Stiffness matrix formulation for double row angular contact ball bearings: Analytical development and validation

Aydin Gunduz, Rajendra Singh *

Acoustics and Dynamics Laboratory, Department of Mechanical & Aerospace Engineering and Smart Vehicles Concepts Center, The Ohio State University, Columbus, Ohio 43210, USA

A R T I C L E   I N F O

Article history:
Received 10 July 2012
Received in revised form 12 March 2013
Accepted 6 April 2013
Handling Editor: H. Ouyang
Available online 13 July 2013

A B S T R A C T

Though double row angular contact ball bearings are widely used in industrial, automotive, and aircraft applications, the scientific literature on double row bearings is sparse. It is also shown that the stiffness matrices of two single row bearings may not be simply superposed to obtain the stiffness matrix of a double row bearing. To overcome the deficiency in the literature, a new, comprehensive, analytical approach is proposed based on the Hertzian theory for back-to-back, face-to-face, and tandem arrangements. The elements of the five-dimensional stiffness matrix for double row angular contact ball bearings are computed given either the mean bearing displacement or the mean load vector. The diagonal elements of the proposed stiffness matrix are verified with a commercial code for all arrangements under three loading scenarios. Some changes in stiffness coefficients are investigated by varying critical kinematic and geometric parameters to provide more insight. Finally, the calculated natural frequencies of a shaft-bearing experiment are successfully compared with measurements, thus validating the proposed stiffness formulation. For double row angular contact ball bearings, the moment stiffness and cross-coupling stiffness terms are significant, and the contact angle changes under loads. The proposed formulation is also valid for paired (duplex) bearings which behave as an integrated double row unit when the surrounding structural elements are sufficiently rigid.

© 2013 Elsevier Ltd. All rights reserved.

1. Introduction

Double row bearings offer certain advantages over single row bearings, as they are capable of providing higher axial and radial rigidity and carrying bi-directional or combined loads. Consequently, double row bearings are widely used in machine tool spindles, industrial pumps, and air compressors, as well as in automotive, helicopter, and aircraft applications such as gear boxes, wheel hubs, and helicopter rotors [1–3]. In certain problems, a double row bearing may be simply represented by two single row bearings attached next to each other. However, in many static and dynamic problems, a double row bearing must be considered as an integrated unit. The stiffness matrices of two single row bearings may not be simply superposed to obtain the stiffness matrix of a double row bearing.

This study focuses on double row angular contact ball bearings which can be categorized into three arrangements according to the organization of their rolling elements as shown in Fig. 1: (i) back-to-back (DB) or ‘O’ arrangement (Fig. 1a), in which the load lines (the lines that pass through the contact point of the rolling elements) meet outside of the bearing; (ii) face-to-face (DF) or ‘X’ arrangement (Fig. 1b), where the load lines converge toward the bore of the bearing; and (iii) tandem (DT) or series arrangement (Fig. 1c), where the load lines act in parallel so they never meet each other as opposed to the other two arrangements. In general the effective load center (spread) of the back-to-back arrangement is

* Corresponding author. Tel.: +1 614292 9044; fax: +1 614292 3163.
E-mail address: singh.3@osu.edu (R. Singh).

0022-460X/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.jsv.2013.04.049
larger; thus, it has higher moment stiffness terms and a higher moment load carrying capacity. The face-to-face arrangement has a smaller load center; however, it has a larger misalignment angle and allows larger misalignments. The tandem arrangement can carry heavier axial loads but only in one direction due to the fact that all rolling elements are organized in the same direction. The vast majority of commercial double row angular contact ball bearings are found in the back-to-back arrangement; however, other arrangements are also utilized [1].

A new, theoretical bearing stiffness matrix ($K_b$) formulation is proposed in this article for double row angular contact ball bearings through a novel extension of the well-known theory for single row [4] and self-aligning (spherical) bearings [5]. Unlike self-aligning bearings [5], the moment stiffness terms of double row angular contact ball bearings are significant and highly dependent on the configuration of the rolling elements [6,7]. Therefore, tilting moments and angular motions bring a major complexity to the proposed formulation.

2. Literature review

2.1. Bearing stiffness

To understand the interfacial characteristics of rolling element bearings in rotating machinery, a wide range of bearing stiffness models of varying complexity have been proposed [1,8–16]. Some of these models describe the bearing as time-invariant translational springs in the axial and radial directions, which can predict only in-plane motions transmitted through the bearing while neglecting out-of-plane or flexural motions. This may result in an inadequate understanding of the bearing as a vibration transmitter, as experimental results have shown that the casing vibrations are typically out-of-plane [13–15]. Although Jones [17] did not define a bearing stiffness matrix, his load–deflection formulations for ball and roller type bearings under static loading conditions could be used to define a fully populated stiffness matrix for single row bearings. Lim and Singh [4] developed a five dimensional symmetric bearing stiffness matrix (which is in fact a six dimensional matrix with the last column and row being all zeros, corresponding to free torsion) for single row ball-type and roller-type bearings. The main advantage of Lim and Singh's [4] stiffness model over previous models was the introduction of flexural and out-of-plane type motions through cross-coupling stiffness terms which clearly explained the vibration transmission through rolling element bearings. Lim and Singh showed the merits of their model in their series of papers [4,18–21] through parametric studies and comparisons with previous analytical and experimental results [12,13]. The importance of moment stiffnesses and cross-coupling stiffness terms are clearly pointed out in these studies. De Mul et al. [22] outlined a general theory for the numerical calculation of the stiffness matrices of loaded bearings by extending Jones' [17] equations. Hernot et al. [23] calculated the stiffness matrix of angular contact ball bearings by integration techniques.

Cermelj and Boltezar [24] used Lim and Singh's [4] stiffness model to further investigate the dynamics of a structure containing ball bearings. Further, Royston and Basdogan [5] studied double row self-aligning (spherical) bearings by proposing a stiffness matrix. As the moment stiffness of self-aligning bearings is negligible, Royston and Basdogan [5] did not consider the effects of angular displacements and tilting moments. Thus, their three dimensional stiffness matrix was, in fact, a simplified version of Lim and Singh's five dimensional model [4], where the last two rows and columns of moment stiffness terms are neglected, but the three dimensional translational motions of each rolling element of both rows are included.
Guo and Parker [25] have recently proposed a numerical method for calculating the bearing stiffness matrix of single-row bearings with a commercial finite element based contact mechanics code [26]. Their numerical approach, however, is not attractive since at least 50 executions of the finite element code (hence excessive computing times) are needed to estimate $K_b$; and also, utilization of finite difference approximation [25] may create problems. Additionally it is not clear if the double row bearings can be analyzed with this method. Thus, analytical methods must be preferred over such numerical methods for the theoretical calculation of $K_b$ as well as insight into the significance of various terms.

Experimental techniques have also been utilized to measure radial and axial bearing stiffness coefficients. For example, Walford and Stone [27,28] used a two-degree-of-freedom model to extract representative stiffness values from measurements. Stone [29] reviewed various efforts to measure the stiffness and damping coefficients of rolling element bearings with changes in preload, speed, or lubrication. Kraus et al. [13] designed an in-situ measurement test to determine the translational bearing stiffness measured from vibration spectra using a single-degree-of-freedom model. Marsh and Yantek [30] extracted translational bearing stiffness coefficients by measuring the resulting responses under known excitation forces. Tiwari and Vyas [31] suggested a method for determining discrete design variables of an automotive wheel assembly that contained a double row angular contact ball bearing. Their optimization algorithm attempted to maximize the bearing service life while satisfying various design constraints including limited mounting space.

### 2.2. Double row bearings

Although single row bearings have been extensively studied, publications specific to double row bearings are sparse. For instance, Bercea et al. [6] formulated the relative displacement between the bearing rings (also termed as the ‘ring approach’) for various double row bearing types such as tapered, spherical, cylindrical roller, and angular contact ball bearings. Their bearing deflection formulation is only valid for a back-to-back arrangement and did not include any stiffness formulation. Then Nelias and Bercea [7] used their double row tapered rolling bearing model for case studies. Cao and Xiao [33] developed a dynamic model for double row spherical roller bearings based on energy principles. Later Cao [34] improved this model by including the effects of rotational motions and shaft misalignments. Choi and Yoon [35] proposed a method for determining discrete design variables of an automotive wheel assembly that contained a double row angular contact ball bearing. Their optimization algorithm attempted to maximize the bearing service life while satisfying various design constraints including limited mounting space.

### 3. Stiffness formulations of a double row bearing vs. two single row bearings

Two approaches may be utilized in the stiffness modeling of double row or duplex angular contact ball bearings as shown in Fig. 2(a-b). The first approach (with two single row bearings) essentially yields 2 separate mean displacement $\mathbf{q}_m = [\phi_{xm}, \phi_{ym}, \phi_{zm}, \phi_{ym}, \phi_{zm}]^T$ and load vectors $\mathbf{f}_m = [F_{xym}, F_{ym}, F_{zm}, M_{xm}, M_{ym}]^T$ (here z-axis is the rotational axis) and two separate stiffness ($K_b^i$) matrices (i.e. one for each individual bearing) where

$$K_b^i = \begin{bmatrix} k_{xx}^i & k_{xy}^i & k_{xz}^i & k_{x\beta_x}^i & k_{x\beta_y}^i \\ k_{yx}^i & k_{yy}^i & k_{yz}^i & k_{y\beta_x}^i & k_{y\beta_y}^i \\ k_{zx}^i & k_{zy}^i & k_{zz}^i & k_{z\beta_x}^i & k_{z\beta_y}^i \\ k_{x\beta_x}^i & k_{y\beta_x}^i & k_{z\beta_x}^i & k_{\beta_x\beta_x}^i & k_{\beta_x\beta_y}^i \\ k_{x\beta_y}^i & k_{y\beta_y}^i & k_{z\beta_y}^i & k_{\beta_x\beta_y}^i & k_{\beta_y\beta_y}^i \end{bmatrix}$$

(1)

Conversely, the second approach results in single mean displacement $\mathbf{q}_m^D = [\phi_{xym}^D, \phi_{ym}^D, \phi_{zm}^D, \phi_{ym}^D]^T$ and load vectors $\mathbf{f}_m^D = [F_{xym}^D, F_{ym}^D, F_{zm}^D, M_{xm}^D, M_{ym}^D]^T$ vectors. Note that a single stiffness matrix ($K_b^D$) needs to be defined for the double row bearing where

$$K_b^D = \begin{bmatrix} k_{xx}^D & k_{xy}^D & k_{xz}^D & k_{x\beta_x}^D & k_{x\beta_y}^D \\ k_{yx}^D & k_{yy}^D & k_{yz}^D & k_{y\beta_x}^D & k_{y\beta_y}^D \\ k_{zx}^D & k_{zy}^D & k_{zz}^D & k_{z\beta_x}^D & k_{z\beta_y}^D \\ k_{x\beta_x}^D & k_{y\beta_x}^D & k_{z\beta_x}^D & k_{\beta_x\beta_x}^D & k_{\beta_x\beta_y}^D \\ k_{x\beta_y}^D & k_{y\beta_y}^D & k_{z\beta_y}^D & k_{\beta_x\beta_y}^D & k_{\beta_y\beta_y}^D \end{bmatrix}$$

(2)

Translational stiffness elements of two side by side bearings with the same shaft and housing are in parallel [36]; thus, they can be superposed to obtain the stiffness values of the integrated unit (i.e. $k_{pq}^D = k_{pq}^x + k_{pq}^z$ where $p,q=x, y, z$). However,
the rotational stiffness or rotational coupling terms \(k_{rs}\) where \(r\) or \(s = \theta_x, \theta_y\) of two single row bearings and a double row bearing may not be easily correlated. In fact, there is simply no way of obtaining \(k_{Drs}\) from \(k_{i rs}\) \((i = 1, 2)\), as \(k_{Drs}\) terms depend on geometric parameters and the organization of rolling elements; thus, they must be derived from basic load–deflection relations.

To illustrate the above-mentioned issues, consider an illustrative example with a 148 mm long uniform solid shaft with a diameter of 50 mm as shown in Fig. 2(a–b). First, two single row bearings (in all three arrangements) are placed on the shaft with \(z\)-coordinates \(z_{BRG1} = 64\) mm and \(z_{BRG2} = 84\) mm, respectively. These two bearings are then replaced with a double row bearing with its geometric center located at \(z_G = (z_{BRG1} + z_{BRG2})/2\). The kinematic properties of the bearing are given in Table 1. The external load is applied at point \(O\) with magnitudes \(F_{xext} = 1\) kN and \(F_{zext} = 3\) kN. Observe that \(F_{xext}\) imposes a bending moment \((M_{yym})\) on both bearings. A combined asymmetrical loading case ensures that the analysis presents a general case. The diagonal stiffness elements of the bearings (i.e. \(k_{pp}\) where \(p = x,y,z,\theta_x, \theta_y\)) are calculated using a commercial code [8]. This particular code [8] contains a large bearing library of commercial single and double row bearings from major manufacturers, and yields only the diagonal elements of \(K_b\) which is sufficient and convenient for the comparative analysis. The calculated elements of \(K_b\) under the combined load are listed in Table 2(a–c). It is clear from the tables that \(k_{pp} \approx k_{pp1}^1 + k_{pp}^2\) is valid for \(p = x,y,z\) but not for \(p = \theta_x, \theta_y\) for all arrangements. Note that the overall tilting characteristics of a shaft-bearing system are dictated by \(k_{i \theta_x \theta_x}\) and \(k_{i \theta_y \theta_y}\) terms but not the \(k_{i \theta_x \theta_y}\) and \(k_{i \theta_x \theta_y}\) \((i = 1, 2)\) terms individually. Since it is not possible to obtain the rotational stiffness or rotational coupling terms of a double row bearing \(k_{rs}\),
where \( r \) or \( s = \theta_x, \theta_y \) from those of two single row bearings, a separate analytical formulation for double row bearings is absolutely essential. Yet another advantage of using a double row formulation is that it could eliminate the need for the solution of multidimensional statically indeterminate problems for simple cases (such as in Fig. 2(b)) by assuming the entire external load is carried by the double row bearing.

### Table 2
Single row versus double row bearing stiffness analyses for the example case of Fig. 2. Bearings are organized in (a) back-to-arrangement; (b) face-to-face arrangement; and (c) tandem arrangement.

<table>
<thead>
<tr>
<th></th>
<th>Bearing 1</th>
<th>Bearing 2</th>
<th>Sum(^*)</th>
<th>Double Row</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{xx} ) (kN/mm)</td>
<td>290.5</td>
<td>110.6</td>
<td>401.9</td>
<td>401.1</td>
</tr>
<tr>
<td>( k_{yy} ) (kN/mm)</td>
<td>293.7</td>
<td>33.9</td>
<td>327.6</td>
<td>327.6</td>
</tr>
<tr>
<td>( k_{zz} ) (kN/mm)</td>
<td>238.4</td>
<td>44.2</td>
<td>282.6</td>
<td>282.6</td>
</tr>
<tr>
<td>( k_{\theta x} ) (MN mm/rad)</td>
<td>153.5</td>
<td>13.1</td>
<td>166.6</td>
<td>342.2</td>
</tr>
<tr>
<td>( k_{\theta y} ) (MN mm/rad)</td>
<td>154.2</td>
<td>43.9</td>
<td>198.1</td>
<td>410.1</td>
</tr>
</tbody>
</table>

**Figure 3.** Mean loads and moments on the bearing and resulting translational and rotational displacements. The coordinate system is also shown.

\( \theta_x, \theta_y \) are the angular displacements of the bearing in the horizontal and vertical directions, respectively. The coordinate system is shown in the figure.

<table>
<thead>
<tr>
<th></th>
<th>Bearing 1</th>
<th>Bearing 2</th>
<th>Sum(^*)</th>
<th>Double Row</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{xx} ) (kN/mm)</td>
<td>144.2</td>
<td>297.8</td>
<td>441.9</td>
<td>442.3</td>
</tr>
<tr>
<td>( k_{yy} ) (kN/mm)</td>
<td>53.6</td>
<td>287.7</td>
<td>341.3</td>
<td>341.7</td>
</tr>
<tr>
<td>( k_{zz} ) (kN/mm)</td>
<td>86.1</td>
<td>256.2</td>
<td>342.3</td>
<td>342.2</td>
</tr>
<tr>
<td>( k_{\theta x} ) (MN mm/rad)</td>
<td>28.3</td>
<td>155.1</td>
<td>183.4</td>
<td>65.7</td>
</tr>
<tr>
<td>( k_{\theta y} ) (MN mm/rad)</td>
<td>82.9</td>
<td>175.6</td>
<td>258.5</td>
<td>100.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Bearing 1</th>
<th>Bearing 2</th>
<th>Sum(^*)</th>
<th>Double Row</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{xx} ) (kN/mm)</td>
<td>166.2</td>
<td>162.6</td>
<td>328.8</td>
<td>329.0</td>
</tr>
<tr>
<td>( k_{yy} ) (kN/mm)</td>
<td>128.8</td>
<td>144.5</td>
<td>273.3</td>
<td>272.8</td>
</tr>
<tr>
<td>( k_{zz} ) (kN/mm)</td>
<td>134.7</td>
<td>82.5</td>
<td>217.2</td>
<td>216.9</td>
</tr>
<tr>
<td>( k_{\theta x} ) (MN mm/rad)</td>
<td>70.2</td>
<td>56.2</td>
<td>126.4</td>
<td>156.1</td>
</tr>
<tr>
<td>( k_{\theta y} ) (MN mm/rad)</td>
<td>103.7</td>
<td>50.3</td>
<td>153.9</td>
<td>211.4</td>
</tr>
</tbody>
</table>

\* : Arithmetic sum of the stiffness elements of bearings 1 and 2

4. **Problem formulation: scope, assumptions and objectives**

There are two primary differences between the ball and roller type rolling element bearings. First, the ball type bearings have a point contact under unloaded conditions and an elliptical contact under loaded conditions between the bearing races and the rolling elements, whereas this contact is a line contact under unloaded conditions and a rectangular contact under loaded conditions for roller type bearings. Due to this difference in contact geometry, the load–deflection relationship (defined by the Hertzian contact theory) is expressed by different exponential coefficients for ball and roller type bearings [1]. The second primary difference is that the contact angle \( \alpha \) of ball type bearings changes under a load, whereas it remains relatively constant for roller type bearings. Thus, a change in \( \alpha \) under loaded conditions must be taken into account. Besides, the moment
stiffnesses and the cross-coupling stiffness coefficients of the angular contact ball bearings are significant. Therefore, some of the simplifying assumptions made for some other bearing types do not hold for angular contact ball bearings.

For the sake of analytical development, consider the double row angular contact ball bearing with a mean bearing load vector \( \mathbf{f}_m = (F_{xm}, F_{ym}, M_{xm}, M_{ym})^T \) and the resulting mean displacement vector \( \mathbf{q}_m = (\delta_{xm}, \delta_{ym}, \delta_{zm}, \beta_{xm}, \beta_{ym})^T \), as illustrated in Fig. 3 (beyond this point, all symbols are relevant to a double row bearing, and thus the superscript 'D' is simply omitted for the sake of convenience). Here \( \delta_{xm}, \delta_{ym}, \delta_{zm} \) and \( F_{xm}, F_{ym}, M_{xm}, M_{ym} \) are the mean displacements and loads in the \( x, y \), and \( z \) directions, and \( \beta_{xm}, \beta_{ym} \) are the mean angular displacements and tilting moments about the \( x \) and \( y \) coordinates, respectively. The shaft is allowed to rotate freely about the \( z \)-axis so the corresponding angular displacement and torsional terms are zero.

In general, both the inner and outer rings of a rolling element bearing may deflect under a load. However, for the purposes of this study it is sufficient to consider only the relative displacement between the bearing rings. Thus, assuming the outer ring is fixed in space and the inner ring is displaced under a load permits a tractable approach which is also pointed out by previous investigators [4,22]. The mean bearing load vector \( \mathbf{f}_m \) will be considered as an effective point load on the inner ring and acting along the geometrical center of the bearing. Similarly, the mean displacement vector \( \mathbf{q}_m \) corresponds to the displacement of the bearing center. The total displacement vector is defined as \( \mathbf{q}(t) = \mathbf{q}_m + \mathbf{q}_a(t) \), where \( \mathbf{q}_a(t) \) is the alternating displacement vector that fluctuates about the mean point \( \mathbf{q}_m \). For small mean loads or sufficiently high dynamic displacements, the bearing stiffness is time-varying, and nonlinear models should be used for the dynamic analysis. In this analysis it is assumed that \( |\mathbf{q}_a(t)| \) is much smaller than \( \mathbf{q}_m \); thus, linearized and time-invariant values of bearing stiffness can be determined about an operating point. In this study, it is assumed that both rows of the bearings are identical in terms of the structural and kinematic parameters. Since the bearings are often mounted on rigid shafts, and within robust housings the structural deformation of bearing rings is assumed to be negligible, and only the elastic deformation of the rolling elements at contact points is considered. It is assumed that the elastic deformation of the rolling elements occurs according to the Hertzian contact stress theory [1], and the load experienced by each rolling element is defined by its angular position in the bearing raceway. The internal friction is also assumed to be negligible compared with the normal loads on rolling elements. Further, the angular position of each rolling element relative to one another remains the same due to rigid cages and retainers. It is also assumed that the bearing does not operate at overcritical speeds; therefore, the gyroscopic moments and centrifugal effects can be neglected. It is assumed that the lubrication does not directly affect the bearing stiffness coefficients except at very high speeds, as explained in the earlier work [4,5]; this assumption is also utilized in some commercial codes [8]. Thus, tribological issues are beyond the scope of this paper. Nevertheless, the authors have shown that the effect of lubrication can be successfully included via bearing (viscous) damping elements in their recent work [36,37]. Finally, the proposed \( K_0 \) is also valid for duplex (paired) bearings mounted in face-to-face, back-to-back, or tandem arrangements, assuming all structural elements (such as the shaft and bearing rings) are sufficiently rigid; thus, the inner rings of the two rows do not move independently under external loads (except initial preloading).

The specific objectives of this study can be listed as follows: (1) develop a systematic and comprehensive approach to determine the fully populated five dimensional stiffness matrix \( K_0 \) for double row angular contact ball bearings of back-to-back, face-to-face, and tandem arrangements (which may be extended for other types of double row bearings in later studies); (2) develop a numerical scheme to compute \( K_0 \) given the mean displacement \( \mathbf{q}_m \) or the mean loading \( \mathbf{f}_m \); (3) verify the diagonal elements of the proposed stiffness matrix with a commercial code for all arrangements for various load cases; (4) conduct parametric studies to observe the effect of different loading scenarios, unloaded contact angle, and angular position of the bearing on the stiffness coefficients; and (5) validate the proposed model by comparing the natural frequencies of a vehicle wheel bearing assembly with the measurements on a double row angular contact ball bearing (in the back-to-back arrangement).

**Fig. 4.** Elastic deformation of a rolling element of the bearing. Elastic deformation of the rolling element is defined as the relative displacement between the inner and outer raceway groove curvature centers due to mean bearing loads and displacements.
5. New analytical formulation: load–deflection relations of a rolling element

To define the bearing stiffness, the relationship between \( f_m \) and \( q_m \) will first be derived. Consider the \( j \)th rolling element of the \( i \)th row of a double row angular contact ball bearing shown in Fig. 4. Assuming the outer ring is fixed, the total elastic deformation \( \delta (\psi_i^j) \) of this rolling element can be calculated by

\[
\delta (\psi_i^j) = \begin{cases} 
A_j^i - A_0 & \text{for } \delta (\psi_i^j) > 0 \\
0 & \text{for } \delta (\psi_i^j) \leq 0
\end{cases}
\]  

(3)

Here, \( \psi_i^j \) is the angular distance of the rolling element from the \( x \)- (horizontal) axis, and \( A_0 \) and \( A_j^i \) are the unloaded and loaded relative distances between the inner and outer raceway groove curvature centers, respectively. If \( A_0 \) is greater than \( A_j^i \), the rolling element is unloaded, and the elastic deformation is zero. To calculate \( A_j^i \), the radial and axial displacements \((\delta_r^i)^j \) and \((\delta_z^i)^j \) of the rolling element are first expressed in terms of the elements of bearing displacement vector \( q_m \)

\[
(\delta_r^i)^j = [\delta_x + c_1 \beta ym e] \cos (\psi_i^j) + [\delta_y - c_1 \beta ym e] \sin (\psi_i^j) - r_L 
\]

(4a)

\[
(\delta_z^i)^j = \delta_x + r_L \beta ym (\sin (\psi_i^j) - \beta ym \cos (\psi_i^j)) 
\]

(4b)

Here \( R \) is the radius of the inner raceway groove curvature center (pitch radius), \( \alpha_o \) is the unloaded contact angle, \( r_L \) is the radial clearance, and \( e \) is the axial distance between the geometric center of the bearing and bearing rows. The effective load center \( (E) \), which controls the tilting stiffness terms of a double row arrangement, can be defined as \( E = 2(e + c_2 R \tan (\alpha_o)) \), and coefficients \( c_1 \) and \( c_2 \) are given as follows:

\[
c_1 = \begin{cases} 
-1 & \text{for } i = 1 \text{ (Left row)} \\
1 & \text{for } i = 2 \text{ (Right row)}
\end{cases}
\]

(5)

\[
c_2 = \begin{cases} 
1 & \text{back to back (DB) arrangement} \\
-1 & \text{face to face (DF) arrangement} \\
0 & \text{tandem (DT) arrangement}
\end{cases}
\]

(6)

The net (effective) radial \( (\delta_r^i)^j \) and axial \( (\delta_z^i)^j \) displacements of the rolling element are then expressed including the effects of initial displacement due to the unloaded contact angle \( \alpha_o \)

\[
(\delta_r^i)^j = (\delta_r^i) + A_0 \cos (\alpha_o) 
\]

(7a)

\[
(\delta_z^i)^j = (\delta_z^i) + c_3 (A_0 \sin (\alpha_o) + (\delta_{\alpha_0})^i) 
\]

(7b)

Here \((\delta_{\alpha_0})^i\) defines an axial displacement preload on the \( i \)th row obtained by bringing the inner and outer raceways closer together by a distance \((\delta_{\alpha_0})^i\) (for instance in the case of split inner rings [35]). \( (\delta_z^i)^j \) can have a positive value only if the radial clearance is eliminated (i.e. \( r_L = 0 \)). Here, \( c_3 \) is a constant dependent on the type of rolling element bearing

Back – to – back (DB) arrangement : \( c_3 = \begin{cases} 
1 & \text{for } i = 1 \text{ (Left row)} \\
-1 & \text{for } i = 2 \text{ (Right row)}
\end{cases} 
\]

(8a)

Face – to – face (DF) arrangement : \( c_3 = \begin{cases} 
-1 & \text{for } i = 1 \text{ (Left row)} \\
1 & \text{for } i = 2 \text{ (Right row)}
\end{cases} 
\]

(8b)

Tandem (DT) arrangement : \( c_3 = 1^{(*)} \)  

(Both rows)  

(8c)

(*) : Assuming the tandem arrangement is axially loaded in the direction it can hold.

The loaded relative distance between the inner and outer raceway groove curvature center \( A(\psi_i^j) \) is finally obtained by vectorial addition of the net displacements

\[
A(\psi_i^j) = \sqrt{((\delta_r^i)^j)^2 + ((\delta_z^i)^j)^2} 
\]

(9)

Then, the elastic deformation \( \delta (\psi_i^j) \) of the rolling element can be determined by Eq. (3) to be used in conjunction with the Hertzian contact stress theory in order to obtain the resultant normal load \( (Q_i^j) \) on the element

\[
Q_i^j = K_n \delta (\psi_i^j)^n. 
\]

(10)

Here \( K_n \) is the rolling element load–deflection stiffness constant, which is a function of material properties and geometry [1]. The exponent \( n \) is equal to 1.5 for point/elliptical type contact (for ball type bearings). The loaded contact angle \( \alpha_i^j \) for the
same element is determined by trigonometry

\[
\tan (\alpha_i^j) = \frac{(\delta_i^j)^y}{(\delta_i^j)^x} = \frac{(\delta_i^j)^y + c_3(A_0 \sin (\alpha_0) + (\delta_0^j)^y)}{(\delta_i^j)^x + A_0 \cos (\alpha_0)}
\]

(11)

where \((\delta_i^j)^y\) and \((\delta_i^j)^x\) are given by Eq. (4a and b).

6. Stiffness matrix

6.1. Formulation

The bearing stiffness matrix is a comprehensive representation that combines the bearing’s kinematic and elastic properties and the effect of each rolling element [4]. To apply the mathematical definition of the stiffness matrix, the mean load vector \(f_m\) is first represented through vectorial sums of \(Q_i^j (i = 1, 2; j = 1, \ldots, Z)\) for each loaded \((\delta_i^j) > 0\) rolling element as follows:

\[
f_m = \begin{pmatrix} F_{xm} \\ F_{ym} \\ F_{zm} \\ M_{xm} \\ M_{ym} \end{pmatrix} = \frac{1}{2} \sum_{i=1}^{Z} \sum_{j=1}^{Z} Q_i^j \begin{pmatrix} \cos (\alpha_i^j) \cos (\psi_i^j) \\ \cos (\alpha_i^j) \sin (\psi_i^j) \\ \sin (\alpha_i^j) \end{pmatrix} \begin{pmatrix} \cos (\alpha_i^j) \sin (\psi_i^j) \\ \sin (\psi_i^j) \\ \cos (\alpha_i^j) \end{pmatrix} \begin{pmatrix} R \sin (\alpha_i^j) - c_1 \sin (\alpha_i^j) \sin (\psi_i^j) \\ -R \cos (\alpha_i^j) + c_1 \sin (\alpha_i^j) \cos (\psi_i^j) \end{pmatrix}
\]

(12)

Expressing \(Q_i^j\) and \(\alpha_i^j\) in Eq. (12) in terms of mean deflections gives the explicit expressions between \(f_m\) and \(q_m\):

\[
f_m = \begin{pmatrix} F_{xm} \\ F_{ym} \\ F_{zm} \\ M_{xm} \\ M_{ym} \end{pmatrix} = K_n \sum_{i=1}^{Z} \sum_{j=1}^{Z} \left( \frac{(\delta_i^j)^y + A_0 \cos (\alpha_0)) \cos (\psi_i^j)}{(\delta_i^j)^x + A_0 \cos (\alpha_0)) \cos (\psi_i^j) + ((\delta_i^j)^y + A_0 \sin (\alpha_0) + (\delta_0^j)^y) - A_0 \right)^n \left( \frac{(\delta_i^j)^y + A_0 \sin (\alpha_0) + (\delta_0^j)^y)}{(\delta_i^j)^x + A_0 \sin (\alpha_0) + (\delta_0^j)^y) - A_0 \right)^n \begin{pmatrix} R((\delta_i^j)^y + A_0 \sin (\alpha_0) + (\delta_0^j)^y)) - c_1 \sin (\alpha_i^j) \sin (\psi_i^j) \\ -R((\delta_i^j)^y + A_0 \sin (\alpha_0) + (\delta_0^j)^y)) + c_1 \sin (\alpha_i^j) \cos (\psi_i^j) \end{pmatrix}
\]

(13)

Substituting \((\delta_i)\) and \((\delta_i^j)\) terms from Eq. (4a-b) into Eq. (13), and assuming \(q_m(t) = \{q_m^i\}\), the five dimensional stiffness matrix \(K_n\) around the operating point can be defined

\[
K_n = \begin{pmatrix} K_{xx} & K_{xy} & K_{xz} & K_{xs, x} & K_{xs, y} \\ K_{xy} & K_{yy} & K_{yz} & K_{ys, x} & K_{ys, y} \\ K_{xz} & K_{yz} & K_{zz} & K_{zs, x} & K_{zs, y} \\ K_{xs, x} & K_{ys, x} & K_{zs, x} & k_{xx} & k_{xy} \\ K_{xs, y} & K_{ys, y} & K_{zs, y} & k_{xx} & k_{xy} \end{pmatrix}
\]

(14)

Here, \(p, q = x, y, z\) and \(r, s = x, y\). The explicit expressions of each stiffness term are symbolically calculated and given in their simplest form as follows:

\[
k_{xx} = K_n \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{(\delta_i^j)^y \cos^2 \psi_i^j ((\delta_i^j)^y)^2 / (A_j^y - A_0) + ((\delta_i^j)^y)^2}{(A_j^y)^2}
\]

(15a)

\[
k_{xy} = K_n \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{(\delta_i^j)^y \sin \psi_i^j (((\delta_i^j)^y)^2 / (A_j^y - A_0) + ((\delta_i^j)^y)^2)}{(A_j^y)^2}
\]

(15b)

\[
k_{xz} = K_n \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{(\delta_i^j)^y (\delta_i^j)^x \cos \psi_i^j (((\delta_i^j)^y)^2 / (A_j^y - A_0) - 1)}{(A_j^y)^2}
\]

(15c)
\[
\begin{align*}
k_{ax} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \cos (\psi_j^i) \left[ \frac{(nA_i^j/(A_i^j - A_0))(R(\delta_j^i)^2/\delta_j^i) - c_1 e((\delta_j^i)^2)}{(A_i^j)^3} \right] \quad (15d) \\
k_{cx} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \cos^2(\psi_j^i) \left[ \frac{-(nA_i^j/(A_i^j - A_0))(R(\delta_j^i)^2/\delta_j^i) - c_1 e((\delta_j^i)^2)}{(A_i^j)^3} \right] + \left( R(\delta_j^i)^2/\delta_j^i - c_1 e((\delta_j^i)^2) \right) \quad (15e) \\
k_{yy} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin^2(\psi_j^i) \left[ \frac{((nA_i^j/(A_i^j - A_0)) + ((\delta_j^i)^2)^2}}{(A_i^j)^3} \right] \quad (15f) \\
k_{yz} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{(nA_i^j/(A_i^j - A_0))(R(\delta_j^i)^2/\delta_j^i) - c_1 e((\delta_j^i)^2)}{(A_i^j)^3} \right] \quad (15g) \\
k_{wy} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{(nA_i^j/(A_i^j - A_0))(R(\delta_j^i)^2/\delta_j^i) - c_1 e((\delta_j^i)^2)}{(A_i^j)^3} \right] \quad (15h) \\
k_{wy} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{(nA_i^j/(A_i^j - A_0))(R(\delta_j^i)^2/\delta_j^i) - c_1 e((\delta_j^i)^2)}{(A_i^j)^3} \right] \quad (15i) \\
k_{zz} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{(nA_i^j/(A_i^j - A_0))(R(\delta_j^i)^2/\delta_j^i) - c_1 e((\delta_j^i)^2)}{(A_i^j)^3} \right] \quad (15j) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{(R(\delta_j^i)^2 - (R(\delta_j^i)^2)^2/2 + c_1 e((\delta_j^i)^2)/2) + c_1 e((\delta_j^i)^2)/(A_i^j)^2}{(A_i^j)^3} \right] \quad (15k) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{-(R(\delta_j^i)^2/2 - c_1 e((\delta_j^i)^2)/2) - c_1 e((\delta_j^i)^2)/(A_i^j)^2}{(A_i^j)^3} \right] \quad (15l) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{(R(\delta_j^i)^2 + c_1 e((\delta_j^i)^2)/(A_i^j)^2)}{(A_i^j)^3} \right] \quad (15m) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{-R(\delta_j^i)^2 - c_1 e((\delta_j^i)^2)/(A_i^j)^2}{(A_i^j)^3} \right] \quad (15n) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \sin (\psi_j^i) \left[ \frac{(R(\delta_j^i)^2 + c_1 e((\delta_j^i)^2)/(A_i^j)^2)}{(A_i^j)^3} \right] \quad (15o) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \cos (\psi_j^i) \left[ \frac{R(\delta_j^i)^2 + c_1 e((\delta_j^i)^2)/(A_i^j)^2}{(A_i^j)^3} \right] \quad (15p) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \cos (\psi_j^i) \left[ \frac{R(\delta_j^i)^2 + c_1 e((\delta_j^i)^2)/(A_i^j)^2}{(A_i^j)^3} \right] \quad (15q) \\
k_{ox} & = K_n \sum_{i=1}^{2} \sum_{j=1}^{Z} \left( \delta_j^i \right)^n \cos (\psi_j^i) \left[ \frac{R(\delta_j^i)^2 + c_1 e((\delta_j^i)^2)/(A_i^j)^2}{(A_i^j)^3} \right] \quad (15r) \\
k_{x} & = k_{xy}, k_{zx} = k_{ez}, k_{zy} = k_{yz} \quad (15s)
\end{align*}
\]
\[ k_{\theta_x} = k_{\theta_y} = k_{\theta_z} = k_{\theta} \]

\[ k_{\theta_x} = k_{\theta_y} = k_{\theta_z} = k_{\theta,\theta} \]

6.2. Numerical estimation of \( K_b \)

If the mean displacement vector \( q_m \) is known, the stiffness coefficients \( k_{pq} \) (\( p, q = x, y, z, \theta_x, \theta_y \)) can be calculated by direct substitution into Eq. 15(a–y). However, in general, \( f_m \) is known, and the resulting displacement vector \( q_m \) is unknown. In this case, the coupled nonlinear equations as described by Eqs. (12) and (13) are numerically solved to determine \( q_m \) for a given \( f_m \). To implement this method, Eq. (12) is rearranged as follows:

\[
g = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{pmatrix} = \begin{pmatrix} F_{xm} \\ F_{ym} \\ F_{zm} \\ M_{xm} \\ M_{ym} \end{pmatrix} - \sum_{i=1}^{2} \sum_{j=1}^{2} Q_j \begin{pmatrix} \cos (\alpha_i) \cos (\psi_j) \\ \cos (\alpha_i) \sin (\psi_j) \\ \sin (\alpha_i) \\ [R \sin (\alpha_i) - c_1 e \cos (\alpha_i)] \sin (\psi_j) \\ [-R \sin (\alpha_i) + c_1 e \cos (\alpha_i)] \cos (\psi_j) \end{pmatrix} = 0 \quad (16)\]

Here \( g = (G_1 G_2 G_3 G_4 G_5)^T \) are the functions to be minimized. In this study, a secant method as described by Xu et al. [38] is employed to solve the multidimensional minimization problem due to its strong convergence characteristics. A summary of the proposed method, with focus on model inputs and outputs, is illustrated in Fig. 5.

7. Computational verification of proposed stiffness model

The proposed model is verified with a commercial code [8]. Since the algorithm used by the code is not published, the verification will be limited to a numerical comparison of the diagonal elements of \( K_b \) for an example case under various loading scenarios. All three configurations of the rolling elements (DF, DB, and DT) are analyzed.

Consider the commercial double row angular contact ball bearing with properties given in Table 1; this example case will be used in the whole article. The bearing is initially unpreloaded, and its outer ring is fixed. First, the shaft is subjected to an axial load \( F_{ax} \) that is increased from 1 kN to 10 kN in 1 kN increments (refer to Fig. 2(b)). Assuming the shaft is rigid, the entire axial load is supported by the double row bearing (i.e. \( F_{ax} = F_{zm} \)). In the absence of any radial or moment load, the radial stiffness terms \( (k_{xx} \text{ and } k_{yy}) \) as well as the tilting stiffness terms \( (k_{\theta_x,\theta_x} \text{ and } k_{\theta_y,\theta_y}) \) of \( K_b \) are equal, yielding three independent diagonal stiffness elements \( (k_{xx}, k_{yy} \text{ and } k_{\theta_y,\theta_y}) \) that are illustrated in Fig. 6 for all three configurations (the proposed model is given by discrete markers, and solid lines denote predictions by the commercial code). As seen from the figure, all stiffness elements of the proposed model show an excellent match with those of the commercial code.

![Fig. 5. Summary of the proposed stiffness matrix formulation with focus on model inputs and outputs.](image-url)
Next, a radial shaft load of \( F_{\text{xext}} = 1 \) kN is applied in positive \( x \)-direction with an axial distance of \( d_x = 74 \) mm away from the bearing center (according to Fig. 2(b)) which imposes a moment load on the bearing \( M_{\text{ym}} = -74 \) kNmm and results in a mean load vector \( \mathbf{f}_m = (1000 \text{ N}, 0, 0, -74,000 \text{ N mm})^T \). Due to the absence of axial load, the solution for \( q_m \) in this case is quite sensitive to the selection of the initial guess, especially for the tandem arrangement. Thus, a more stable loading case with \( \mathbf{f}_m = (1000 \text{ N}, 0, 3000 \text{ N}, 0, -74,000 \text{ N mm})^T \) is also considered. Since \( k_{xx} \neq k_{yy} \) and \( k_{\theta_x,\theta_x} \neq k_{\theta_y,\theta_y} \) for both cases, five distinct diagonal stiffness terms are calculated and presented in Tables 3 and 4, respectively. As seen from both tables, the stiffness elements of the proposed model show an excellent match with the commercial code, with minor errors for both loading cases. In the absence of axial loading (Table 3), stiffness elements in unloaded directions (\( y, z, \) and \( \theta_x \) for this particular case) are unconventionally small, especially for a tandem arrangement (while both models still being consistent).

**Table 3**
Comparison of diagonal stiffness elements of the proposed model and the commercial code [8] for DF, DB, and DT arrangements for the example case given \( \mathbf{f}_m = (1000 \text{ N}, 0, 0, -74,000 \text{ N mm})^T \).

<table>
<thead>
<tr>
<th>Bearing arrangement</th>
<th>Face-to-face (DF)</th>
<th>Back-to-back (DB)</th>
<th>Tandem (DT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiffness element</td>
<td>Commercial code</td>
<td>Proposed model</td>
<td>Commercial code</td>
</tr>
<tr>
<td>( k_{xx} ) (kN/mm)</td>
<td>407</td>
<td>401</td>
<td>360</td>
</tr>
<tr>
<td>( k_{yy} ) (kN/mm)</td>
<td>302</td>
<td>306</td>
<td>267</td>
</tr>
<tr>
<td>( k_{zz} ) (kN/mm)</td>
<td>311</td>
<td>318</td>
<td>226</td>
</tr>
<tr>
<td>( k_{\theta_x,\theta_x} ) (MN mm/rad)</td>
<td>55</td>
<td>50</td>
<td>264</td>
</tr>
<tr>
<td>( k_{\theta_y,\theta_y} ) (MN mm/rad)</td>
<td>95</td>
<td>89</td>
<td>354</td>
</tr>
</tbody>
</table>

**Fig. 6.** Comparison of axial, radial and tilting stiffness coefficients of the proposed model and the commercial code [8] for the example case of Table 1 with respect to axial load. Key: Discrete points, proposed model; \( \times \), face-to-face arrangement; \( \circ \), back-to-back arrangement; \( \bigcirc \), tandem arrangement; solid line, commercial code.
After the application of \( F_{ext} = 3 \) kN, the values become more conventional, and the agreement between the two models is still excellent.

8. Examination of stiffness coefficients

8.1. Effect of mean bearing loads on stiffness coefficients

Typical variations in the stiffness elements under various loads are analyzed with the proposed model and compared with those of the single row bearing with identical kinematic properties as described by Lim and Singh’s model [4]. First, the effect of pure axial load \( (\delta_{zm} \neq 0; F_{zm} \neq 0 \text{ while other elements of } q_m \text{ and } f_m \text{ are zero}) \) on \( k_{zz} \) is analyzed. The relationship between \( F_{zm} \) and \( \delta_{zm} \) is shown in Fig. 7(a). Here, the slope of the tangent (about an operating point) defines \( k_{zz} \), which is plotted with respect to \( \delta_{zm} \) (Fig. 7(b)) and \( F_{zm} \) (Fig. 7(c)). As seen from Fig. 7(b and c), \( k_{zz} \) of the DT arrangement is significantly higher than those of the DB and DF arrangements, as expected. In fact, the DF and DB arrangements tend to act as a single row bearing, showing a behavior identical to Lim and Singh’s model [4], since only one row of these arrangements is loaded under a pure axial load [1]. Note that at a given \( \delta_{zm} \), \( k_{zz} \) of the DT arrangement is twice that of the DB or DF arrangements (i.e. DT arrangement acts like two parallel springs in the axial direction). However, at a given \( F_{zm} \), \( k_{zz} \) of the DT arrangement is less than twice the DB or DF arrangements due to the stiffening nature of the load–deflection curve.

Second, the bearing is loaded in a radial \( (x) \) direction \( (\delta_{xm} \neq 0; F_{xm} \neq 0 \text{ while all other } q_m \text{ and } f_m \text{ elements are zero}) \). The relationship between \( F_{xm} \) and \( \delta_{xm} \) is shown in Fig. 8(a), and its slope \( (k_{xx}) \) is plotted with respect to \( \delta_{xm} \) in Fig. 8(b). Here, the load distribution for all configurations are identical; thus, DF, DB, and DT arrangements show an identical \( k_{xx} \) behavior, which is exactly twice that of a single row bearing for a given \( \delta_{xm} \) (i.e. all double row configurations act like two parallel springs in the radial direction). Next, a misalignment about the y-axis \( (\beta_{ym}) \) is applied, and \( k_{\theta y \theta y} \) elements are investigated. Plots of \( M_{ym} \) vs. \( \beta_{ym} \) and \( k_{\theta y \theta y} \) vs. \( \beta_{ym} \) are given in Fig. 9(a) and (b), respectively. As expected, \( k_{\theta y \theta y} \) of the DB arrangement is significantly higher than those of the DF or DT arrangements due to its larger effective load center.

Since a bearing load affects all diagonal and some off-diagonal elements of \( K_m \), one can generate a large number of plots similar to Figs. 7–9 considering different loading and stiffness coefficient combinations. In general, it is more difficult to

<table>
<thead>
<tr>
<th>Bearing arrangement</th>
<th>Face-to-face (DF)</th>
<th>Back-to-back (DB)</th>
<th>Tandem (DT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiffness element</td>
<td>Commercial code</td>
<td>Proposed model</td>
<td>Commercial code</td>
</tr>
<tr>
<td>( k_{xx} ) (kN/mm)</td>
<td>442</td>
<td>465</td>
<td>402</td>
</tr>
<tr>
<td>( k_{yy} ) (kN/mm)</td>
<td>341</td>
<td>357</td>
<td>328</td>
</tr>
<tr>
<td>( k_{zz} ) (kN/mm)</td>
<td>342</td>
<td>359</td>
<td>283</td>
</tr>
<tr>
<td>( k_{\theta x \theta x} ) (MN mm/rad)</td>
<td>66</td>
<td>61</td>
<td>342</td>
</tr>
<tr>
<td>( k_{\theta y \theta y} ) (MN mm/rad)</td>
<td>100</td>
<td>96</td>
<td>410</td>
</tr>
</tbody>
</table>

After the application of \( F_{ext} = 3 \) kN, the values become more conventional, and the agreement between the two models is still excellent.
Fig. 8. Variation of the radial force and the radial stiffness with respect to radial displacement of the bearing for the example case. Key: (○), back-to-back arrangement; (△), face-to-face arrangement; (◇), tandem arrangement; (●), single row bearing. (a) $F_{x_{m}}$ vs. $\delta_{x_{m}}$; (b) $k_{x_{m}}$ vs. $\delta_{x_{m}}$.

Fig. 9. Variation of the bending moment and the tilting stiffness with respect to angular displacement of the bearing for the example case. Key: (○), back-to-back arrangement; (△), face-to-face arrangement; (◇), tandem arrangement; (●), single row bearing. (a) $M_{y_{m}}$ vs. $\beta_{y_{m}}$; (b) $k_{\theta_{y}}$ vs. $\beta_{y_{m}}$.

Fig. 10. Variation of some off-diagonal elements of $K_{b}$ under various loads for the example case. Key: (○), Back-to-back arrangement; (△), face-to-face arrangement; (◇), tandem arrangement; (●) single row bearing. (a) $k_{xz}$ vs. $\beta_{y_{m}}$; (b) $k_{x_{m}}$ vs. $\delta_{x_{m}}$; (c) $k_{z_{m}}$ vs. $\delta_{z_{m}}$. 
observe clear stiffness trends under combined loading cases. It is also rather difficult to predict which bearing loads might specifically affect off-diagonal elements of $K_b$. For instance, for a pure axial load, the only significant cross-coupling terms are $k_{x\theta y}$ and $k_{y\theta x}$, whereas the matrix becomes fully populated if all elements of $q_m$ are nonzero. Fig. 10(a–c) illustrates sample variations in some cross-coupling terms under various loading cases. Note that off-diagonal terms are also highly dependent on the organization of the rolling elements.

8.2. Effect of unloaded contact angle (under radial load) on stiffness coefficients

A radial displacement of $\delta_{xm}=0.05$ mm is applied to the bearing of Table 1, and the effect of unloaded contact angle ($\alpha_0$) on the stiffness coefficients are investigated and shown in Figs. 11 and 12 for diagonal and off-diagonal elements of $K_b$, respectively. These figures are plotted up to $\alpha_0=90^\circ$ as numerical issues seem to occur around $\alpha_0=90^\circ$ in some cases. The translational stiffness coefficients ($k_{xx}$, $k_{yy}$, $k_{zz}$) of all three arrangements are found to be identical under a pure radial load as shown in Fig. 11(a). Here, $k_{xx}$ and $k_{yy}$ follow a similar trend, as they have a maximum value for $\alpha_0=0^\circ$ (deep groove ball bearing), which nonlinearly converge to zero as $\alpha_0$ approaches $90^\circ$. On the other hand, $k_{zz}$ is minimum when $\alpha_0=0^\circ$, reaches its maximum value around $\alpha_0=65^\circ$, and then decreases for higher $\alpha_0$ values. Recall that bearings with $\alpha_0\geq45^\circ$ are often viewed as "thrust bearings", and thus, an increase in $k_{zz}$ for higher $\alpha_0$ is an expected result. Fig. 11(b and c) illustrates $k_{x\theta y}$ and $k_{y\theta x}$ coefficients, which depend on the bearing configuration. As expected $k_{x\theta y}$ and $k_{y\theta x}$ of the DB arrangement are higher than the other two arrangements at all $\alpha_0$ values. Also $k_{x\theta y}$ is greater than $k_{y\theta x}$ for all arrangements and $\alpha_0$ values.

The dominant off-diagonal terms of $K_b$ ($k_{xz}$, $k_{x\theta y}$, $k_{y\theta x}$, and $k_{z\theta y}$) for a given $\delta_{xm}$ are plotted with respect to $\alpha_0$ in Fig. 12(a–d). Other off-diagonal terms are negligible under pure radial load. As seen from Fig. 12(a), $k_{xz}$ is identical for all three

![Fig. 11. Variations in the diagonal stiffness elements of $K_b$ with $\alpha_0$ given $\delta_{xm}=0.05$ mm for the example case. (a) $k_{xx}$, $k_{yy}$, and $k_{zz}$, (b) $k_{x\theta y}$, (c) $k_{y\theta x}$. Key: (●), Back-to-back arrangement; (○), face-to-face arrangement; (▲), tandem arrangement.](image-url)
arrangements at all $\alpha_0$ and has a maximum value when $\alpha_0 = 45^\circ$. On the other hand, $k_{x\theta y}$, $k_{y\theta x}$, and $k_{z\theta y}$ are dependent on the bearing arrangement and $\alpha_0$. One can also observe that $k_{x\theta y}$ and $k_{z\theta y}$ are negative, and $k_{y\theta x} = -k_{x\theta y}$.

8.3. Effect of angular position of the bearing on stiffness coefficients

So far, the stiffness elements have been calculated by assuming that the rolling elements are equally spaced along the bearing races, and the first rolling element of both rows are coincident with the $x$-axis (i.e. $\psi_i = 0$ rad, $i=1,2$). This assumption should be verified since the load distribution among the rolling elements changes with angular position under radial and moment loads; thus, the bearing may exhibit considerable stiffness variations.

Again, consider the same example case (of Table 1) subjected to $\delta_{xm} = 0.05$ mm. Three diagonal stiffness elements of $K_b$ ($k_{xx}$, $k_{zz}$, $k_{x\theta y}$) are normalized with respect to their values at $\psi_i = 0$ rad and plotted against the angular position of the bearing over a complete ball passage period as shown in Fig. 13(a–c). These results are illustrated for the DB arrangement, though all arrangements show a similar behavior. As seen from Fig. 13(a–c), the stiffness elements vary between minimum and maximum values over a ball passage period. In particular, $k_{zz}$ shows the maximum variation (about 6 percent) over the
ball passage period. The variations in $k_{xx}$ and $k_{\theta y \theta y}$ on the other hand are negligible. These results suggest that the angular position does not have a significant effect on the stiffness elements, as the maximum variation in a stiffness coefficient is less than 6 percent even for a fairly large radial displacement (such as $\delta_{xm} = 0.05$ mm).

When the variations in the stiffness elements are not small, the accuracy of the linear stiffness model decreases, and certain kinematic parameters have a significant role in such variations. For instance, the number of rolling elements is especially critical as it changes the ball passage period and affects the load zone of a bearing. A reduction in the number of (loaded) rolling elements extends the ball passage period and results in higher variations in the stiffness elements. To illustrate this issue, consider an extreme case with only 4 rolling elements in each row (while all other parameters of the example are kept the same), and calculate $k_{xx}$, $k_{zz}$, and $k_{\theta y \theta y}$ for $\delta_{xm} = 0.05$ mm. These results are shown in Fig. 13(d–f). As seen from the figures, $k_{xx}$, $k_{zz}$, and $k_{\theta y \theta y}$ are now around 18 percent and 16 percent, respectively. These results clearly show that the angular position is an important parameter; however, its effect can often be neglected in the presence of a sufficient number of rolling elements.

Although the variation of the stiffness elements of all arrangements are similar for a given radial load, they occur differently for a given moment load. Fig. 14 shows the variations in $k_{xx}$, $k_{zz}$, and $k_{\theta y \theta y}$ for $\beta_{ym} = 0.03$ rad (for $Z = 14$). As seen from the figure, the variations are arrangement specific for a given $\beta_{ym}$; however, they are all within 2 percent; hence, they are negligible.

9. Experimental validation

An experiment consisting of a vehicle wheel bearing assembly with a double row angular contact ball bearing (in the back-to-back arrangement) is designed, instrumented, and tested to experimentally validate the proposed formulation; details of this experimental study are described by the authors in a recent article [37]. The shaft-bearing experiment is analytically described by a five degree-of-freedom model that consists of a rigid shaft (with three translational ($x, y, z$) and
two rotational (θx, θy) dimensions). The double row angular contact ball bearing is modeled using the proposed five-dimensional bearing stiffness matrix (Kb) with associated viscous damping (Cb). An impulse hammer test is conducted at an intermediate axial preload. Table 5 summarizes measured and predicted natural frequencies of the system (where r is the
modal index). Observe that predictions match well with measurements, with small errors. When there is no external radial or moment load applied in the model, natural frequencies at \( r = 1 \) and \( r = 2 \), as well as \( r = 4 \) and \( r = 5 \), are repeated. With an application of a slight amount of load in the radial \((x)\) direction \((F_{xm} = 0.3 \text{ kN})\), these repeated roots separate as shown in the fourth column of Table 4. Now, the second, third, and fifth natural frequencies show a better match with the measurements, but the estimation of the first natural frequency deviates further from experiments.

The last column of Table 5 shows a case where a diagonal \( K_b \) (with zero cross-coupling terms) is employed in the analytical model. In this case, the natural frequency calculations deviate significantly from measurements. These results clearly highlight the importance of cross-coupling stiffness terms of \( K_b \) and verify the need for a bearing stiffness matrix when analyzing the vibration transmission paths [4].

### 10. Conclusion

The chief contribution of this study is the analytical development of the fully populated five dimensional stiffness matrix \((K_b)\) for double row angular contact ball bearings of back-to-back, face-to-face, and tandem arrangements. First, calculations show that it is not possible to obtain the rotational stiffness or rotational coupling terms of a double row bearing from those of two single row bearings, and thus a separate analytical formulation for double row bearings is absolutely essential. Using the proposed analytical expressions, the diagonal and non-diagonal (cross-coupling) stiffness coefficients of a double row angular contact ball bearing are determined given either the mean displacement vector \( q_m \) (by direct substitution) or the mean load vector \( f_m \) (by numerical solution of the nonlinear system equations). A secant method is utilized to solve the multidimensional minimization problem and the convergence issues seen in prior work [4] are minimized. The diagonal elements of the stiffness matrix are verified with a commercial code through a detailed comparison of the stiffness elements for an example case. Excellent agreement between the proposed model and the commercial code has been obtained under three different loading scenarios. Then, some changes in \( K_b \) elements are further investigated with the proposed model by varying bearing loads, unloaded contact angle, and angular position of the bearing to provide some insight. The experimental results (as briefly presented in this article) clearly suggest that the proposed stiffness model is valid and can be confidently utilized for modeling double row angular contact ball bearings; further validation at this stage is not possible since the stiffness elements (especially the off diagonal terms) cannot be directly measured.

Considering the lack of publications on double row bearings, the new formulation provides a useful tool in the static and dynamic analyses of double row angular contact ball bearings, such as the vibration analysis of shaft-bearing assemblies. The proposed formulation is also valid for paired (duplex) bearings which behave as an integrated (double row) unit when the surrounding structural elements (such as the shaft and bearing rings) are sufficiently rigid. The proposed theory could be extended to the analyses of double row cylindrical and tapered roller bearings as a part of the future work. In fact, the mathematical formulation of angular contact ball bearings (as presented) is the most comprehensive as some of the simplifying assumptions made for other bearing types (e.g. constant contact angle for roller type bearings) do not hold for angular contact ball bearings.

### Acknowledgments

We are grateful to the member organizations (such as the Army Research Laboratory and Honda R&D) of the Smart Vehicle Concepts Center (www.SmartVehicleCenter.org) and the National Science Foundation Industry/University Cooperative Research Centers program (www.nsf.gov/eng/iip/iucrc) for supporting this work.

### References
