groan may also occur on brake application as reported by Brecht et al.; it typically can be a long event for brake release on takeoff and a short event for vehicle braking to stop. Jang et al. suggest approaches to minimize the vibration by increasing damping or inertia of the rotating bodies and by reducing system compliance. Kim et al. examined the effect of particle sizes of pad constituents and concluded that groan was more severe with smaller abrasive particles with respect to filler or lubricant particle sizes (such as rubber and graphite). Further studies of the material compositional effects on groan, such as the static to kinetic ratio and velocity dependence of the friction coefficient are given in Ref. 2. By using a brake test bench they made correlations between a reduction in amplitude of friction force oscillations with reduced friction ratio and proposed new pad material formulations to give these characteristics. Bettella et al. examined the path to receiver and the influence of panels, windscreen and acoustic cavity. By examining the transmission losses in the airborne path they concluded that the groan noise is structure-borne. Their sound pressure and acceleration data show a fundamental frequency at 60 Hz (stick-slip) and multiple orders, due to the discontinuity in the friction force; these modes should be separated from those of the acoustic cavity. Donley and Riesland examined the significant

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flexible bodies of a McPherson strut system by using a multi-body dynamics software package. They calculated frequency response functions for relative displacement between the rotor and the pad (with unit force applied at the rotor/pad interface); several modes in the range of 30–100 Hz are evident in the compliance function. This is of relevance to our estimate of brake subsystem modes. Our approach to understanding brake groan draws inspiration from many other friction-induced problems such as those reviewed by Martins et al.\textsuperscript{6}.

Brake groan remains a complex engineering problem that has yet to be fully understood or analyzed. To aid this void in the literature, we propose an analytical investigation of four-degree-of-freedom torsional model (as shown in Fig. 1). It includes driveline and brake torsional subsystems, with friction interface through the brake rotor and pad. This model should conceptually describe the brake creep-groan phenomenon and several types of stick-slip motions, and yet it could be easily extended. Note that two dynamic subsystems are coupled by a friction interface, an important aspect of the non-linear model (as illustrated by Figs. 1 and 2. The frequency ratio for the subsystem modes dominating the stick-slip motions is not near zero (or infinite), hence neither the brake or powertrain subsystems may be considered as a rigid body. Many parameters and conditions affect stick-slip orbit(s) including, but not limited to, system modes, friction characteristics, driving torque magnitude, rate and magnitude of brake release, rate and magnitude of brake reapplication and hydraulic system dynamics.

Fig. 1—Torsional model utilized for brake groan analysis: (a) Driveline subsystem, (b) Brake subsystem and (c) Rotor/tire side view. System parameters are as follows: Inertia, $J_d = 9.54$, $J_r = 0.652$, $J_b = 0.189$ and $J_d = 73.1$ kg m$^2$; Stiffness, $k_d = 8790$, $k_b = 68000$, $k_t = 22000$ N m rad$^{-1}$; Damping, $c_d = 1.15$, $c_b = 4.53$, $c_t = 13$, $d_d = 23.9$ and for Case I brake force $d_t = 600$, for Case II, $d_t = 200$ N ms rad$^{-1}$.

Fig. 2—Brake friction force as a function of sliding velocity.
2 PROBLEM FORMULATION

We establish the following objectives for an AT equipped vehicle: a) Obtain and analyse example set of real system data; b) Develop a groan model in terms of two dynamic subsystems coupled with friction interface and briefly identify relevant system parameters; c) Examine the computational issues associated with stick-slip motion simulation and obtain solutions to the creep-groan response; and d) Show a preliminary correlation between theory and experiment. The lumped torsional model is employed to approximate the dynamics of the real mechanical system. The model assigns inertia to the drive, brake rotor, brake caliper and tire/vehicle, and by appropriate algorithms the simulation time histories may include effects of the friction non-linearity (Fig. 2) coupling brake and rotor. We consider how to force this system, from what physical state, and finding appropriate parameters and solutions. In doing so the driveline subsystem (Fig. 1(a)) has parameters determined from a sensible reduction and the brake subsystem (Fig. 1(b)) parameters are estimated from modal testing. For the initial value problems there are important computational issues, as related to the stick-slip or slip-stick transitions, which must be addressed before programming solution algorithms. To relate solutions to the vehicle test we attempt to reproduce roughly a function for brake pressure as controlled by the driver (Fig. 3(a)) and compare the time domain characteristics of the transient stick-slip motions to those measured. Next we seek an analytical insight by implementing a brake release function (Fig. 3(b)), that leads to steady state stick-slip orbits, which have been found to have various types of motion depending on the controlling brake actuation parameters. That the groan features a discontinuous friction force and finite-repeating motions allows for comparisons in the frequency domain for the expected stick-slip frequency and multiple orders. Thus the frequency content is examined for simulated and test signals including the transient ebb and flow of real groan signals via both time and frequency domain analyses.

3 MEASURED BRAKE GROAN EVENTS

3.1 Vehicle Experiment

A typical medium sized passenger vehicle is used to duplicate the driver behavior of creeping forward in almost stationary traffic or at traffic lights. The vehicle is stationary on flat ground, the engine is idling and the applied brake is resisting a small amount of torque from the torque converter. The driver releases the brake very slowly until groan occurs and with careful actua-

Fig. 3—Functions for transient brake normal force, \( F_n(t) \): a) Case I with various profiles—1. \( \beta =300 \text{ N}, \tau =0.05 \text{ s}, \alpha =0.1 \text{ s}, 2. \beta =240 \text{ N}, \tau =0.1 \text{ s}, \alpha =0.15 \text{ s}, 3. \beta =180 \text{ N}, \tau =0.15 \text{ s}, \alpha =0.2 \text{ s and 4.} \beta =100 \text{ N}, \tau =0.2 \text{ s}, \alpha =0.25 \text{ s}; (b) Case II with \( \tau =1, \beta =30, \gamma_1 =0.95, \gamma_2 =0.90, \gamma_3 =0.995. \) Note: \( F_n(t) =F_m -\beta \tanh \sigma t, \sigma =-1/\tau \ln([1-\gamma_2])/(1+\gamma_2)^{-1} \text{ and } \alpha =-1/2\sigma \ln((1-\gamma)\gamma^{1}) \). see Ref. 7.

Fig. 4—Accelerometer locations for brake groan vehicle test.
with the driver carefully inducing roughly 15 groan events, each of around 0.5–2 s duration. The brake is fully applied between groan events.

Multiple intermittent brake groan events are shown for measured accelerations in Fig. 5. Events A, B, E and G are a short creak sound, rather than sustained groan; here the driver has released the brake only slightly past the point of initial slip before reapplication. See Fig. 6 for a close up look of event B; evident is an impulse, or two, followed by a decay transient with rich spectra. Events C, D, F, H and I have a more continuous repetition of this motion over a finite duration. Event F is quite sustained, Fig. 7 provides the first 0.8 s (19.8–20.6 s); here the groan is more transient until settling to a fairly steady state for 1 s (20.6–21.6 s), as shown by Fig. 8. The impulses are repeating at close to 75 Hz until the groan dies away fairly abruptly at around 22.1 s.

### 3.2 Quantification of Groan Events

Groan severity could be quantified from time domain data in various ways, such as the length of the groan event, peak to peak range and mean-square quantities. As follows are a set of metrics (given by $Q_i$ with subscript $i$) illustrated via the test data. First we examine the “steady-state” response given by Figs. 8 and 9 and compare to a “no-groan” record of the same length, $T$ (though such a figure is not shown). For the no-groan case the vehicle is idling and stationary and the accelerometers and microphones pick up vibrations from engine firing and background noise. The peak to peak range is $Q_1 = \psi_{\text{max}} - \psi_{\text{min}}$, for all $T$ and mean-square values (a ‘power’ like quantity from the perspective of signal processing) are defined as $Q_2 = \frac{1}{T_2} \int_{t_a}^{t_b} \psi(t)^2 \, dt$, with $T = t_b - t_a$. Assignments for measured signals are, acceleration, $\psi = \ddot{x}$ and sound pressure, $\psi = P$. The metric $Q_1$ (Table 1) shows accelerations for “steady-state” groan are at least an order of magnitude greater than for no groan. For sound pressure the results are not as distinct. Values of $Q_2$, give a similar finding, accelerative “power” is distinctly greater for the groan case and sound “power” (mean-square pressure) gives mixed results. This gives an
understanding of which time domain measurements will give a clear representation for groan events. The accelerometers (say at the suspension strut which is a practical location) give groan readings clearly distinct from engine vibrations. Metric 3 is the length of a groan event, $Q_3 = t_b - t_a$, from the first significant acceleration spike, at $t_a$, (quite clear in Fig. 7 at approximately 19.82 s) to the last acceleration spike at $t_b$. Note, to “quantify” what is a significant acceleration spike we have taken approximately 12.5% of the peak to peak range given in Table 1 as the cutoff magnitude,

![Fig. 7—Acceleration time histories for initial non-steady state section of groan event F: a) Suspension strut fore-aft accelerometer; b) Caliper tangential accelerometer.](image)

![Fig. 8—Acceleration time histories for steady state portion of groan event F: a) Suspension strut fore-aft accelerometer; b) Caliper tangential accelerometer.](image)

![Fig. 9—Sound pressure time histories for steady state portion of groan event F: a) Wheel rim microphone; b) Driver’s right ear position microphone.](image)
i.e. for suspension strut, \( \bar{x}_{cut} = \pm 0.25 \text{ g} \) and for caliper, \( \bar{x}_{cut} = \pm 0.5 \text{ g} \). Assigning such limits allows for easier decision making in data processing. Metric 4 is an ‘energy’ like quantity that is defined as: 
\[
Q_4 = \int_0^T \bar{\psi}^2(t) \, dt
\]
Unlike the “steady-state” sample of Fig. 8, most transient groan events grow and decay and have outlier minimum and maximum values, hence metrics \( Q_1 \) and \( Q_2 \) may be less appropriate measures. In Fig. 10, \( Q_3 \) and \( \sqrt{Q_4} \) are rank ordered by \( \sqrt{Q_4} \) for Events A-I. Event F is the worst case, in terms of \( Q_3 \) and \( \sqrt{Q_4} \), followed by Event D. Next is Event C which has higher “energy” than Event H yet was shorter in length; a longer event does not necessarily equate to more severe groan in terms of vibration energy, however, the receiver perception (driver) could be studied. Comparing with \( \sqrt{Q_4} \) (g s\(^{0.5}\)) rather than \( Q_4 \) (g\(^2\) s) gives units scaled better to \( Q_3 \) (s).

### 3.3 Identification of Groan in Frequency Domain

The example data of Figs. 8 and 9 are chosen for frequency domain analysis given the steady-state nature and Figs. 11 and 12 give corresponding spectra up to 300 Hz for the superimposed with the no-groan case. The engine idling contributes significantly to amplitudes under 50 Hz. The acceleration signals give the clearest account of groan with the highest peak at 76 Hz and near multiples at 152, 226, 303, 378 and 454 Hz, i.e. up to the first six orders of the dominant harmonic, and discernible peaks up to the 11\(^{th}\) order (frequencies above 300 Hz are not shown in the plots). As suggested by earlier authors\(^{1-5}\), brake groan is a stick-slip event and here the stick-slip frequency would be 76 Hz, with multiple orders due to the impulsive behavior shown in accelerations at stick-slip transitions.

This data can be further processed to establish the most significant frequency band for groan. Here we use a frequency domain averaging method to find the band-limited mean-square quantities, defined as 
\[
\bar{\psi}_{ms}(n) = \frac{1}{f_{b}f_{s}} \int_{f_{b}}^{f_{s}} \bar{\psi}(f) \, df,
\]
where \( n = 1, 2, \ldots, \lfloor f_{b}/(2m\Delta f) \rfloor \). The band limit, \( f_{b} \), is selected as \( f_{b}=m\Delta f, \) and we select \( |m\Delta f|=10 \text{ Hz} \). For \( \Delta f = 0.183 \text{ Hz} \), \( m=55 \) and \( f_{b}=10.07 \text{ Hz} \). This could be put as follows: with these parameters, all 55 frequency bins from the mean-square values of Figs. 11 and 12 are averaged into one single bin at \( f_{b}/2 \) and \( n \) multiples thereof. The resulting band-limited spectral comparisons are given in Figs. 13 and 14 for both groan and no-groan cases and these yield several insights. First, it can be seen that the caliper and strut have similar responses, with main peaks at the above mentioned stick-slip frequency and multiple orders, the impulse at every stick and slip is also exciting the system with a broadband effect. The noise is minimal at the wheel rim microphone compared to the driver right ear position. This matches the perception of those present for the test, i.e. for the human ear, brake groan seems louder or more perceptible within the vehicle than standing next to the wheel. The driver’s right ear microphone suggests that the receiver should perceive the transient

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**Table 1—‘Steady-State’ groan and no groan comparison metric \( Q_1 \): Peak to peak range and metric \( Q_2 \): Mean-square values.**

<table>
<thead>
<tr>
<th>Event Type</th>
<th>Suspension Strut</th>
<th>Caliper</th>
<th>Wheel Arch</th>
<th>Drivers Right Ear</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Groan</td>
<td>( Q_1 (g) )</td>
<td>( Q_2 (g^2) )</td>
<td>( Q_1 (g) )</td>
<td>( Q_2 (g^2) )</td>
</tr>
<tr>
<td>‘Steady-State’ Groan Ratio</td>
<td>20.1</td>
<td>430.0</td>
<td>29.9</td>
<td>162.8</td>
</tr>
<tr>
<td>Groan/No-Groan</td>
<td>0.101</td>
<td>4.65e-4</td>
<td>0.127</td>
<td>6.94e-4</td>
</tr>
<tr>
<td></td>
<td>2.038</td>
<td>0.200</td>
<td>3.801</td>
<td>0.113</td>
</tr>
</tbody>
</table>

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**Fig. 10—Groan event comparison with metrics \( Q_3 \) (thin lines) and \( \sqrt{Q_4} \) (thick line) for suspension strut (solid), Caliper (dotted). Events are rank ordered from ‘Best’ to ‘Worst’ based on \( Q_4 \) for suspension strut. Note \( Q_3 \) lines are overlaid at this magnification.**
groan events mostly in the 50 to 250 Hz bandwidth. The results show the noise is likely to be structure born and is exciting the acoustic cavity. A frequency domain metric could be defined as the area under the frequency curve over 50–250 Hz (or some other span) and used for comparative evaluations. Further the time domain metrics could be applied after signals are band pass filtered with suitable cutoffs to reduce extraneous affects from engine and background noise sources. For sound pressure measurements this would improve calculations of $Q_{1-4}$, but thought should be spared for the effect of filtering in the case of transient signals.

The complex values from short time FFTs may be used to construct waterfall plots (Fig. 15) showing the time-frequency content. The transient signals do bring to the spectral calculation process an averaging effect on frequencies and their amplitudes; yet the results are quite illustrative. To improve clarity for the figures, the lower cutoff frequency is selected as 50 Hz, removing high amplitude content from engine idling. The time-frequency characteristics of the five most major groan events C, D, F, H and I are captured. For each event, the variation in brake pressure leading to groan would be different, depending on how the driver actuated the pressure, also the length and mean square quantities are different (Table 1 and Fig. 10). Note that

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**Fig. 11**—Acceleration spectra for steady state portion of groan Event F (thick line) and no groan sample (thin line): a) Suspension strut fore-aft accelerometer; b) Caliper tangential accelerometer. Record length, $T=1$ s with a Hanning window. Sampled at 48 kHz, zero padded to $2^{18}$ points for a frequency resolution of $\Delta f=0.183$ Hz.

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**Fig. 12**—Sound pressure spectra for steady state portion of groan Event F (thick line) and no groan sample (thin line): a) Wheel rim microphone; b) Driver’s right ear position microphone. Record length, $T=1$ s with a Hanning window. Sampled at 48 kHz, zero padded to $2^{18}$ points for a frequency resolution of $\Delta f=0.183$ Hz.
the dominant spectral energy stays within a fairly narrow band around 76 Hz during the transient and steady state motions.

4 DEVELOPING A MODEL AND IDENTIFYING BRAKE SYSTEM PARAMETERS

4.1 Dynamic Model Reduction

Careful reduction of the powertrain is important for proper selection of parameters for the reduced model. For clarity, the process is illustrated in steps as shown in Fig. 16. The powertrain is initially discretised with the most significant inertias and compliant elements (Fig. 16(a)). Here we have $J_e$, the engine inertia, $J_{tc}$, the torque converter inertia and $J_{tr1}$, the inertia of the transmission input components, these three inertias rotate at engine speed (ignoring the torque converter dynamics). The reduced order model of the driveline is given by three inertias, $J_d$, representing the combined inertia of the powertrain elements from the engine to the final drive ring gear, $J_r$, the inertias of the brake rotor, wheel hubs and rims and $J_t$, the tire inertia and equivalent inertia of the vehicle mass. These inertias are connected via lumped stiffness elements, $k_d$, an equivalent drivetrain stiffness and $k_r$, the tire stiffness. Brake groan occurs at low speeds where the first gear is engaged, hence the transmission gears down with speed ratio, $n_{tr}$, the inertias of transmission output side components, $J_{tr2}$ and the final drive pinion, $J_{fd1}$, rotate at the corresponding speed (here the driveshaft shaft

Fig. 13—Band limited mean-square acceleration for steady state portion of groan Event F (dotted line) and no-groan steady state sample (solid line): a) Suspension strut fore-aft accelerometer; b) Caliper tangential accelerometer. Record length, $T=1$ s with a Hanning window. Sampled at 48 kHz, zero padded to $2^{18}$ points for a frequency resolution of $\Delta f=0.183$ Hz. Band limit, $f_b=m\Delta f=10.07$ Hz, where $[m\Delta f]=10$ Hz with $m=55$.


Fig. 14—Band limited mean-square sound pressure for steady state portion of groan Event F (dotted line) and no-groan steady state sample (solid line): a) Wheel rim microphone; b) Driver’s right ear position microphone. Record length, $T=1$ s with a Hanning window. Sampled at 48 kHz, zero padded to $2^{18}$ points for a frequency resolution of $\Delta f=0.183$ Hz. Band limit, $f_b=m\Delta f=10.07$ Hz, where $[m\Delta f]=10$ Hz with $m=55$. 

\begin{align*}
\text{Frequency Hz} &
\end{align*}

\begin{align*}
\text{Band Limited Power (g}^2\text{T}) &
\end{align*}

\begin{align*}
\text{Band Limited Power (Pa}^2\text{T}) &
\end{align*}

\begin{align*}
\text{Frequency Hz} &
\end{align*}
inertia can be split and lumped evenly to \( J_{tr2} \) and \( J_{fd1} \). Then the final drive unit gears down with speed ratio \( n_{fd} \) to the axle speed, where there are the rotating elements, \( J_{fd2} \), the final drive ring gear, \( J_{rl} \) and \( J_{rr} \), the left and right brake rotors and wheel rims and \( J_{tl} \) and \( J_{tr} \), the left and right tires combined with an equivalent vehicle inertia, \( J_v \). Note that \( J_{tl} = J_{tr} = J_{tire} + 0.5m_vr_t^2 \), where \( J_{tire} \) is the inertia of one tire, \( m_v \) the vehicle mass and \( r_t \) the tire radius. Two wheel braking is assumed for simplicity in later transient simulations, thus the engine drives and the brakes slow the whole vehicle mass via one pair of wheels. The stiffness connections are \( k_{is} \), transmission input shaft, \( k_{os} \), trans-

Fig. 15—Waterfall plot showing ‘averaged’ spectral contents of transient groan Events A-I as measured via suspension strut fore-aft accelerometer. Short time record length, \( T = 0.2 \) s with a Hanning window. Sampled at 48 kHz, zero padded to \( 2^{16} \) points for a frequency resolution of \( \Delta f = 0.732 \) Hz.

Fig. 16—Model reduction process for the driveline subsystem. System parameters for (a) are: Inertia, \( J_e = 0.2, J_{tr1} = 0.1, J_{fr1} = 0.02, J_{fr2} = 0.01, J_{fd1} = 0.005, J_{fd2} = 0.05; J_{rl} = J_{rr} = 0.65 \) and \( J_{tl} = J_{tr} = 73 \) kg m\(^2\) (inc. vehicle mass); Stiffness, \( k_{is} = 20000, k_{os} = 35000, k_{ps} = 30000, k_{al} = k_{ar} = 10000 \) and \( k_{tl} = k_{tr} = 22000 \) N m rad\(^{-1}\).
mission output shaft, $k_{ps}$, driveshaft, $k_d$, and $k_r$, left and right axles and $k_d$ and $k_r$, left and right tires. The driving torques include engine torque, $T_e(t)$, brake torque $T_b(t)$ and vehicle resistance torque $T_v(t)$ (see Sec. 5.3).

The first reduction step (Fig. 16(b)), modifies the components rotating at engine speed to equivalent parameter values at transmission output speed. Only braking of the right wheel is considered, hence the left axle and tire subsystem are now omitted, giving $J_r = J_{rr1} + J_{rr2} + J_{rr3}, J_i = J_{ir}$ and $k_a = k_{ar}, k_i = k_{ir}$; note $J_{rr}$ is composed of brake rotor, $J_{rr1}$, wheel hub, $J_{rr2}$ and wheel rim, $J_{rr3}$ inertias. The second reduction step, shown as Fig. 16(c), sums in series the spring elements between engine and final drive pinion and lumps engine, torque converter and transmission lumps together, giving:

$$J_{eq1} = n_{e1}^2(J_e + J_{lc} + J_{n1}) + J_{n2}, \quad (1a)$$

$$k_{eq1} = \left[\frac{1}{n_{e1}^2k_{lc}} + \frac{1}{k_{as}} + \frac{1}{k_{ps}}\right]^{-1} \quad (1b)$$

The final reduction step, shown as Fig. 16(d) and corresponding to Fig. 1(a), modifies the components now rotating at driveshaft speed to equivalent parameter values at the final drive ring gear speed. These parameters are then divided by two, considering that the model is for only the right axle, brake and tire subsystems:

$$J_d = 0.5[n_{d1}^2(J_{eq1} + J_{fd1}) + J_{fd2}] = 0.5[n_{d1}^2n_{e1}^2(J_e + J_{lc} + J_{n1}) + J_{n2}^2(J_{rr1} + J_{rr2} + J_{rr3})] \quad (2a)$$

$$k_d = \left[\frac{2}{n_{d1}^2k_{eq1}} + \frac{1}{k_{as}}\right]^{-1} \quad (2b)$$

$$T_d = 0.5n_{r1}n_{d1}T_e \quad (2c)$$

Note the brake subsystem includes $J_b$, an approximated inertia for the caliper and knuckle and $k_b$, the torsional stiffness of the suspension arms. Given the complexity of the suspension/brake knuckle assembly and for the sake of analysis regarding the stated objectives, these parameters will be defined via $w_b = \sqrt{J_b/k_b}$, the natural frequency of the first torsional mode of the caliper/knuckle/suspension system (see Sec. 4.2.2).

The brake groan model is designed so as to be applicable to both transient and stick-slip periodic analyses. Since brake groan is induced by the stick-slip phenomenon the equations of motion are given by the following set of differential equations where $J$, $C$ and $K$ represent inertia, viscous damping and stiffness matrices respectively, $T$ is the external torque vector and $\theta$ is the angular displacement vector.

$$\ddot{\theta} + C \dot{\theta} + K\theta = T(\dot{\theta}, t). \quad (3)$$

Note that the friction torque would introduce a piecewise non-linearity. Under the slipping condition the governing system is of dimension four and is given by $\theta = \{\theta_d \theta_r \theta_b \theta_t\}^T$ and

$$J = \text{diag}[J_d J_r J_b J_t]; \quad (4a)$$

$$K = \begin{bmatrix} k_d & -k_d & 0 & 0 \\ -k_d & k_d + k_t & 0 & -k_t \\ 0 & 0 & k_b & 0 \\ 0 & -k_t & 0 & k_t \end{bmatrix}; \quad (4b)$$

$$C = \begin{bmatrix} c_d + d_d & -c_d & 0 & 0 \\ -c_d & c_d + c_t & 0 & -c_t \\ 0 & 0 & c_b & 0 \\ 0 & -c_t & 0 & c_t + d_t \end{bmatrix}; \quad (4c)$$

$$T = \begin{bmatrix} T_d \\ -T_b \\ T_b \\ -T_v \end{bmatrix} \quad (4d)$$

The viscous damping is included via $c$ terms along with stiffness elements and inertial damping terms, $d_d$ and $d_b$, that are applied on the powertrain equivalent inertia and tire/vehicle inertia. For the sticking condition the dimension is reduced by one, giving, $\theta = \{\theta_d \theta_r \theta_b \theta_t\}^T$, where $\theta_{rb}$ indicates brake and rotor coordinates, and

$$J = \text{diag}[J_d J_r J_b J_t]; \quad (5a)$$

$$K = \begin{bmatrix} k_d & -k_d & 0 \\ -k_d & k_d + k_t & -k_t \\ 0 & -k_t & k_t \end{bmatrix}; \quad (5b)$$

$$C = \begin{bmatrix} c_d + d_d & -c_d & 0 \\ -c_d & c_d + c_t & -c_t \\ 0 & -c_t & 0 \end{bmatrix}; \quad (5c)$$

$$T = \begin{bmatrix} T_d \\ k_b \delta \\ -T_v \end{bmatrix}. \quad (5d)$$

The brake and rotor travel together when sticking and will become offset for slipping motions between sticking points. Defining $\delta = \theta_r - \theta_b$ as the relative displacement offset gives a torque offset $k_b \delta$ in Vector (5d). The vehicle displacement may be determined as $x_v = r_i \theta_i$. 

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4.2 Identification of System Parameters and Eigensolutions

4.2.1 Driveline subsystem

The powertrain system parameters are given similar values to those provided for models in Ref. 8. These parameters are given with Fig. 16. The vehicle mass is \( m_v = 1500 \text{ kg} \) while transmission and final drive speed ratios are \( n_{tr} = 2.4 \) and \( n_{fd} = 3.2 \), respectively. The tire radius is \( r_t = 0.31 \text{ m} \). Using the reduction method the driveline parameters are obtained as given with Fig. 1.

4.2.2 Brake subsystem

A floating caliper type brake system (Fig. 17) of a common vehicle is used for obtaining approximate parameters for our model. Note that it is not the same brake as for the experiment of Sec. 3 but it is a similarly sized vehicle. With experimental modal analysis the brake system is found to have several modes, in the region of 20 to 150 Hz. Distinctive motions include bending around a longitudinal axis (fore-aft vehicle), bending around a vertical axis and twisting around a transverse axis (though the vehicle). Naturally the brake and suspension system is complex to model and thus it is desirable to develop a small lumped system applicable for non-linear analyses. Given the moment arm transmitting the brake friction force as torque on the suspension arms, it is suspected that the twisting motion of the caliper and knuckle around some centre of rotation (on a transverse axis) would be excited significantly by stick-slip dynamics at the rotor/pad interface. With this in mind, four particular results from the modal tests are considered (Fig. 18). The accelerometer mounting positions are shown in Fig. 17: a) Location A: mounted in the vertical plane atop the fixed part of the brake caliper; b) Location B: mounted in the transverse plane on the upper control arm. An impact hammer is used to excite the system in the vertical plane (by Location A) and the acceleration type frequency response functions are obtained with a digital signal analyzer.

First examine Fig. 18(a) for the result of Test I, here the floating part of the caliper is removed (as shown in Fig. 17) and the measurement is at Location A. Two main response frequencies we consider are around 42 and 120 Hz, believed to be bending about a longitudinal axis and twisting about a transverse axis, respectively (also evident are several other modes such as at 56, 110 and 134 Hz). We can identify that the mode at 120 Hz (as well as 110, 114 and 134 Hz) is associated with the twisting motion; examine Fig. 18(b) for Test II, with measurement at Location A and measurement at Location B for Test II (Brake released) and Test IV (Brake applied).

![Fig. 17—Typical Floating-Caliper Brake System with Acceleration Measurement Locations A and B.](image)

![Fig. 18—Experimental modal analysis for typical floating-caliper brake system: a) Accelerance with impact and measurement at Location A for Test I (Brake released) and Test III (Brake applied); b) Coherence for Test I; c) Coherence for Test III; d) Accelerance with impact at Location A and measurement at Location B for Test II (Brake released) and Test IV (Brake applied).](image)
conducted with the fixed part of the caliper reattached and the brake firmly applied. Test III (Fig. 18(a)) is for measurement at Location A. Comparing with Test I we see that the peak at 40 Hz is much smaller and the 120 Hz mode (or a nearby mode) has shifted significantly down to about 90 Hz. This illustrates an important point for the stick-slip dynamic analysis: some modes will change significantly during stick-slip or slip-stick transitions. Similarly Test IV, with measurement at Location B, also shows this mode shifting to 90 Hz. The coherence plots show expected characteristics; Test I close to the impact point has better coherence than Test III, and coherence is good at resonances and poor at anti-resonances. Test II and IV show the same coherence characteristics (though the plots are not shown). The shift could be attributed to one or more of the following modal couplings: a) increased inertia and mass when the fixed part of the caliper is reattached; b) an addition to the rotor inertia due to the torsional motion of the brake subsystem via the brake clamping force; and c) an addition to the driveshaft stiffness coupling the brake subsystem to the essentially grounded driveline gearing (by large inertia and static friction).

We can use this understanding to estimate the effective \( k_b \) as follows. First, modal measurements are made of the brake system to approximate inertia values (given the lack of availability of solid models). The dimensions of the floating and fixed parts of the caliper are assumed (Table 2a) giving a total mass of about 5.5 kg for the caliper; similar the to manufacturer’s figure. Inertia about the centroid, \( J_{yy} \), is calculated via \( J_{yy} = m(h^2 + w^2) / 12 \) (rectangular block), where \( m, h \) and \( w \) are mass, height and width respectively. Then inertia about the tire rolling axis is obtained with an approximated radius of rotation and by applying the parallel axis theorem. The brake rotor and hub are assumed to be simple discs, with measurements, masses and inertias as given in Table 2b. Finally, the brake knuckle (the most complex shape) is also approximated as a disc, providing a mass of about 4 kg, again similar to manufacturer’s figure.

Under the condition of Test I, the sticking equation for the brake, as extracted from Eqn. (3), gives the natural frequency for sticking brake subsystem as \( f_{s1} = \frac{1}{2\pi}(k_d / J_d)^{0.5} \). The combined inertia is that of the fixed part of the caliper plus knuckle, \( J_b = 0.03 \text{ kg m}^2 \), hence given the test result \( (f_{s1} = 120 \text{ Hz}) \), an approximated stiffness is found as \( k_d = 17000 \text{ N mrad}^{-1} \). Test III has the caliper reattached and the brake applied, so now the combined inertia consists of the whole caliper, knuckle, wheel hub and rotor, \( J_b + J_r = 0.184 \text{ kg m}^2 \); note the wheel rim inertia, \( J_r3 \), is removed for the test. This condition is considered to be governed by the sticking system equations; however the tire is removed and hence \( k_t = 0 \). Also, \( J_d \gg (J_b + J_r) \), so for the purpose of linear modal analysis, the drive inertia may be considered grounded, giving \( (J_r + J_b) \dot{\theta}_{r/b} + (k_d + k_b) \dot{\theta}_{r/b} = 0 \). Now the resulting natural frequency of the coupled brake-axle torsional system is \( f_{s2} = \frac{1}{2\pi}[(k_d + k_b) / (J_b + J_r)]^{0.5} = 60 \text{ Hz} \). This illustrates the mode shift as shown in Fig. 18(a), however, it is significantly far from an observed shift to 90 Hz as found in Test III. With such approximations we obviously do not expect the linear system theory to closely match the modal tests.

Table 2—Estimated brake subsystem geometry and inertia parameters: a) Caliper, b) Wheel hub, rotor and knuckle (original and adjusted).

<table>
<thead>
<tr>
<th>Component</th>
<th>Height (m)</th>
<th>Width (m)</th>
<th>Depth (m)</th>
<th>Mass (kg)</th>
<th>Radius of Rotation (m)</th>
<th>Inertia about Centroid (kg m²)</th>
<th>Inertia about Tire Rolling Axis (kg m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Caliper (fixed part)</td>
<td>0.14</td>
<td>0.03</td>
<td>0.065</td>
<td>2.205</td>
<td>0.004</td>
<td>0.1</td>
<td>0.023</td>
</tr>
<tr>
<td>Caliper (floating part)</td>
<td>0.14</td>
<td>0.05</td>
<td>0.065</td>
<td>3.276</td>
<td>0.006</td>
<td>0.13</td>
<td>0.061</td>
</tr>
<tr>
<td>Wheel hub ( J_{r,2} )</td>
<td>—</td>
<td>—</td>
<td>0.06</td>
<td>0.025</td>
<td>—</td>
<td>2.20</td>
<td>0.004</td>
</tr>
<tr>
<td>Brake rotor ( J_{r,1} )</td>
<td>0.14 (outer)</td>
<td>0.07 (inner)</td>
<td>—</td>
<td>0.02</td>
<td>7.20</td>
<td>0.088</td>
<td>—</td>
</tr>
<tr>
<td>Knuckle</td>
<td>0.12</td>
<td>—</td>
<td>0.045</td>
<td>3.97</td>
<td>—</td>
<td>3.97</td>
<td>0.007</td>
</tr>
<tr>
<td>Knuckle (adjusted)</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.096</td>
</tr>
</tbody>
</table>

Note: Table 2a provides the estimated brake subsystem geometry and inertia parameters for a) Caliper, b) Wheel hub, rotor and knuckle (original and adjusted).
and an improvement must be made. Looking at the assumptions made in geometry, the most inaccurately described is likely the knuckle inertia, given its complex shape. This inertia may be understated, given asymmetry and since a proportion of the suspension linkage inertias (if known) must be added. Increasing to a value similar to the brake rotor (the adjusted value in Table 2b) gives $J_b = 0.119 \text{ kg m}^2$ (fixed part of caliper and adjusted knuckle) for Test I and with $f_{b1} = 120 \text{ Hz}$, we now estimate a more reasonable $k_b = 68000 \text{ N mrad}^{-1}$ (perhaps a more likely value for the torsional rigidity of the suspension linkages). For Test III, $J_b + J_r = 0.272 \text{ kg m}^2$ (whole caliper, adjusted knuckle, wheel hub and rotor) and the torsional mode in the sticking condition is $f_{b2} = 85 \text{ Hz}$; closer to the 90 Hz measured. These calculations of brake subsystem parameters are quite approximate and serve the purpose of providing order of magnitude values rather than just “guessing” some generic values. A significant challenge would be to add additional degrees of freedom to the brake subsystem of Fig. 1 and thus one must describe only those modes that have most significance to the stick-slip dynamics.

### 4.2.3 Eigensolutions

Eigensolutions yield the natural frequencies, damping ratios and mode shapes under the slipping and sticking system conditions. These are given in Table 3; for further discussion on these modes see Ref. 7.

### 5 NON-LINEAR FRICTION AND TRANSIENT EXCITATION MODELS

#### 5.1 Overview

Two braking cases are applied for the transient force. Case I (Fig. 3(a)) is an attempt to approximate the driver behavior and will see a finite end to motions. Case II (Fig. 3(b)) will see either steady-state stick-slip orbits or steady sliding solutions. Case I simulations may be used to examine trends with time domain metrics for various brake force profiles and some qualitative comparisons with experimental results are made. Case II simulations demonstrate the effectiveness of our computational methods and bring an understanding of steady-state stick-slip orbits. Friction laws, initial forcing functions, initial conditions, and stick to slip transitions are discussed next.

#### 5.2 Friction Laws

The brake torque is a function of the average radius of contact, $r_p$, between the brake pad and rotor and brake friction force, $F_b(\delta, t)$, as shown in Fig. 1(c). A key simplifying assumption is that the dry friction force is described with Coulomb’s law with the additional consideration of a static friction coefficient, $\mu_s$, different from kinetic, $\mu_k$. Figure 2 illustrates the friction law for an instantaneous or constant value of the brake normal force, $F_n(t)$. With consideration of the direction of sliding and the sticking regime, the brake friction force or torque may be expressed with a piece-wise function defined in terms of the slipping velocity, $\dot{\theta} = \dot{\theta}_r - \dot{\theta}_b$. 

---

### Table 3—Damped frequencies, damping ratios, modal and phase (rad) vectors for the model of Fig. 1 ($d_t = 200 \text{ Nmsrad}^{-1}$): a) Under the slipping condition, b) Under the sticking condition.

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>$f_{di}^{SL}$ (Hz)</th>
<th>$\xi_i^{SL}$</th>
<th>$f_{di}^{ST}$ (Hz)</th>
<th>$\xi_i^{ST}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive ($\theta_d$)</td>
<td>1.00</td>
<td>n/a</td>
<td>2.38</td>
<td>9.45%</td>
</tr>
<tr>
<td>Rotor ($\theta_r$)</td>
<td>1.00</td>
<td>n/a</td>
<td>4.63</td>
<td>4.52%</td>
</tr>
<tr>
<td>Brake ($\theta_b$)</td>
<td>0.00</td>
<td>n/a</td>
<td>54.6</td>
<td>3.25%</td>
</tr>
<tr>
<td>Tire ($\theta_t$)</td>
<td>1.00</td>
<td>n/a</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 3 Continued

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>$\Phi_i^{SL}$</th>
<th>$\psi_i^{SL}$</th>
<th>$\Phi_i^{ST}$</th>
<th>$\psi_i^{ST}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive ($\theta_d$)</td>
<td>1.00</td>
<td>n/a</td>
<td>1.00</td>
<td>-3.09</td>
</tr>
<tr>
<td>Rotor ($\theta_r$)</td>
<td>1.00</td>
<td>n/a</td>
<td>1.00</td>
<td>-3.09</td>
</tr>
<tr>
<td>Brake ($\theta_b$)</td>
<td>0.00</td>
<td>n/a</td>
<td>1.00</td>
<td>n/a</td>
</tr>
<tr>
<td>Tire ($\theta_t$)</td>
<td>1.00</td>
<td>n/a</td>
<td>1.00</td>
<td>0.11</td>
</tr>
</tbody>
</table>

---

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At some instantaneous time, \( \dot{\delta} > 0 \), the slipping friction force, \( F_b \), the breakaway friction force (or threshold) and \( T_b \), the actual sticking force or the shear force between sticking elements during sticking motions. In terms of the friction torque, given by Eqn. (6b), we have corresponding definitions, \( T_b^{ST} \), \( T_b^\theta \), and \( T_b^\beta \), for slipping, breakaway and sticking friction torques, respectively. The slipping condition is valid for all \( t \), and the sticking condition is for all \( t \) when \( \dot{\delta} = 0 \). The stick to slip transition occurs at the instant when the sticking friction force exceeds the breakaway force (for subsequent analysis we also define here the torque expressions which provide identical conditions):

\[
\begin{align*}
F_b^ST &> |F_b^\beta| = \mu_s F_n; \\
T_b^ST &> |T_b^\beta| = r_d \mu_s F_n. 
\end{align*}
\]  

At some instantaneous time, \( t \), the motions may be at the stick-slip boundary, which is defined by the following, where \( \dot{\delta} = 0^\pm \):

\[
\begin{align*}
|F_b^ST| & = |F_b^\beta| = \mu_s F_n; \\
T_b^ST & = |T_b^\beta| = r_d \mu_s F_n. 
\end{align*}
\]  

Note that \( F_b^ST \) is multi-valued when sticking, considering Eqn. (7a), i.e., \( F_b^ST \in \{ -\mu_s F_n \leq 0 \leq \mu_s F_n \} \), and may lie anywhere on the vertical axis between these points (Fig. 2). Finding the exact intersection point, \( \dot{\delta} = 0 \), for the above Eqn. (9a), is impractical, whether solving via numerical integration or when assembling piecewise analytical solutions. Various approaches such as bisection methods require a bracket within which a value close to zero may be taken as zero. We illustrate this bracket in Fig. 2, which is referred to as the zero velocity tolerance:

\[
\dot{\delta} = 0 \quad \text{where} \quad \delta \in [ -\varepsilon_5 \leq 0 \leq \varepsilon_5 ].
\]  

In our simulations we apply a time-varying tolerance as a function of the slope of relative velocity.

5.3 Engine Load and Brake Actuation Functions

The engine torque is assumed as \( T_e = 20 \) N⋅m, which for the reduced model (Fig. 1) gives a drive torque, \( T_d = 0.5 n_d n_\delta T_e = 76.8 \) N⋅m. Vehicle resistance is simplified with \( d_i \), providing a drag torque of \( -d_i \dot{\theta}_i \) N⋅m, as shown in Eqns. (4c) and (5c) and as \( T_v \) in Fig. 1. This simplifies complicating friction laws governing the dynamic interactions between the tire and road surface and between axle and wheel bearings. It is considered that initially there is sufficient brake torque to hold the vehicle stationary and to aid in formulating brake forces we introduce \( \tilde{F}_n = T_d/r_d \mu_s \) as the brake slip line. Its intersection with brake force is the time of initial slip.

Brake Force Case I: Two hyperbolic tangent functions are assembled together with a delay region. Three parameters, \( \beta, \sigma \) and \( \alpha \) respectively control the amplitude, slope, and delay between functions; a few examples are illustrated in Fig. 3(a). The first function, \( F_{n1}(t) = \tilde{F}_n - \beta \tanh(\sigma(t - \tau)) \), crosses \( \tilde{F}_n \) at its mean after time, \( \tau \), which is related to slope by, \( \sigma = -0.5 \pi^4 \ln[(1 - \gamma)/(1 + \gamma)^{-1}] \). Once the solution reaches \( t > 2 \tau \) in the delay function is \( F_{n2}(t) = \tilde{F}_n - \gamma \beta \) till \( t \) reaches \( 2\tau + \alpha \) where a second hyperbolic function is applied as \( F_{n3}(t) = \tilde{F}_n + \beta - 2\beta \tanh[\sigma(3\tau + \alpha - t)] \). This is \( F_{n1} \) inverted, offset and scaled in amplitude. It brings a fairly abrupt end toward the groan events in simulations by crossing well above the slip line (consider too that \( \mu_s < \mu_\ast \) and a brake stick line is less determinable).

\[
F_n(t) = \begin{cases} 
F_{n1}(t) & 0 \leq t \leq 2\tau \\
F_{n2}(t) & 2\tau < t < 2\tau + \alpha \\
F_{n3}(t) & t \geq 2\tau + \alpha 
\end{cases}
\]

Note that to truly control the “length” of the groan event, the time between crossings of the brake slip line
is a useful quantity to fix or vary, and depends on slope and delay parameters:

\[ L = 2 \tau + \alpha - 0.5 \sigma^{-1} \log [(1 + 0.5)(1 - 0.5)^{-1}] \]

(12)

The function in Eqn. (11) has limits in smoothness and if values are assigned sufficiently small for \( \tau \) (or sufficient large for \( \sigma \)) then there will a significant discontinuity to the brake force.

Brake Force Case II: Here the force (Fig. 3(b)) crosses the brake slip line near the maximum of the hyperbolic function. Parameters \( \beta \) and \( \tau \) still control amplitude and slope, however, Case I and Case II profiles are quite different for the same \( \beta \) and \( \tau \). It is intended that the function is applied as \( 0 \leq F_n(t) \leq \hat{F}_n + \varepsilon \) for all \( t \) and \( F_n(t) \) approaches a finite value as \( t \to \infty \) so as to lead to steady state solutions of some type. Specific details are provided in our recent article.

5.4 Initial Conditions

Initial conditions are determined so that at \( t=0 \) s the system has no motion and is under the stick state, i.e., with matrices Eqns. (5a)–(5d) and coordinate vector \( \theta = (\theta_d \theta_r \theta_b \theta_d \dot{\theta}_b \dot{\theta}_b) \). Given the above mentioned forcing functions the initial torque vector is \( \mathbf{T}(0) = [T_d 0 0]^T \), yielding initial conditions,

\[ \theta(0) = \mathbf{K}^{-1} \mathbf{T}(0), \quad (13a) \]
\[ \dot{\theta}(0) = 0. \quad (13b) \]

6 COMPUTATIONAL ISSUES

6.1 State Vectors and Stick-Slip Transitions

An algorithm for stick-slip transitions is proposed next. Two Runge-Kutta solvers with adaptive step size control are used. These are designated in this paper as “RK23”\(^{11} \) and the higher order “RK45”\(^{12} \), which utilize 2/3 and 4/5 orders respectively. The governing second order differential equations are reduced to the first order form as, \( \dot{\mathbf{U}} = \mathbf{BU} \), where \( \mathbf{U} = (\theta_d \theta_r \theta_b \dot{\theta}_d \dot{\theta}_r \dot{\theta}_b)^T \). Matrix \( \mathbf{B} \) is not calculated explicitly, rather each equation is determined separately. On each successful time step, \( t_n \), the solver returns values of \( \mathbf{U}(t_n) \), then velocities within \( \mathbf{U} \) are set as \( \dot{\mathbf{U}}(t_n) = \dot{\mathbf{U}}_{i+1}(t_n) \), \( i = 1 \) to 4. Under the slipping condition, accelerations, \( \ddot{\mathbf{U}}_{i}(t_n), \) \( i = 5 \) to 8, are determined with the system described by Eqn. (3) with Eqns. (4a)–(4d). For the sticking condition they are determined in similar fashion using the expanded form of Eqn. (3) with Eqns. (5a)–(5d), for \( \ddot{\theta}_d \) and \( \ddot{\theta}_b \). So as to negate the need for applying the brake/rotor displacement offset (and associated programming), i.e., \( \ddot{\theta}(t_n) = 0 \) for all \( t_n \), the brake and rotor coordinates are kept distinct, yielding:

\[ \ddot{\theta}_r = \ddot{\theta}_b = (J_r + J_b)^{-1} [c_d (\ddot{\theta}_d - \dot{\theta}_d) - c_r (\ddot{\theta}_r - \dot{\theta}_r) - c_b \dot{\theta}_b + k_d (\theta_d - \theta_r) - k_b (\theta_b - \theta_r)] \quad (14) \]

Since the rotor and brake have the same acceleration, \( \ddot{\theta}_b = \ddot{\theta}_r \) and velocity, \( \dot{\theta}_b = \dot{\theta}_r \), during the sticking motions these two components of \( \dot{\mathbf{U}} \) are equal. However, their displacement solutions remain distinct and this approach allows the fixed dimension of \( \mathbf{U} \) for sticking and slipping states. During sticking motions these displacements will change at the same rate and thus \( \ddot{\theta}_r = \ddot{\theta}_b \) holds. Programming the solution in such a way may slightly increase the computational time of the solver, but significantly decreases the programming complexity; the solver does not need to stop and start at each crossing of the stick-slip boundary to change the dimension of \( \mathbf{U} \). Additionally, the small computational cost associated with writing and storing a separate overall solution vector and in initializing the solver at each transition is avoided.

6.2 Algorithm for Stick to Slip Transitions

Assume the system in stick at \( t_0 = 0 \) s and record its state as \( V(t_0) = 0 \). First, solve the sticking system equations until the sticking torque is greater than the breakaway torque, i.e., Condition given by Eqn. (7b). Then switch to the slipping system equations. Knowing \( \ddot{\theta}_r = 0 \) during sticking allows the sticking torque to be determined via subtraction of slipping equations for rotor and brake as:

\[ \ddot{\delta} = 0 = \ddot{\theta}_r - \ddot{\theta}_b = \frac{1}{J_r} [c_d (\ddot{\theta}_d - \dot{\theta}_d) - c_r (\ddot{\theta}_r - \dot{\theta}_r) + k_d (\dot{\theta}_d) - k_r (\theta_r - \theta_r) - T_b^{ST}] + \frac{1}{J_b} [k_b \ddot{\theta}_b + c_b \dot{\theta}_b - T_b^{ST}] \quad (15) \]

Introduce an inertia ratio parameter, \( \Gamma = J_r / (J_r + J_b) \), and rewrite Eqn. (15) as:

\[ T_b^{ST} = (1 - \Gamma) [c_d (\ddot{\theta}_d - \dot{\theta}_d) - c_r (\ddot{\theta}_r - \dot{\theta}_r) + k_d (\dot{\theta}_d - \theta_r) - k_r (\theta_r - \theta_r) + \Gamma [k_b \ddot{\theta}_b + c_b \dot{\theta}_b] \quad (16) \]

As stepping time is reduced, \( \Delta t \to 0 \), the accuracy improves for predicting the crossing of threshold given by Eqn. (7b). Thus the employment of a fine time step is essential. A measure of error for how far the prediction exceeds the threshold is specified as \( E_{\delta_b}[\tau] = 1 - T_b^{ST} (t_b^{ST} - \Delta t) / T_b^{ST} (t_b^{ST} - \Delta t) \), where \( t_n = t_b^{SL} \) are the first slip time(s) after the transition. Note that the state is not
changed until the step after crossing the threshold; this is discussed in Sec. 6.4. We define an acceptable bound within which this error must lie, \(-\varepsilon_{\delta} \leq E_{SL}[r] \leq \varepsilon_{\delta}\) and instruct the solver to either stop or leave a flag if \(E_{SL}\) falls out of the bound.

Once the system reaches the stick-slip transition, several considerations must be made. First, simulations with \(\dot{\delta}(0) = 0\) will have the sign of the sliding friction torque undetermined for the initial slip step, as \(\text{sgn}[\dot{\delta}(0)] = 0\). However, from the condition given by Eqn. (7b) we know which direction the solution has progressed we determine a switching condition given by 
\[
\dot{\delta}(t_{n}^{Sl} + \Delta t) > T_{h}^{ST}(r_{n}^{Sl} - \Delta t) = r_{n}d_{n}F_{n}
\]
for the positive boundary and 
\[
\dot{\delta}(t_{n}^{Sl} - \Delta t) < -T_{h}^{ST}(r_{n}^{Sl} - \Delta t) = -r_{n}d_{n}F_{n},
\]
for the negative boundary. Hence the first slip step after transition requires the brake friction torque to be assigned as:
\[
T_{h}^{ST}(r_{n}^{Sl}) = \text{sign}[T_{h}^{ST}(r_{n}^{Sl} - \Delta t)]r_{n}d_{n}F_{n}
\]
(17)

For later stick-slip transitions, the relative velocity is constant before slipping, but will not be zero, as at the last instant of slip (before the prior stick) it fell within the tolerance \(-\varepsilon_{\delta} \leq \dot{\delta} \leq \varepsilon_{\delta}\). This residual value may be misleading to the actual direction of transition through the boundary at the first instant of slip, hence Eqn. (17) is still applied in the same manner. Residual values are best left unchanged, rather than set to zero; any change will add/subtract small energy to the system through the damping parameters. The next consideration is that the zero velocity tolerance must be set as \(\varepsilon_{\delta} = 0\), for even though the next step will give relative velocity, \(\dot{\delta}(t_{n}^{Sl} + \Delta t) > 0\), this value may still lie within the bracket, \(-\varepsilon_{\delta} \leq \dot{\delta}(t_{n}) \leq \varepsilon_{\delta}\). This would satisfy the first item of the slip to stick transition, condition given by Eqn. (9a), and if Eqn. (9c) is also satisfied then the system will immediately return to the sticking state. This results in an unsatisfactory stick-slip “chatter,” repeating at the period, \(\Delta t\), until clear sticking or slipping motions are achieved. This problem is resolved by temporarily setting the velocity tolerance as \(\varepsilon_{\delta} = 0\). At some time after transition, the tolerance needs to be returned to a non-zero value, otherwise the solution will never find the approximate \(\dot{\delta} = 0\) intersection. If the relative velocity increases/decreases after breakaway, it naturally must decrease/increase again before reaching \(\dot{\delta} = 0\). Hence as \(t\) progresses we determine a switching value, \(S(t_{n}) = \text{sgn}[\ddot{\delta}(t_{n-1})\ddot{\delta}(t_{n})]\). A value \(S(t_{n}) = 1\) indicates \(\dot{\delta}\) is increasing/decreasing and some at \(t_{n}\), the switch becomes \(S(t_{n}) = -1\), indicating the step where \(\dot{\delta}\) changes direction. The zero velocity tolerance is then reset as \(\varepsilon_{\delta} > 0\) at this step and the switch is not queried again until after the next stick-slip transition.

6.3 Algorithm for Slip to Stick Transitions

Once the system is clearly slipping the state is recorded as \(V(t_{n}) = 1\), i.e. slip. Considering that \(\varepsilon_{\delta}\) is switched back on, we must assign a value. One method is to choose a fixed value, run solutions, check the intersection \(\dot{\delta} = 0\) is not overshoot, reduce \(\Delta t\) if necessary and repeat until acceptable solutions are reached; this is a time consuming approach. Instead, we define a time-varying tolerance, \(\varepsilon_{\delta}(t_{n}) = 0.5\ddot{\delta}(t_{n-1})[t_{n} - t_{n-1}]\), representing approximately half the change in \(\dot{\delta}\) between previous and current time steps. The solution now cannot cross over \(\dot{\delta} = 0\) without falling within the bracket, \(-0.5\ddot{\delta}(t_{n-1})[t_{n} - t_{n-1}] \leq \dot{\delta} \leq 0.5\ddot{\delta}(t_{n-1})[t_{n} - t_{n-1}]\). Initial stick times are defined as \(t_{n} = t_{n}^{Sl}\), the residual values are defined at these times as \(\ddot{\delta}_{ST}[m]\).

6.4 Critical Computational Considerations

A critical consideration is to record prior states, \(V(t_{n-1})\), switches and values: \(\ddot{\delta}(t_{n-1})\) and \(t_{n-1}\), for prior successful steps but not for mid-interval or unsuccessful steps. The methodology given is only appropriate to single step solvers, multi-step methods will require \(V\) and \(\ddot{\delta}\) values for additional times such as \(t_{n-2}\), \(t_{n-1}\), \(t_{n}\) and \(t_{n+1}\), bringing additional complications. Further, improvements can be made to results and speed by careful judging of timing for state transitions. For transitions in state found during evaluations of mid-interval steps, it is best to force that evaluation and those subsequent to be for the original state. The transition is recorded and then applied after the step has been successful. This is most easily explained with an example using the classical second-order Runge-Kutta (midpoint) method, \(^{10}\) where \(h\) is the interval, \(k\) the estimation and \(O(h^{m+1})\) the error term:
\[
k_{1} = hf(x_{n}y_{n}),
\]
(18a)
\[
k_{2} = hf(x_{n} + 0.5h,y_{n} + 0.5k_{1}),
\]
(18b)
\[ y_{n+1} = y_n + k_2 + O(h^3) \]  \hspace{1cm} (18c)

With reference to Fig. 19(a), examine a transition from slipping to sticking (note that the equivalent parameters for Eqn. (3) would be \( y = \dot{\delta} \) and \( h = \Delta t \)). The first estimation, \( k_1 = hf(x_n, y_n) \) over the whole interval, \( h \), is for the slipping state, with a certain slope of the derivative, as indicated in Fig. 19(a) by the arrow at \((x_n, y_n)\). The estimation is applied to a trial step in the midpoint of the interval, \( y_n + 0.5k_1 \). This trial step now falls within the zero velocity tolerance and the midpoint derivative is evaluated for sticking as \( f(x_n + 0.5h, y_n + 0.5k_1) = 0 \) (assuming a sticking is satisfied). This yields \((x_{n+1}, y_{n+1}) = (x_n + h, y_n)\), consequently the sticking transition has been found without actually having a component of \( y \) fall within the tolerance. Additionally the two derivatives are non-smooth over the interval \( h \).

In practice, the Runge-Kutta routines (in Matlab) use adaptive step size control and if mid-interval state transitions occur then the solutions most often fail to meet solver error tolerances. Then the step size gets smaller, yet the same set of circumstances can repeat; the result is the step size becoming unnecessarily small before the solution is accepted, hence wasting computational effort. Figure 19(b) illustrates how this is avoided. Here the midpoint derivative is evaluated for the original state, but the transition is held off till the next step. The solution is smooth over the interval \( h \), giving \((x_{n+1}, y_{n+1}) = (x_n + h, y_n + k_2)\) and then sticking applies to all evaluations for \( y_{n+2} \). The solution is still

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**Fig. 20** — Comparison of transition points predicted by Runge-Kutta methods: a) Breakaway torque, \( T^B_b(t) \), brake friction torque, \( T_b(t) \), composed of sticking torque, \( T^ST_b(t) \) and slipping friction torque, \( T^SL_b(t) \); (b) Relative Velocity, \( \dot{\delta} \) and time-varying zero velocity tolerance, \( \varepsilon \dot{\delta} \) (multiplied by 50); (c, d) Enlargement for critical regions in (a, b), here \( \varepsilon \dot{\delta} \) is not multiplied by 50. Simulation is for Case II brake rorcing with \( \beta = 30 \) N, \( \tau = 0.4 \) s, \( \mu_k = 0.4 \) and \( \mu_s = 0.6 \). (Key: dotted—RK45, solid—RK23).

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**Fig. 21** — Prediction of state transition points for \( 0 \leq t_r \leq t_f \): a) Error estimation parameter, \( |E_{ST^r}[r]| \), measuring overshoot of breakaway torque threshold at last stick times, \( t^{ST}_r - \Delta t \) and b) Residual values \( \dot{\delta}_{ST}[m] \) at initial stick times, \( t^{ST}_m \).
with immediate slip, the knowledge of state $V(t_0)$ at $t_1$ is important to ensure $\varepsilon_{\dot{\theta}} = 0$ until the time-varying tolerance is switched on. For initial conditions starting in the slip state, $V(t_0) = 1$ and an initial value for zero tolerance must be set as $\varepsilon_{\dot{\theta}}(t_1) < \dot{\varepsilon}(0)$. After the first step $\varepsilon_{\dot{\theta}}$ is switched back to the time-varying tolerance.

### 6.6 Comparison of Lower and Higher Order Runge-Kutta Routines

Here we compare the stick-slip transition points using the lower and higher order Runge-Kutta routines. Solutions are run until steady state orbits for stick-slip motion are found. Both solutions use a maximum time step of $\Delta t_{\text{max}} = 2^{-15}$ s which although Matlab documentation recommends using as a last resort with their solvers, is helpful for accurate predictions of state transitions. All results in this paper use this $\Delta t_{\text{max}}$ and while variable stepping is applied by the solvers we specify output times for later frequency domain analysis at $\Delta t = \Delta t_{\text{max}}$. For an example case (Fig. 20) the RK23 ran nearly 2.1 s and RK45 nearly 4 s. From visual inspection the solution could have been considered steady-state after 1.6 s. The higher order solver took much longer to meet the strict conditions for steady-state stick-slip. To compare typical orbits we shift the start of the second last orbit back to $t_0 = 0$ s and provide overlays of the results for the last two orbits. Figure 20(a) provides $T_b(t)$ and the enlargement (Fig. 20(c)), shows $T_b^{ST}(t)$ (in stick) approaching the breakaway threshold, which at the point exceeding the threshold drops to $T_b^{SL}(t)$. Both lower and higher order methods find the threshold crossing with reasonable accuracy. Figure 21(a) provides the error values, $E_{ST}[\varepsilon_{\dot{\theta}}]$, plotted versus solution time, at times immediately before slipping, $t_n = t_{SL} - \Delta t$. The achievement of $E_{ST} < 0.41\%$ and mean values, $E_{ST} = 0.186\%$ (RK23), $E_{ST} = 0.206\%$ (RK45), shows reasonable bounds for error. Figure 20(b) provides $\dot{\varepsilon}(t)$ along with $\varepsilon_{\dot{\theta}}(t) \times 50$ (to be on the same scale). The enlargement (Fig. 20(d)) illus-

Fig. 23—Sample simulation for Case I brake forcing, $\beta = 300$ N, $\sigma = 50$ and $L = 2$ s: Acceleration signature showing entire groan event. Note the similarity in shape to Event $F$ of Fig. 5.

Fig. 22—Sample simulation for Case I brake forcing, $\beta = 300$ N, $\sigma = 50$ and $L = 2$ s: a) Sticking and slipping velocity; b) Initial transient of sticking and slipping velocity on brake release; c) Final transient of sticking and slipping velocity on brake rise; d) Initial brake force, $T_b^B$ (composed of sticking and slipping torques) vs. breakaway force, $T_b^B$. e) Final brake force, $T_b^S$ (composed of sticking and slipping torques) vs. breakaway force, $T_b^S$. Note the central region of (a) features steady-state stick-slip orbits for a finite time.

non-smooth over two-intervals of $h$ and the new step will likely fail to meet tolerances of error, but the solver will carry on after only a few reductions in $h$, as the mid-interval transitions are avoided. Additional computational switches need to be implemented for this forcing of stick or slip state.

### 6.5 Adaptation to Various Initial Conditions

The above discussion has been framed around the given initial conditions and application to alternate conditions or transitions in time may be made. For this study, commencing in the stick state the initial conditions yield a solution with a known characteristic to not immediately slip. There are no special considerations, however by setting $V(t_0) = 0$ the original state is recorded. In the case of initial conditions where $\dot{\varepsilon} = 0$...
Fig. 24—Acceleration signatures for sample simulations for Case I brake forcing with a) $\beta=550$ N, $\sigma=50$ and $L=2$ s; b) $\beta=225$ N, $\sigma=50$ and $L=2$ s and c) $\beta=550$ N, $\sigma=10$ and $L=0.45$ s.

trates the relative velocity approaching the zero crossing and the last few variations in $\epsilon$. When $\dot{\delta}$ falls within the bracket it flattens to the residual value, $\delta_{ST}[m]$ and $\dot{\delta}(t)=\delta_{ST}[m]$ for $t_{m}^{ST}\leq t\leq t_{m}^{SL}$. The $\delta_{ST}[m]$ values are given in Fig. 21(b) for all $m$ (all slip-stick transitions). Notice the maximums have significant difference, $\delta_{ST}[m]=1.57\times10^{-3}$ (RK23) and $\delta_{ST}[m]=2.18\times10^{-3}$ rad/s (RK45), although either one is acceptable considering the orders of magnitude difference to peak relative velocity, $\dot{\delta}=0.55$ rad/s, of the orbit. The mean values are found as $\ddot{\delta}[m]=5.55\times10^{-3}$ (RK23) and $\ddot{\delta}[m]=6.31\times10^{-4}$ rad/s (RK45). It is clear that the two methods yield similar results, with little divergence over the entire solution. The lower order (2/3) method is indeed faster and as it performs better for all of the above indices is used for further studies.

7 CONCLUSION

Brake Force Case I (Fig. 3(a)) is designed to simulate the driver input in the experiment while allowing for parameters to describe amplitude of brake release, rate and the delay before brake reaplication. The actual input has not been measured; this function is an approximation. Responses for a sample simulation are shown in Figs. 22 and 23, where $\beta=300$, $\sigma=50$ and $L=2$. Figure 22(a) shows the relative velocity between brake and rotor (slipping velocity) for the entire groan event. Stick-slip responses are evident. Figures 22(b)–22(e) plot together the initial and final transients of velocity and brake torque. With reference to conditional statements Eqns. (7)–(9), the brake is initially sticking, $\dot{\delta}=0$, with sticking torque, $T_b^{ST}=T_b^{ST}\leq T_b^{SS}$, then at $t=0.05$ s the actuating brake force is reduced to the point where $T_b^{ST}>T_b^{SS}$ giving an initial slip, $\dot{\delta}>0$ and now $T_b=T_b^{SL}$. Accordingly the torsional responses at slipping frequencies (an initial value problem) bring a return of slipping velocity at $t=0.06$ s to within tolerance of zero velocity, i.e. $-\varepsilon\dot{\delta}\leq\delta\leq\varepsilon\dot{\delta}$. The sticking torque is within the breakaway value, $T_b^{ST}\leq T_b^{SS}$ so the bodies stick again and now $\dot{\delta}=0$ and $T_b=T_b^{ST}$. This first stick is short, however, the motions fall into a steady orbit by $t\approx 1$ s, with sticking period, $\tau^{SS}=0.0196$ s, slipping period $\tau^{SL}=0.0076$ s and stick-slip period $\tau^{SS}=0.0272$ s. The motions are periodic where at each transition there is an initial value problem for response with a certain homogeneous and particular solution. These solutions are governed by Eqn. (3) with the values $\theta$ and $\theta'$ at transition points and the shape of the response dependent on the modal properties of slipping given by Eqns. (4a)–(4c) and sticking given by (5a)–(5c) matrices and $T_{ns}$, $T_{ns}$ and $T_{ns}$. Referring Figs. 22(c) and 22(e) the rise in brake force (on reapplication by the driver) sees an abrupt end to the orbit and the brake then sticks for all $t$ and now $\dot{\delta}=0$ and $T_b=T_b^{ST}$. Figure 23 is for qualitative comparison to Event F (and sized as such). The overall shape compares well but this is not to say a direct comparison may be made as the system parameters and forcing conditions are not matched. The simulated stick-slip frequency is 37 Hz compared to the dominant 76 Hz inferred from experiment. Yet, the forcing conditions can lead to different stick-slip frequencies*. Other researchers have reported stick-slip frequencies in this order (30–60 Hz) for test and simulation. Some simulation results show more of a transient tail (than an abrupt end), for example see Fig. 24 with three variations of $\beta, \alpha$ and $L$. This was also seen in some of experimental data (e.g. Event I of Fig. 5). Influences on this shape include both damping and the timing of the stick-slip response with respect to the brake force rise (see Fig. 22(e) and note that the brake nearly slipped again at $t\approx 2.06$ s).

Figure 25 shows the acceleration spectra for the rotor and brake, calculated from a steady-state portion of the signal with 19 stick-slip cycles for a sample length $t_s=0.5$. The brake groan, in both simulation and experiment (see Figs. 11 and 13 for experiment) exhib-
its a primary stick-slip frequency and then multiple harmonics. For the simulation we know the harmonics are due to discontinuities in calculated accelerations changing at stick to slip and slip to stick points. We can infer that the same behavior is occurring in the experiment.

Brake Force Case II has $F_n(t)$ approaching a finite value as $t \to \infty$ (see Fig. 3(b)), hence solutions fall into a steady-state motion; either stick-slip orbits or steady sliding. Four different motion types are illustrated (Fig. 26), each has different slipping, sticking and stick-slip periods, which lie within defined ranges and may vary with $\beta$ and $\tau$ (see Ref. 7). The stick-slip frequencies range from 25–95 Hz. In these simulations, for a given $\beta$, the slope, governed by $\tau$, was shown to be able to influence which orbit was found due to variation in the transient solution flow leading to the orbit. The time domain result of Figs. 22 and 23 (of Brake Force Case I) features a finite portion of Type I Motions as illustrated by Fig. 26(a). For the experiment, the time-frequency content (Fig. 15) of the measured groan events A-I do not indicate any significant variation in the dominant harmonic (stick-slip frequency), it could be construed that the real system is operating within the bounds of a condition where there is only one motion type. This however is for the successful inducement of groan in a test; if the driver releases the pedal enough a steady-sliding (Fig. 26(d)) motion is quickly obtained. Hence at least two motion types are easily identified in this one experiment. In driving several automatic and manual transmission vehicles (in at least two continents) we have observed (from our own audible perception) that most vehicles seem to have one tone though multiple tones could be heard depending on the actuation (pedal) pressure. It would be of interest to see if future test data reveals multiple motion types. A detailed nonlinear analysis of the manual transmission equipped vehicle is also needed; here investigation is warranted for brake groan driven by mass, particularly in slow braking while reversing down a steep slope.
8. REFERENCES