Order domain analysis of speed-dependent friction-induced torque in a brake experiment

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Article info
Article history:
Received 16 December 2011
Received in revised form 17 May 2012
Accepted 19 June 2012
Handling Editor: H. Ouyang
Available online 20 July 2012

Abstract
A friction-induced forced vibration problem, as excited by the geometric distortions of the brake rotor, is studied in this article. The focus is on the order domain analysis, as the speed-dependent behavior of friction torque is not well understood. First, a new laboratory experiment is constructed to simulate vehicle brake judder in a scientific and yet controlled manner. The variations in pressure and torque are measured as the rotor slows down, and the order domain tracking is used to construct shaft torque vs. speed diagrams. A quasi-linear model of the laboratory experiment is then developed to obtain an analytical solution and to estimate the torque envelope function. A nonlinear model of the laboratory experiment (with a clearance) is also investigated to examine the resonant amplitude growth. Finally, predictions are successfully compared with measurements. Several contributions emerge over the prior literature. In particular, the experimental data clearly show that multiple-orders of the rotor surface distortion profile excite the friction-induced torque, and a clearance in the torsional system controls the resonant amplitude regime. New analytical and numerical solutions provide much insight into the speed-dependent resonant amplitude growth process.

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1. Introduction

Vehicle brake judder is a friction-induced forced vibration problem, and typically speed-dependent motions are felt by the driver through the steering wheel, brake pedal, or floor. This low frequency problem is excited by geometric distortions of the brake rotor such as disc thickness variation and lateral run-out. The excitation frequency of the external forcing function is proportional to the vehicle or wheel speed, and the first few orders of geometric disturbance are usually dominant [1–7]. Jacobsson [1] has discussed plausible causes and effects in a comprehensive pre-2003 literature review. In particular, Jacobsson [1–3] associates judder with the dynamic amplification of brake torque variation (T(t)) while passing through a critical vehicle speed based on simplified vehicle models. In a more recent study, Duan and Singh [4] investigate the judder problem by using a source-path-receiver model that includes the brake rotor, suspension system, and steering wheel. An amplification of the steering wheel angular displacement is calculated as a function of the speed. Table 1 summarizes selected speed-dependent peak-to-peak values from prior studies in terms of the measured angular acceleration of the caliper [2] and the predicted steering wheel angular displacement [4]. Values are normalized with peak-to-peak resonant amplitude as the reference. The amplitudes are lower at higher speeds, but they grow as the vehicle speed excites the resonance and then decrease as the rotor slows down. The system resonance is due to the combined...
torsional stiffness of the tie-rod and rack-pinion subsystem of the steering system, as reported by Duan and Singh [4]. They also discuss the effects of nonlinear pad and suspension bushing stiffness characteristics.

In related studies, Kang and Choi [5] use a simplified lumped parameter caliper model to predict $T(t)$; however, they do not address the torque amplification issue. Leslie [6] describes a detailed caliper model where the torque amplitude is related to the stiffness of the brake pads, caliper body, and hydraulic system, though amplitude growth is not calculated. Kim et al. [7] use the multi-body dynamics approach (via a commercial code) to suggest that a lower stiffness in the normal direction of the brake rotor and a higher stiffness in the rotational direction should lead to lower $T(t)$ amplitudes. Overall, the speed-dependent behavior of $T(t)$ is not well understood, and thus physical mechanisms leading to this response are experimentally, analytically, and numerically examined in this article. The focus is on the order domain analysis, and the envelope function of $T(t)$ is examined, though it is first introduced by Jacobsson [2,3].

2. Problem formulation

The time-varying torque, $T(t)$, is given by two components [2]. The rotational part, with time-varying $T_r(t)$, is related to the pure rotational (spinning) motion of the brake rotor as controlled by the time-varying actuation pressure $p(t)$. The alternating part ($T_a(t)$) is caused by the vibratory motion of the rotor which leads to judder. Further, Jacobsson [2] defines the alternating part as $T_a(t) = B(t) \sin(n \text{TH}_R(t))$ where $B$ is time-varying amplitude, $n$ is the dominant order of speed as given by the rotor surface distortion, and $\text{TH}_R(t) = \theta_i(t) - \theta_c(t)$ is the relative angular displacement between the rotor ($\theta_i(t)$) and caliper ($\theta_c(t)$). Since the caliper is connected to the chassis, it is reasonable to assume that $\theta_i(t) \gg \theta_c(t)$ which simplifies it to $\theta_i(t) \approx \theta_i(t)$. Essentially, the frequency of $T_a(t)$ varies with speed (or time) and is proportional to $\theta_i(t)$. Given a wide range of rotor speed, there would be an amplification of $T(t)$ if the system has one or more natural frequencies.

The main objectives of this study are: (1) design a laboratory experiment to simulate the vehicle-like judder source and measure $T(t)$, hydraulic pressure $p(t)$, $\theta_1(t)$, and rotor surface distortions on finger and piston sides of the caliper ($\zeta^r(t)$ and $\zeta^n(t)$) as the rotor slows down; (2) develop a quasi-linear model of the laboratory experiment, obtain an analytical solution, and estimate the envelope function of $T(t)$; and (3) develop a nonlinear mathematical model of the laboratory experiment to investigate its transient dynamics and compare $T(t)$ predictions with measurements with focus on the order domain analysis.

3. Experimental studies

The laboratory experiment of Fig. 1 is designed based on a scaled kinetic energy or inertia concept, and it includes only one brake corner without the suspension system and wheel-tire assembly. It consists of two torsional inertia elements, a brake rotor and a flyer wheel that are connected through a shaft, and a caliper assembly located on the rotor (Fig. 1). Since the total kinetic energy in the experiment is always less than that of a vehicle, lower actuation pressures are used. The maximum rotor temperature after a braking event remains around 50°C. Accordingly, the scope is limited to an examination of the “cold judder” problem where the effect of temperature on geometric distortions or friction coefficient is negligible. The geometric distortions of the brake rotor are assumed to be the only source of judder response. The rotor is allowed to have only the rotational motion in the experiment, and thus no wobbling motion is possible. The laboratory experiment has a torsional resonance, which allows vehicle-like variations in $T(t)$ to be obtained.

The actuation (hydraulic) pressure $p(t)$ is an input in the laboratory experiment. As explained later in this section, the total measured hydraulic pressure $p(t)$ has another component besides $p(t)$. This second component ($p_a(t)$) is the alternating part of $p(t)$, and it is assumed to be caused by $\zeta^r(t)$ and $\zeta^n(t)$. It is assumed the actuation pressure $p$ does not vary with time, and thus it leads to time-invariant $T_r$. The alternating part, $p_a(t)$, leads to oscillations in $T(t)$, but no feedback from $T(t)$ to $p(t)$ is considered in this article.

During the braking event, the friction torque is generated at the pad–brake rotor interface as the rotational system is decelerated from a high speed. The following sensors are utilized to capture speed-dependent events: a pressure $p(t)$ transducer at the hydraulic connection of the brake caliper, strain-gage based torque sensing $T(t)$ patch output to a telemetry system, tri-axial accelerometer on the caliper body, non-contact displacement probes (to measure $\zeta^r(t)$, $\zeta^n(t)$) close to the rotor surface at the leading edge of the caliper, and a quadrature encoder connected to the rotor (to measure $\theta_i(t)$).
Given the speed-dependent nature of problem, the time domain formulation is converted to the spatial domain. Due to the constant and alternating parts of \( T(t) \), one should expect the same behavior for angular displacement \( y_1(t) \) and angular velocity \( \dot{y}_1(t) \); i.e. \( \theta_1(t) = \theta_{r1}(t) + \theta_{a1}(t) \) and \( \dot{\theta}_1(t) = \dot{\theta}_{r1}(t) + \dot{\theta}_{a1}(t) \). Note that \( \theta_{a1}(t) \) and \( \dot{\theta}_{a1}(t) \) are the instantaneous parts, and \( \theta_{r1}(t) \) and \( \dot{\theta}_{r1}(t) \) are essentially the short-term time averages of \( \theta_1(t) \) and \( \dot{\theta}_1(t) \), i.e. \( \theta_{r1}(t) = \langle \theta_1(t) \rangle_v \) and \( \dot{\theta}_{r1}(t) = \Omega_{1}(t) = \langle \dot{\theta}_1(t) \rangle_v \), where \( v \) is a dummy variable in the short-term time averaging process. This claim is examined using the measurements of Fig. 2(a), where both measured \( \theta_1(t) \) (with solid line) and its short-term time average \( \theta_{a1}(t) \) (with circles) are displayed. Thus, one may calculate \( \theta_{a1}(t) \) given \( \theta_1(t) - \theta_{r1}(t) \). Fig. 2(b) represents \( \theta_{a1}(t) \) with respect to \( t \), and it shows that \( \theta_{a1}(t) \) is significant, especially in the vicinity of a resonance. Thus, \( \Omega_{1} \) is used in this study to highlight the importance of the speed (or order) domain instead of the time domain.

![Fig. 1. Schematic of the brake experiment and instrumentation. Signal conditioning items are not shown here.](image1)

![Fig. 2. Angular displacements of rotor. (a) \( \theta_1(t) \) and \( \theta_{r1}(t) \). Key: - measured \( \theta_1(t) \); O, calculated \( \theta_{a1}(t) \). (b) Calculated \( \theta_{a1}(t) \) with \( \theta_1(t) - \theta_{r1}(t) \).](image2)
especially in the peak (resonant) amplitude formation for $\Omega_2$, constant oscillation amplitude for $\Omega_0 < \Omega_1 < \Omega_c$, constant oscillation amplitude beyond the system resonance for $\Omega_t < \Omega_1 < \Omega_c$, and yet another amplification for $\Omega_1 < \Omega_1 < \Omega_c$. Further, observe that the lower bound of $T_1$ amplitudes, especially in the $\Omega_c < \Omega_1 < \Omega_t$ region, follows the zero torque line which is indicated by a horizontal line. This behavior seems to suggest the existence of a torsional clearance.

4. Order domain analysis

In order to gain more insight regarding a transient signal, say $x(t)$, the short-time Fourier transform $X(t,\omega)$ is calculated where $t$ is the time variable, and $\omega$ (in rad/s) is the continuous frequency variable [8].

$$X(t,\omega) = \int_{-\infty}^{\infty} x(t)w(t-t)\exp(-i\omega t)dt. \tag{1}$$

Here, $x(t)$ and $w(t)$ are the one-dimensional time signal and window function, respectively, and $\tau$ indicates the location of the sliding window. This equation is essentially the Fourier transform of $x(t)$, but viewed through a sliding window $w(t-t)$. Eq. (1) is implemented using the discrete time event process with 4096 points. The Hamming window is applied with an overlap of 97.6%. Typical short-time Fourier magnitude of this transient event is given in Fig. 3(b). First, observe that the higher orders ($n$) of the rotational speed ($\Omega_1$) arise as the slanted lines in this figure. This happens due to the rotor surface distortions, $\zeta^s_1(\Omega_1)$ and $\zeta^p_1(\Omega_1)$. For a perfectly smooth rotor with $\zeta^s_1 = \zeta^p_1 = 0$, none of these order lines should be visible. However, at least the first three orders are found to be significant in the experimental studies. Second, a system resonance is seen around 22 Hz (as a vertical line). This resonance causes an amplification in $T_1$ amplitudes, and orders beyond $n=1$ go through this resonance as $\Omega_1$ decreases. Finally, Fig. 3(b) shows that the super-harmonics of the excitation frequency also get excited while one of the orders passes through the torsional resonance. The amount of backlash (about 0.4°) is measured with the quadrature encoder, which is used to measure $\theta_1(t)$ by fixing the shaft and rotating the rotor as the backlash would permit. In the experiment of Fig. 1, $I_2$ is directly connected to $I_1$ through the constant velocity joint, and therefore backlash in this component plays a significant role on the response. However, in vehicle judder responses, the brake torque is transferred to the chassis through the knuckle and suspension tower. Nevertheless, other nonlinearities are still relevant; examples include steering system clearance, nonlinear bushing stiffness, and stoppers in the suspension system.

The amplitudes of normalized $T_1$ ($T_1$), $p_1$ ($p_1$), and disc thickness variation ($\Delta \zeta_1(\Omega_1) = |\zeta^s_1(\Omega_1) - \zeta^p_1(\Omega_1)|$) are compared in Table 2 where the values are normalized with $T_n$, $p_n$, and the mean value of $\Delta \zeta_1(\Omega_1)$, respectively, as the references. Selected values are tabulated for two different speeds: high speed before the resonance ($\Omega_1 = 975$ rev/min) and moderate speed at the resonance ($\Omega_1 = 635$ rev/min). At $\Omega_1 = 975$ rev/min $T_1$ and $p_1$ have the same trend, and amplitudes decrease as $n$ increases. The same behavior is again valid for $p_1$ at $\Omega_1 = 635$ rev/min, but $T_1$ has its maximum value when $n=2$. The main reason for this result is that at $\Omega_1 = 635$ rev/min, the second order of rotor surface distortion excites the torsional resonance, thus $T_1$ has a peak for $n=2$. For the sake of comparison, the comparable orders of $\Delta \zeta(\Omega_1)$ are listed as well in Table 2. Observe that the second order of the rotor surface distortion is indeed the dominant order. The higher orders of $p_1$ seem to diminish due to the overdamped characteristics of the hydraulic system, as the actuation pressure may not respond to the displacement excitations sufficiently fast enough at higher frequencies. (The dynamic investigation of the hydraulic system is beyond the scope of this paper.)

Fig. 4 shows the $n=1$, 2, and 3 lines as projected onto the speed axis. Here, the $T_1$ amplitudes are normalized with $T_n$. These order lines are extracted from Fig. 3(b) using a constant bandwidth filter. Usually, the $X(t,\omega)$ based order tracking
is an integer multiple of the frequency of the first-order oscillates with constant amplitude around an almost constant mean value during the braking event. From this observation, the bandwidth of 5 Hz is used. Essentially, the areas bounded by the solid lines in Fig. 3(b) are extracted for lead to high integrated to get the total energy at each order. As seen in Fig. 4, orders beyond only 1050 rev/min. Furthermore, Fig. 4 clearly shows a shift in the peak amplitude location to a lower amplitudes, especially at the critical speeds. An angular speed of about 1300 rev/min would be required for the first order to excite the torsional resonance; however, the initial speed of the laboratory experiment is limited to provided the frictional contact properties are known. Some simplifying assumptions must be made to develop a tractable analytical model of the laboratory experiment. requires an order track detection process, especially when the signal-to-noise ratio is low. Various algorithms such as the maximum likelihood estimator, image processing technique, neural network, and statistical model type order track detection have been utilized depending on specific applications [9]. However, the shape of the track in the current study is known from the measured \( \theta_1(\Omega_1) \); thus, a true order track detection algorithm is not necessary. Instead, the signal is processed with a constant bandwidth filter at center frequency \( f_0 \) with a bandwidth \( \Delta f \). The center frequency at each order is an integer multiple of the frequency of the first-order \( f_1 (f_0=hf_1) \), where \( f_1 \) is proportional to \( \Omega_1 \). In this study, a constant bandwidth of 5 Hz is used. Essentially, the areas bounded by the solid lines in Fig. 3(b) are extracted for \( n=1, 2, 3 \) and integrated to get the total energy at each order. As seen in Fig. 4, orders beyond \( n=1 \) excite the torsional resonance and lead to high \( \tilde{T}(\Omega_1) \) amplitudes, especially at the critical speeds. An angular speed of about 1300 rev/min would be required for the first order to excite the torsional resonance; however, the initial speed of the laboratory experiment is limited to only 1050 rev/min. Furthermore, Fig. 4 clearly shows a shift in the peak amplitude location to a lower \( \Omega_1(t) \) as \( n \) increases. In addition, the super-harmonics of the excitation frequency are found to be excited due to the clearance nonlinearity.

The measured (normalized) pressure \( p(\Omega_1) = p(\Omega_1)/p_0 \) is plotted in Fig. 5(a) with respect to \( \Omega_1(t) \). It is seen that \( p(\Omega_1) \) oscillates with constant amplitude around an almost constant mean value during the braking event. From this observation and Fig. 3(a), it is concluded that there is little feedback from \( T(\Omega_1) \) to \( p(\Omega_1) \). Also, the oscillation frequency of \( p(\Omega_1) \) decreases as the rotor slows down, which shows the need for order domain analysis in terms of \( \theta_1(t) \) and \( \Omega_1(t) \). To illustrate this effect, \( p(\Omega_1) \) and the (normalized) disc thickness variation \( \Delta \tilde{D}(\Omega_1) \) are shown in Fig. 5(b) for two revolutions of the rotor. It is clear that \( p(\Omega_1) \) and \( \Delta \tilde{D}(\Omega_1) \) follow the same trend. This relationship between \( p(\Omega_1) \) and \( \Delta \tilde{D}(\Omega_1) \) is summarized in Table 2. As seen from the tabular values, as \( \Omega_1 \) decreases, amplitudes of \( \Delta \tilde{D}(\Omega_1) \) slightly increase for \( n=4 \) and 5. Especially, the \( \Delta \tilde{D}(\Omega_1) \) amplitude at \( n=5 \) becomes higher than the amplitude at \( n=4 \). This same behavior is clearly seen from the \( p(\Omega_1) \) amplitudes at \( n=4 \) and 5.

5. Analytical formulation

From the experimental measurements, it is concluded that \( \Delta \tilde{D}(\Omega_1) \) acts as a speed-varying multi-order displacement excitation on the brake rotor, and it can be used to calculate \( p(\theta_1,\Omega_1) \) provided the frictional contact properties are known. Some simplifying assumptions must be made to develop a tractable analytical model of the laboratory experiment.

**Table 2**

Measured \( T(\Omega_1), \tilde{p}(\Omega_1) \) and \( \Delta \tilde{D}(\Omega_1) \) amplitudes for the first five orders at two different speeds: at high speed before resonance (975 rev/min) and at resonance (635 rev/min).

<table>
<thead>
<tr>
<th>Order number</th>
<th>( \Omega_1 = 975 \text{ rev/min} )</th>
<th>( \Omega_1 = 635 \text{ rev/min} )</th>
<th>( \Omega_1 = 975 \text{ rev/min} )</th>
<th>( \Omega_1 = 635 \text{ rev/min} )</th>
<th>( \Omega_1 = 975 \text{ rev/min} )</th>
<th>( \Omega_1 = 635 \text{ rev/min} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n=1 )</td>
<td>0.237</td>
<td>0.310</td>
<td>0.140</td>
<td>0.163</td>
<td>0.036</td>
<td>0.018</td>
</tr>
<tr>
<td>( n=2 )</td>
<td>0.114</td>
<td>3.659</td>
<td>0.092</td>
<td>0.103</td>
<td>0.157</td>
<td>0.142</td>
</tr>
<tr>
<td>( n=3 )</td>
<td>0.012</td>
<td>0.229</td>
<td>0.024</td>
<td>0.032</td>
<td>0.081</td>
<td>0.082</td>
</tr>
<tr>
<td>( n=4 )</td>
<td>0.002</td>
<td>0.085</td>
<td>0.003</td>
<td>0.007</td>
<td>0.003</td>
<td>0.019</td>
</tr>
<tr>
<td>( n=5 )</td>
<td>0.001</td>
<td>0.104</td>
<td>0.002</td>
<td>0.011</td>
<td>0.001</td>
<td>0.020</td>
</tr>
</tbody>
</table>

**Fig. 4.** Measured speed-dependent torque (\( T(\Omega_1) \)) amplitudes for \( n=1, 2, 3 \) and 4. Key: \( n=1; \cdots \cdots \), \( n=2; \cdots \cdots \), \( n=3 \).
First, thermal effects that induce disc warping, coning, and hot spots are ignored. This is a valid assumption, and the rotor temperature increases about 5–10 °C during a braking event because of the reduced inertia in the experiment. Second, the coefficient of friction ($\mu$) at the pad–rotor contact interface and the effective brake radius ($R$) are assumed to be temporally and spatially invariant. In the experiments, great care is taken to ensure a constant $\mu$, and thus the same burnished rotor and pads are used for all tests. This assumption is also consistent with Jacobsson [2], who concludes that spatial variations in $\mu$ and $R$ could be ignored for some rotors with high $D_x(t)$, which is the main excitation for $T(t)$ in this study. Third, the effect of brake caliper dynamics on $T(t)$ is not taken into account. Fourth, the brake pad–rotor interface is simplified and described with a single normal (point) force instead of a distributed force (or line or area contact). Moreover, the contact model is represented with a linear spring and a viscous damper. Finally, all the torsional springs and dampers in this study are assumed to be linear time-invariant.

The experiment of Fig. 1 is modeled with a two degree of freedom torsional system as shown in Fig. 6. Here, the brake rotor ($I_1$) and flywheel ($I_2$) are represented by pure inertial elements, and the shaft in between is described by a torsional spring ($K_{12}$) and a viscous damper ($C_{12}$). The friction-induced torque ($T_f(\theta_1, \dot{\theta}_1)$) and drag torque ($T_d$) are assumed to act as excitation and load, respectively. Note that $T_f(\theta_1, \dot{\theta}_1)$ is a function of both $\theta_1(t)$ and $\dot{\theta}_1(t)$, as explained later, and $T_d$ is constant. Even though drag torque usually varies with speed, the $T_d$ term in this study represents the effective dissipative load that includes friction at the bearings, off-brake drag torque at the pad/rotor interface and structural damping. Nevertheless, the contribution of $T_d$ is minimal when compared with $T_f(\theta_1, \dot{\theta}_1)$. The experiments confirm this as the deceleration rate during the brakes-on case is an order of magnitude higher than that observed during the brakes-off case. The governing equations for the torsional model are derived as follows, where $\delta_{12} = \theta_1 - \theta_2$ is the relative displacement between $I_1$ and $I_2$, and $\theta_2$ represents the angular displacement of $I_2$.

$$I_1\ddot{\theta}_1 + C_{12}(\delta_{12})\dot{\delta}_{12} + K_{12}h(\delta_{12}) = -T_f(\theta_1, \dot{\theta}_1),$$  \hspace{1cm} (2)$$

$$I_2\ddot{\theta}_2 - C_{12}(\delta_{12})\dot{\delta}_{12} - K_{12}h(\delta_{12}) = -T_d.$$  \hspace{1cm} (3)

Fig. 5. Measured actuation pressure and disc thickness variation as a function of angular displacement: (a) speed-dependent pressure $\bar{p}(\Omega_1)$ and (b) $|\bar{p}(\Omega_1)|$ and $|\Delta T(\Omega_1)|$ over two revolutions of the rotor. Key: $\bigcirc$ $\bar{p}(t)$, $\bigtimes$ $|\bar{p}(t)|$, $\times$ $|\Delta T(t)|$. 

Here, arbitrary nonlinear dissipative \(g(\delta t_{12})\) and elastic \(h(\delta t_{12})\) functions that arise due to the clearance (as explained later in Section 8) are employed. The frictional torque \(T_f(\theta_1, \dot{\theta}_1)\) is also defined in terms of rotational \((T_{rf})\) and alternating \((T_{af}(\theta_1, \dot{\theta}_1))\) parts. The rotational part is calculated using the actuation pressure \(p_a\), which is constant, and thus \(T_{rf}\) is constant as well. The alternating part is calculated from \(p_a(\theta_1, \dot{\theta}_1)\) which arises due to \(\Delta \xi(\theta_1, \dot{\theta}_1)\). The surface distortions are separately defined on both sides of the rotor as

\[
\xi^f(\theta_1) = \sum_{n=1}^{N_p} \Xi^f_n \sin(n\theta_1 + \phi^f_n),
\]

\[
\xi^p(\theta_1) = \sum_{n=1}^{N_p} \Xi^p_n \sin(n\theta_1 + \phi^p_n),
\]

where \(N_p\) represents the total number of orders used for the Fourier expansion, and \(\Xi_n\) and \(\phi_n\) are the distortion amplitude and the phase for the finger (superscript \(f\)) and piston (superscript \(p\)) sides at order \(n\), respectively. From these equations, \(T_{rf}\) and \(T_{af}(\theta_1, \dot{\theta}_1)\) are calculated as given below:

\[T_{rf} = 2\mu R_p A,\]

\[T_{af}(\theta_1, \dot{\theta}_1) = \mu R_p \xi^f(\theta_1, \dot{\theta}_1) + \mu R_p \xi^f(\theta_1, \dot{\theta}_1) + \mu R_p \xi^p(\theta_1, \dot{\theta}_1) + \mu R_p \xi^p(\theta_1, \dot{\theta}_1),\]  

where \(A\), \(k_p\), and \(c_p\) represent the master cylinder area, pad contact stiffness, and pad viscous damping terms for finger and piston sides, respectively. Note that Duan and Singh [4] assume a nonlinear \(k_p\), which is a function of \(p_a\). Additionally, Kang and Choi [5] define \(k_f\) and \(c_p\) as functions of \(p_a\) and \(\dot{\theta}_1\), respectively; however, in this study these parameters are linearized about an operating point and then assumed not to vary with \(p_a\) and/or \(\dot{\theta}_1\).

The torsional natural frequency is obtained by calculating the torsional parameters. The total inertia of the system \(I_1 = I_1 + I_2 = I_1 + I_2 = I_1\) is closer to \(I_1\) since \(I_2 \gg I_1\). The torsional stiffness of the steel shaft (of length 1.2 m and 13.6 mm radius) is calculated using \(K_{12} = GJ/L\) where \(G\), \(J\), and \(L\) are shear modulus, polar moment of inertia, and the length of the shaft, respectively. The cast iron rotor has a radius of 160 mm and is 22 mm thick. Consequently the torsional natural frequency is \(\omega_1 = \sqrt{K_{12}/I_1} = 146.6\text{ rad/s} = 23.3\text{ Hz}\), which is reasonably close to the frequency observed in the measurements. If a wheel were to be placed in the experiment, this frequency would decrease.

6. Simplified analytical solution using a quasi-linear model

Before solving the nonlinear model given by Eqs. (2) and (3), the quasi-linear model (without the clearance nonlinearity) is analytically solved. As defined with Eqs. (4) and (5), there are two displacement inputs to the system as applied on the finger and piston sides of the caliper. By assuming identical elastic and dissipative properties on both sides, these inputs are combined, and the amplitude and phase of the combined displacement input are expressed as follows:

\[\Xi_n = \left|\Xi^f_n \exp(i\phi^f_n) + \Xi^p_n \exp(i\phi^p_n)\right|,\]

\[\phi_n = \tan^{-1}\left(\frac{\text{Im}(\Xi^f_n \exp(i\phi^f_n) + \Xi^p_n \exp(i\phi^p_n))}{\text{Re}(\Xi^f_n \exp(i\phi^f_n) + \Xi^p_n \exp(i\phi^p_n))}\right).\]

Thus \(T_f(\theta_1, \dot{\theta}_1)\) reduces to

\[T_f(\theta_1, \dot{\theta}_1) = \mu R_p A + \mu R_p \sum_{n=1}^{N_p} n\theta_1 \Xi_n \cos(n\theta_1 + \phi_n) + \mu R_p \sum_{n=1}^{N_p} \Xi_n \sin(n\theta_1 + \phi_n).\]
By ignoring the clearance nonlinearity in Eqs. (2) and (3), linear system equations are obtained as follows:

\[ I_1 \ddot{\theta}_1 + C_{12}(\dot{\theta}_1 - \dot{\theta}_2) + K_{12}(\theta_1 - \theta_2) = -T_f(\theta_1, \dot{\theta}_1), \tag{11} \]

\[ I_2 \ddot{\theta}_2 + C_{12}(\dot{\theta}_2 - \dot{\theta}_1) + K_{12}(\theta_2 - \theta_1) = -T_d, \tag{12} \]

Note that the right hand side of Eq. (11) is defined in terms of the motion variables. To get an explicit expression of \( T_f(\theta_1, \dot{\theta}_1) \), it is assumed that \( \theta_1(t) \) and \( \dot{\theta}_2(t) \) are composed of mean and alternating parts which are being affected by \( T_r \) and \( T_{af}(\theta_1, \dot{\theta}_1) \), respectively. Mathematically they are \( \theta_1(t) = \theta_1(t) + \theta_1(t) \) and \( \dot{\theta}_2(t) = \dot{\theta}_2(t) + \dot{\theta}_2(t) \). It is reasonably safe to assume that \( \theta_2(t) \) is negligible since \( I_2 \gg I_1 \). Furthermore, it is observed that the angular deceleration of the system during the braking experiment is almost constant, \( \dot{\theta}_2 = -A \). Accordingly, \( \dot{\theta}_2 = -A t + \Omega_0 \), and \( \dot{\theta}_2 = -A t^2 / 2 + \Omega_0 t + \Psi_0 \). In these equations, \( A, \Omega_0, \) and \( \Psi_0 \) represent the angular deceleration, initial angular velocity, and initial position of \( I_2 \), respectively. To find an analytical solution for \( \theta_2(t) \), rewrite Eq. (11) for only the mean part as follows:

\[ I_1 \ddot{\theta}_1 + C_{12}\ddot{\theta}_1 + K_{12}\theta_1 = -\mu R_p A + C_{12}(\dot{\theta}_1 + \dot{\theta}_2) + K_{12}\left( -\frac{A}{2} t^2 + \Omega_0 t + \Psi_0 \right). \tag{13} \]

From Eq. (13), \( \theta_1(t) \) is

\[ \theta_1(t) = -\frac{A}{2} t^2 + \Omega_0 t + \frac{\Psi_0 K_{12} + I_1 A - \mu R_p A}{K_{12}} \cos \left( \sqrt{\frac{I_1 K_{12} - (C_{12}^2 / 4)}{I_1}} t \right) + \frac{C_{12}}{2 \sqrt{I_1 K_{12} - (C_{12}^2 / 4)}} \sin \left( \sqrt{\frac{I_1 K_{12} - (C_{12}^2 / 4)}{I_1}} t \right). \tag{14} \]

Eq. (14) shows that \( \theta_1(t) \) follows \( \theta_2(t) \) with a lag that also has both mean and oscillating parts. The oscillating part is neglected by assuming that \( I_1 \) and \( I_2 \) rotate as a rigid body, except that \( I_1 \) follows \( I_2 \) with this lag. This is a reasonable assumption since the amplitude of the oscillating part is indeed very small (as shown in the next section), and it decreases as the time passes due to the exponential function at the denominator. This assumption simplifies \( \theta_1(t) \) to

\[ \theta_1(t) = -\frac{A}{2} t^2 + \Omega_0 t + \frac{\Psi_0 K_{12} + I_1 A - \mu R_p A}{K_{12}}. \tag{15} \]

Assume that only \( \theta_1(t) \) affects \( T_f(\theta_1, \dot{\theta}_1) \). By incorporating this assumption into Eq. (10), the following is obtained:

\[ T_f(t) = \mu R_p A + \mu R_p \sum_{n=1}^{N_s} n(\dot{-A} t + \Omega_0) \xi_n \cos \left( -\frac{\pi A}{2} t^2 + \Omega_0 t + \Phi_n \right) + \mu R_p \sum_{n=1}^{N_s} \xi_n \sin \left( -\frac{\pi A}{2} t^2 + \Omega_0 t + \Phi_n \right), \tag{16} \]

where \( \Phi_n = n \Psi_0 + n A / K_{12} - n \mu R_p A / K_{12} + \Phi_n \) is a combined phase value. To calculate \( \theta_1(t) \), Eq. (11) is rewritten in terms of the alternating parts and transformed to the Laplace domain (s) with zero initial conditions

\[ \Theta_{a_1}(s) = \frac{1}{I_1 s^2 + C_{12} s + K_{12}} \mathcal{L} \left\{ -T_f(t) \right\}. \tag{17} \]

Since Eq. (17) is a product of two functions in the Laplace domain, a convolution of these functions results in the following time domain expression:

\[ \Theta_{a_1}(t) = \int_0^t \exp \left( -\frac{C_{12}(u - w)}{I_1} \right) \mathcal{L} \left\{ -T_f(u) \right\} du, \tag{18} \]

where \( u \) is a dummy variable. The integral for the damped system, as defined by Eq. (18), includes a product of exponential and trigonometric functions. It may be solved with some algebraic manipulations. However, for the sake of simplicity, an undamped system is defined by ignoring \( C_{12} \) and \( \xi_0 \). (As shown in the next section, \( \xi_0 \) does not affect the results, but \( C_{12} \) has a significant effect on \( T(t) \).) This assumption simplifies Eq. (18) to

\[ \Theta_{a_1}(t) = \sum_{n=1}^{N_s} \int_0^t \cos \left( \frac{\pi A}{2} u^2 + (\pi A_0 - \pi \omega_1) u + (\omega_1 t + \Phi_n) \right) \mathcal{L} \left\{ \mu R_p A \sin \left( -\frac{\pi A}{2} t^2 + (\pi A_0 - \pi \omega_1) t + (\omega_1 t + \Phi_n) \right) \right\} du, \tag{19} \]

Note that the order of the integration and summation is changed in Eq. (19). Eq. (19) is written by using the trigonometric identities as

\[ \Theta_{a_1}(t) = -\frac{\mu R_p A}{2 \sqrt{I_1 K_{12}}} \sum_{n=1}^{N_s} \xi_n \int_0^t \cos \left( \frac{\pi A}{2} u^2 + (\pi A_0 - \pi \omega_1) u + (\omega_1 t + \Phi_n) \right) du \]

\[ + \frac{\mu R_p A}{2 \sqrt{I_1 K_{12}}} \sum_{n=1}^{N_s} \xi_n \int_0^t \cos \left( -\frac{\pi A}{2} u^2 + (\pi A_0 - \pi \omega_1) t + \omega_1 t + \Phi_n \right) du. \tag{20} \]
Solutions of the integrals given in Eq. (20) are given by the Fresnel integral operators of cosine (C) and sine (S) type respectively:

\[
\int \cos(ax^2 + 2bx + c) \, dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos\left(\frac{\sqrt{2a}}{a}b\right) C\left(\frac{\sqrt{2a}}{a}b + \frac{\sqrt{2a}}{a}c\right) + \sin\left(\frac{\sqrt{2a}}{a}b\right) S\left(\frac{\sqrt{2a}}{a}b + \frac{\sqrt{2a}}{a}c\right) \right\}
\]

These integrals are defined by the error (erf) functions as [10]

\[
C(z) = \text{Re} \left\{ \frac{1+i}{2} \text{erf} \left( \frac{\sqrt{\pi}}{2} (1-i)z \right) \right\},
\]

\[
S(z) = \text{Im} \left\{ \frac{1+i}{2} \text{erf} \left( \frac{\sqrt{\pi}}{2} (1-i)z \right) \right\}.
\]

By using Eqs. (21)–(23) in (20), \( \theta_{a1}(t) \) is obtained as

\[
\theta_{a1}(t) = [\Theta_{a1}^1(t) - \Theta_{a1}^1(0)] - [\Theta_{a1}^2(t) - \Theta_{a1}^2(0)],
\]

\[
\Theta_{a1}^1(t) = -\frac{\mu Rk_p \sqrt{\pi}}{2 \sqrt{1} K_{12} A \sqrt{1}} \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \left\{ \cos \left( \frac{m(n_0 + c) + c}{2nA} + \omega_1 t + \Phi_n \right) C \left( \frac{1}{\sqrt{nA}} \left( nA t - n \Omega_0 - \omega_1 \right) \right) \right\} + \sin \left( \frac{m(n_0 + c) + c}{2nA} - \omega_1 t + \Phi_n \right) S \left( \frac{1}{\sqrt{nA}} \left( nA t - n \Omega_0 - \omega_1 \right) \right),
\]

\[
\Theta_{a1}^2(t) = -\frac{\mu Rk_p \sqrt{\pi}}{2 \sqrt{1} K_{12} A \sqrt{1}} \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \left\{ \cos \left( \frac{m(n_0 + c) + c}{2nA} + \omega_1 t + \Phi_n \right) C \left( \frac{1}{\sqrt{nA}} \left( nA t - n \Omega_0 + \omega_1 \right) \right) \right\} + \sin \left( \frac{m(n_0 + c) + c}{2nA} + \omega_1 t + \Phi_n \right) S \left( \frac{1}{\sqrt{nA}} \left( nA t - n \Omega_0 + \omega_1 \right) \right).
\]

The arguments of Fresnel integrals given by Eqs. (25) and (26) represent slow and fast time scales. It is obvious that the slow time scale is related to the envelope function concept as initially described by Jacobsson [2,3]. The most dominant term in these arguments is \( n \Omega_0 \). Since the term \( nA t \) has the same sign for both arguments, the last term, which is \( \omega_1 \), should be checked to determine the fast and slow time scales. Since \( n \Omega_0 \) and \( \omega_1 \) have the same sign for the argument of Eq. (25), this argument should be related to the fast time scale. Thus, the argument of Eq. (26) is used to propose a refined envelope function:

\[
E(t) = \sum_{n=1}^{N} \sqrt{[C(\rho_n(t) - C(\rho_n(0))]^2 + [S(\rho_n(t) - S(\rho_n(0))]^2},
\]

\[
\rho_a(t) = \frac{1}{\sqrt{nA}} (nA t - n \Omega_0 + \omega_1).
\]

Compare this with Jacobsson’s expression [2], where the angular displacement of the caliper is defined as \( \theta_a(t) = a(t) \sin(n \Omega_0(t)) + a(t) \cos(n \Omega_0(t)) \), where \( a(t) \) and \( d(t) \) are time varying functions with a slow time scale compared to the \( \sin(n \Omega_0(t)) \) and \( \cos(n \Omega_0(t)) \) terms. Hence, the envelope function of \( \theta_a(t) \) is obtained as \( E(t) = \sqrt{a(t)^2 + d(t)^2} \). In the current article, \( a(t) \) and \( d(t) \) are numerically determined. In yet another paper, Jacobsson [3] uses a similar method and obtains the envelope function as \( E(t) = \sqrt{Y_1(t)^2 + Y_2(t)^2} \) where \( Y_1(t) = \text{Re}(V_1(t)) \) and \( Y_2(t) = \text{Im}(V_1(t)) \). Here, Re and Im denote real and imaginary parts. In Jacobsson’s paper [3], the expression for \( V_1(t) \) is similar to Eq. (26) as derived above, except that a multi-order surface distortion profile is assumed in our study. The approach adopted in the current paper differs from an older study where Lewis [11] had approximated the envelope of the transient response for a single degree of freedom system in the vicinity of the resonance by using contour integration procedure for selected cases.

7. Comparison of analytical and numerical solutions

The analytical expression of Section 6 is verified by using the numerical solution of the quasi-linear model. Fig. 7 compares the analytical and numerical predictions of normalized torque \( \tilde{T}(t) \). First, the same trend is seen for both methods: constant \( \tilde{T}(t) \) amplitude for \( \Omega_0 < \Omega_1 < \Omega_0 \); peak amplitude formation for \( \Omega_c < \Omega_1 < \Omega_0 \); yet another constant \( \tilde{T}(t) \) amplitude region for \( \Omega_0 < \Omega_1 < \Omega_c \); and another peak amplitude formation between \( \Omega_0 < \Omega_1 < \Omega_c \). Moreover, the amplitudes do not decay after the resonance, due to the lack of damping \( C_{12} \) in the analytical model. Since it is difficult to visually compare the results in Fig. 7, normalized peak-to-peak torque amplitudes \( (T_{pp}(t)) \) for the regions mentioned above are compared in Table 3. Both methods are in good agreement before the resonance for \( \Omega_0 < \Omega_1 < \Omega_0 \). However, when \( \Omega_1 < \Omega_0 \), the analytical and numerical solutions begin to differ. This is because \( \theta_{a1}(t) \) starts to become more and more significant, and its effects on \( \tilde{T}(\theta_1, \theta_1) \) can no longer be ignored. This is shown in Fig. 8(a), where \( \theta_{a1}(t) \) becomes almost 40% of \( \theta_{11}(t) \). Fig. 8(b) shows that the \( \theta_{a2}(t) = 0 \) assumption is reasonable since \( \theta_{22}(t) \) shows a linear decay. Further, the maximum amplitude of the oscillating part of Eq. (14), which is \( AI/K_{12} - \mu R p A/\Omega_{12} \), is about 0.008 rad, which implies
that the assumption of neglecting this part is valid. The importance of $C_{12}$ is clear from the last two columns of Table 3, which compares numerical predictions of $T_{pp}(t)$ for undamped and damped cases. Observe that $C_{12}$ is indeed effective during the entire braking event, not just at the resonance.

Fig. 7(a) and (b) present the envelope functions for $y_{a1}(t)$ and $T(t)$, respectively, where the normalized value is given by $\overline{y}_{a1}(t) = y_{a1}(t)/b$, where $b$ is the backlash. The oscillation amplitudes are successfully predicted. The proposed envelope function method can be used to estimate the non-resonant $T(t)$ amplitudes especially at higher speeds. Such amplitudes are often used by vehicle engineers to set and verify dynamic performance targets.

Table 3
Validation of $T_{pp}(t)$ amplitude predictions (peak-to-peak) based on the quasi-linear model.

<table>
<thead>
<tr>
<th>Rotor speed</th>
<th>Measured</th>
<th>Analytical solution</th>
<th>Numerical solution with $\zeta_1 = 0$</th>
<th>Numerical solution with $\zeta_1 = 0.008$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_6 &lt; \Omega_1 &lt; \Omega_A$</td>
<td>1.37</td>
<td>1.55</td>
<td>1.46</td>
<td>0.59</td>
</tr>
<tr>
<td>$\Omega_D &lt; \Omega_1 &lt; \Omega_B$</td>
<td>15.92</td>
<td>26.51</td>
<td>24.61</td>
<td>11.15</td>
</tr>
<tr>
<td>$\Omega_C &lt; \Omega_1 &lt; \Omega_D$</td>
<td>2.33</td>
<td>23.54</td>
<td>21.53</td>
<td>0.84</td>
</tr>
<tr>
<td>$\Omega_L &lt; \Omega_1 &lt; \Omega_E$</td>
<td>5.40</td>
<td>25.79</td>
<td>23.57</td>
<td>1.76</td>
</tr>
</tbody>
</table>

Fig. 8. Predicted angular velocities based on a numerical solution of the quasi-linear model: (a) $\Omega_1(t)$ vs. $t$ and (b) $\Omega_2(t)$ vs. $t$.

that the assumption of neglecting this part is valid. The importance of $C_{12}$ is clear from the last two columns of Table 3, which compares numerical predictions of $T_{pp}(t)$ for undamped and damped cases. Observe that $C_{12}$ is indeed effective during the entire braking event, not just at the resonance.

Fig. 9(a) and (b) present the envelope functions for $\overline{y}_{a1}(t)$ and $T(t)$, respectively, where the normalized value is given by $\overline{y}_{a1}(t) = y_{a1}(t)/b$, where $b$ is the backlash. The oscillation amplitudes are successfully predicted. The proposed envelope function method can be used to estimate the non-resonant $T(t)$ amplitudes especially at higher speeds. Such amplitudes are often used by vehicle engineers to set and verify dynamic performance targets.

8. Nonlinear model

To describe the clearance in the joint, $g(\delta_{12})$ and $h(\delta_{12})$ of Eqs. (2) and (3) are described in the piecewise linear manner:

$$g(\delta_{12}) = \begin{cases} 
1 & \delta_{12} > b \\
0 & -b \leq \delta_{12} \leq b \\
1 & \delta_{12} < -b 
\end{cases}$$

(29)
Envelope functions.

\[ \tanh d \]

\[ \text{s}gn \]

\[ \text{Eqs. (31) and (32) are approximated with hyperbolic tangent functions \[12\] which smoothen Eqs. (2) and (3).} \]

The approximated functions are:

\[ g(\delta_{12}) = \frac{1}{2} + \frac{\text{sgn}(\delta_{12} - b)(1 + \text{sgn}(\delta_{12}))}{4} - \frac{\text{sgn}(\delta_{12} + b)(1 - \text{sgn}(\delta_{12}))}{4}, \]

\[ h(\delta_{12}) = \delta_{12} + \frac{(\delta_{12} - b)\text{sgn}(\delta_{12} - b)}{2} - \frac{(\delta_{12} + b)\text{sgn}(\delta_{12} + b)}{2}. \]

Consider three regimes based on the above model. The first regime, when \( \delta_{12} > b \), implies that the shaft is in contact with \( I_1 \) and \( I_2 \) drives \( I_1 \). The second regime, when \( -b \leq \delta_{12} \leq b \), represents a loss of contact, and no torque is transmitted by the shaft; both \( I_1 \) and \( I_2 \) rotate as independent rigid bodies. The last regime, when \( \delta_{12} < -b \), \( I_1 \) and shaft are again in contact; however, now \( I_2 \) drives \( I_1 \).

As previously mentioned \( I_2 \approx 100I_1 \), hence \( C_{12} \) is calculated from the single degree of freedom torsional system formulation, and the corresponding damping ratio \( \zeta_1 \) is assumed to be 0.008. Similarly, \( k_p \) and \( c_p \) are estimated based on the single degree of freedom formulation where the brake rotor acts as the mass. For this translational system, the natural frequency is 110 Hz and the damping ratio is assumed to be 0.006. Finally, \( \mu = 0.4 \), \( p_r = 0.17 \text{ MPa} \), and the hydraulic piston radius \( s \) is 12.7 mm.

The nonlinear model of Section 5 with Eqs. (31) and (32) is numerically solved using two different methods. In method A, Eqs. (31) and (32) are approximated with hyperbolic tangent functions [12] which smoothen Eqs. (2) and (3). The approximated functions are:

\[ g_1(\delta_{12}) \approx \frac{1}{2} + \frac{\text{tanh}(\sigma(\delta_{12} - b))(1 + \text{tanh}(\sigma\delta_{12}))}{4} - \frac{\text{tanh}(\sigma(\delta_{12} + b))(1 - \text{tanh}(\sigma\delta_{12}))}{4}, \]

\[ h_1(\delta_{12}) \approx \delta_{12} + \frac{(\delta_{12} - b)\text{tanh}(\sigma(\delta_{12} - b))}{2} - \frac{(\delta_{12} + b)\text{tanh}(\sigma(\delta_{12} + b))}{2}, \]

where \( \sigma \) is a regularizing factor for the hyperbolic tangent function. Approximated functions approach the original functions as \( \sigma \) increases. However, a very high value of \( \sigma \) can induce numerical instabilities for a highly nonlinear system, and a very low value of \( \sigma \) is often not sufficient. For instance, Kim et al. [12] suggest that the \( \sigma \) value must be at least 100.

Method B solves the nonlinear model by numerically integrating the discontinuous equations while tracking the conditions specified with Eqs. (29) and (30). As the integration continues, the system switches states, which requires a change in the governing equations. Thus, before switching from one state to another state, the integration must be stopped, and the equations must be updated. The integration starts again while admitting the final values of the previous state variables as the initial conditions for the next state. Method B is the rigorous way of solving the problem, but it is computationally intensive. Two main issues for this method are the detection and location of specific events that would trigger the state switching based on the \( |\delta_{12}| - b = 0 \) condition. When the integration time step is not sufficiently small, an even number of crossings over the discontinuity could occur, and the event may not be successfully detected. This may require variable time step and event detection algorithms [13,14].

The initial conditions used for the numerical integration are \( \dot{\theta}_1 = \theta_2 = 0, \dot{\theta}_1 = \dot{\theta}_2 = \Omega_1(0) \). In order to check the repeatability of the numerical code, it is run multiple times, and exactly the same results are obtained, which shows numerical stability and convergence. Furthermore, the initial conditions are varied by setting \( \dot{\theta}_1 = 0 \), and the difference in the peak-to-peak resonant amplitudes is found to be less than 0.1%.

![Fig. 9. Predicted speed-dependent angular displacement and torque along with their envelope functions using the quasi-linear model: (a) \( S_{\Omega_1}(\Omega_1) \) vs. \( \Omega_1(t) \) and its envelope; (b) \( T(\Omega_1) \) vs. \( \Omega_1(t) \) and its envelope. Key: \( -\), \( S_{\Omega_1}(\Omega_1) \) and \( T(\Omega_1) \); \( \times \), envelope functions.](image)
9. Experimental validation

Predicted normalized torques $T(t)$ using the nonlinear model with two numerical methods (A and B) are displayed in Fig. 10. Also, Table 4 compares $T_{pp}(t)$ for measured and predicted torque amplitudes over selected speed ranges.

Fig. 10. Predicted $\overline{T}(\Omega_1)$ vs. $\Omega_1(t)$ based on the nonlinear model: (a) with approximate equations (method A) given $s=10^5$; (b) with approximate equations (method A) given $s=100$ and (c) with discontinuous equations (method B). Refer Table 4 for a quantitative comparison.
In particular, Fig. 10(a) and (b) show the results for method A, which uses a smoothening function with a very high \( s = 10^6 \) and a very low \( s = 100 \) regularizing factor. The predictions of Fig. 10(a) and (c) exhibit a close match with the measured torques of Fig. 3(a). However, the prediction is very poor in Fig. 10(b) with \( s = 100 \), especially when \( \Omega_1 < \Omega_E \). The reason for this discrepancy can be understood from Fig. 11 where \( \delta(t) \) is plotted with respect to \( \Omega_1(t) \). Straight horizontal lines in this figure represent the boundaries for a loss of contact, i.e. shaft and rotor are not in contact in between these lines. Like Fig. 3(a), different events are observed: full contact for \( \Omega_B < \Omega_1 < \Omega_A \); single-sided impacts for \( \Omega_D < \Omega_1 < \Omega_C \); full contact beyond the resonance for \( \Omega_C < \Omega_1 < \Omega_D \); single-sided impacts for \( \Omega_F < \Omega_1 < \Omega_E \) as excited by the third order of the rotor profile; and full contact when \( \Omega_1 < \Omega_F \) as the third order passes the torsional resonance. Very slight single-sided impacts occur when \( \Omega_1 < \Omega_E \), and \( T(t) \) barely touches the zero torque line. Since this is a

<table>
<thead>
<tr>
<th>Rotor speed</th>
<th>Measured</th>
<th>Predicted with smoothened equations (method A) for ( \sigma = 10^6 )</th>
<th>Predicted with smoothened equations (method A) for ( \sigma = 100 )</th>
<th>Predicted with discontinuous equations (method B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega_B &lt; \Omega_1 &lt; \Omega_A )</td>
<td>1.37</td>
<td>1.32</td>
<td>1.49</td>
<td>1.32</td>
</tr>
<tr>
<td>( \Omega_C &lt; \Omega_1 &lt; \Omega_B )</td>
<td>4.87</td>
<td>4.53</td>
<td>4.63</td>
<td>4.53</td>
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<tr>
<td>( \Omega_D &lt; \Omega_1 &lt; \Omega_C )</td>
<td>15.92</td>
<td>15.62</td>
<td>15.69</td>
<td>15.60</td>
</tr>
<tr>
<td>( \Omega_E &lt; \Omega_1 &lt; \Omega_D )</td>
<td>2.33</td>
<td>2.18</td>
<td>2.38</td>
<td>2.18</td>
</tr>
<tr>
<td>( \Omega_F &lt; \Omega_1 &lt; \Omega_E )</td>
<td>5.40</td>
<td>4.41</td>
<td>5.38</td>
<td>4.41</td>
</tr>
<tr>
<td>( \Omega_1 &lt; \Omega_F )</td>
<td>1.99</td>
<td>1.50</td>
<td>5.79</td>
<td>1.50</td>
</tr>
</tbody>
</table>

**Fig. 11.** Predicted relative (normalized) displacement \( \delta(\Omega_1) \) as a function of speed \( \Omega_1(t) \). Single- and double-sided impacts occur when \( \delta(\Omega_1) \) crosses the boundaries as shown in the figure by horizontal lines.

**Fig. 12.** Validation of speed-dependent torque \( T(\Omega_1) \) amplitude predictions for \( n = 2 \) and \( n = 3 \): (a) \( n = 2 \) and (b) \( n = 3 \). Key: ○, measured; —, predicted using the nonlinear model.

In particular, Fig. 10(a) and (b) show the results for method A, which uses a smoothening function with a very high \( (\sigma = 10^6) \) and a very low \( (\sigma = 100) \) regularizing factor. The predictions of Fig. 10(a) and (c) exhibit a close match with the measured torques of Fig. 3(a). However, the prediction is very poor in Fig. 10(b) with \( \sigma = 100 \), especially when \( \Omega_1 < \Omega_E \). The reason for this discrepancy can be understood from Fig. 11 where \( \delta(t) = \delta(t)/b \) is plotted with respect to \( \Omega_1(t) \). Straight horizontal lines in this figure represent the boundaries for a loss of contact, i.e. shaft and rotor are not in contact in between these lines. Like Fig. 3(a), different events are observed: full contact for \( \Omega_B < \Omega_1 < \Omega_A \); single-sided impacts for \( \Omega_D < \Omega_1 < \Omega_C \) as a result of the excitation of the torsional resonance by the second order of the rotor profile; double-sided impacts for \( \Omega_D < \Omega_1 < \Omega_C \); full contact beyond the resonance for \( \Omega_C < \Omega_1 < \Omega_D \); single-sided impacts for \( \Omega_F < \Omega_1 < \Omega_E \) as excited by the third order of the rotor profile; and full contact when \( \Omega_1 < \Omega_F \) as the third order passes the torsional resonance. Very slight single-sided impacts occur when \( \Omega_1 < \Omega_E \), and \( T(t) \) barely touches the zero torque line. Since this is a
transition line between two states, the smoothening approximation is crucial. In the $\sigma=100$ case, $g_1(\delta_{12})$ is under-approximated, which results in a smaller dissipative force. Thus, the impacts do not decay for this particular value of $\sigma$.

To further validate numerical method B (which uses discontinuous equations), its results are compared with measurements in the order domain with the technique described in Section 4. Fig. 12(a) and (b) compare the amplitudes of measured and predicted $T(t)$ for $n=2$ and $n=3$ respectively. Again, a close match between measurements and predictions validates the nonlinear model.

10. Conclusion

The speed-dependent response of a brake corner has been investigated in this article using experimental, analytical, and numerical approaches. First, a new laboratory experiment has been constructed to simulate vehicle-like judder behavior. Measurements exhibit a good correlation between $p(t)$ and $\Delta \tilde{z}(t)$, which suggest that $p(t)$ oscillations are proportionally excited by the surface distortions. In addition, the key regimes during the braking event are successfully identified, and their source mechanisms are explained. The order analysis procedures are developed, which clearly show that at least the first three orders of the rotor surface distortion profile $\Delta \tilde{z}(t)$ introduce fluctuations in $p(t)$ and $T(t)$. Second, an improved closed form analytical solution of the quasi-linear model has been obtained, and this solution agrees well with the numerical solution, especially at higher speeds. Further, the analytical solution is used to obtain the envelope function of $T(t)$. The proposed analytical solution and envelope function can be used to set non-resonant $T(t)$ amplitude targets during product development. Third, a nonlinear model (with clearance) of the experiment is developed by extending the quasi-linear model. This model is numerically solved with two different methods and then experimentally validated. The model was validated numerically, and the resonant amplification of $T(t)$ was predicted successfully in the presence of clearance nonlinearity.

Several contributions emerge over the prior literature [1–4]. The dynamic friction experiment (as introduced in this article) simulates the vehicle judder in a simplified yet controlled manner; the path clearance conceptually duplicates the key features of many vehicle system nonlinearities. The controlled experiments yield benchmark data in terms of multiple-orders of the rotor surface distortion profile exciting friction-induced torque. New analytical and numerical solutions provide much insight into the speed-dependent resonant amplitude growth process. Nevertheless, the study assumes a simplified contact model and ignores the contributions of caliper and hydraulic system on $T(t)$. Also, only the numerical methods are utilized to solve the clearance nonlinearity problem. Future research efforts should attempt to improve the system models, as well as seek semi-analytical solutions [15,16].

Acknowledgement

The authors gratefully acknowledge Honda R&D Americas, Inc. for supporting this research. The following individuals are thanked for their contributions: W. Post, B. Nutwell, S. Ebert, F. Howse, P. Bray, D. Thompson.

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