Low frequency dynamics of a translating friction element in the presence of frictional guides, as motivated by a brake vibration problem

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A S T R A C T

A simplified translating friction element system is formulated, designed, and evaluated to simulate practical brake vibration (judder) issues. This simplified translating friction element system includes two frictional guides and two actuators to simulate dynamic interactions between the brake pad and the calipers (including the guide pins). First, a two degree of freedom nonlinear model and then a single degree of freedom linear model of the translating friction element problem are developed and solved for eight cases. Second, a simple laboratory experiment is designed, and the measurements are found to correlate well with the predictions of both mathematical models. Third, experiments and mathematical models are utilized to study the effects of the frictional guide and normal load application locations on a shift in the center of contact force. An effective torsional stiffness element is also defined to incorporate the center of contact force concept. Finally, a two degree of freedom nonlinear model of a simplified brake vibration (judder-like) problem is proposed, and the role of calipers on the dynamic response is discussed.

1. Introduction

Vehicle brake judder is a low frequency, friction-induced forced vibration problem, and it is typically quantified in terms of brake torque variations (T(t)) [1–4]. The primary source to this problem is assumed to be the geometric distortions of brake rotors at single or multiple orders of the excitation frequency, which is proportional to the vehicle speed. Disturbances due to these geometric imperfections of brake rotors are transmitted to the driver through structural paths with nonlinearities, and can get amplified due to an existing path resonance [1–5]. Most mathematical models for “cold” brake judder assume a uniform brake disc–pad contact surface [1–7]. However, a disc that possesses a wavy braking surface will create a non-uniform contact interface between it and the brake pad. This non-uniformity on the interfacial braking surface can affect heat distribution patterns on the disc surface as well as the time-varying effective center of pressure. Changes in the center of pressure can affect several parameters which can ultimately affect T(t). These suggest that judder sensitivity at the caliper can be reduced by reducing the effective stiffness of the caliper system, a change that can adversely affect pedal feel. But the existing literature does not take into account the stiffness distribution from leading to trailing edge of the contact interface in the disc–pad or pad–caliper interfaces during a loaded braking, which may uncover means to reduce judder sensitivity at the caliper without affecting caliper compliance.

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^L, A^T$</td>
<td>internal areas of point actuators at leading and trailing edges</td>
</tr>
<tr>
<td>$c$</td>
<td>effective contact damping coefficient</td>
</tr>
<tr>
<td>$c^x, c^T$</td>
<td>viscous damping coefficient for piston–hydraulic system</td>
</tr>
<tr>
<td>$c_f, c_f^T$</td>
<td>viscous damping coefficients for contact model at leading/trailing edges</td>
</tr>
<tr>
<td>$C^0$</td>
<td>torsional contact damper</td>
</tr>
<tr>
<td>$F_f$</td>
<td>friction force between the friction element and friction material</td>
</tr>
<tr>
<td>$F_{fg}$</td>
<td>friction force between the friction material and frictional guide</td>
</tr>
<tr>
<td>$F_{pp}$</td>
<td>peak-to-peak value of $F_f$</td>
</tr>
<tr>
<td>$F_{pp}^{il}$</td>
<td>normalized $F_{pp}$ with corresponding value of case A1, $F_{pp}^{il}/F_{pp}^{il}$</td>
</tr>
<tr>
<td>$F_I$</td>
<td>total contact force at friction material/element interface, $F_I + F_{fg}^I$</td>
</tr>
<tr>
<td>$F_I^L, F_I^T$</td>
<td>contact forces between translating friction element and friction material at leading and trailing edges of friction material</td>
</tr>
<tr>
<td>$F_N$</td>
<td>total external force, $F_N = F_N^L + F_N^T$</td>
</tr>
<tr>
<td>$F_{Na}$</td>
<td>time varying component amplitude of the total external force</td>
</tr>
<tr>
<td>$F_{Na}^L, F_{Na}^T$</td>
<td>amplitudes of time varying components of $F_N^L$ and $F_N^T$</td>
</tr>
<tr>
<td>$F_{Nm}$</td>
<td>mean component of the total external force</td>
</tr>
<tr>
<td>$F_{Nm}^L, F_{Nm}^T$</td>
<td>mean components of $F_N^L$ and $F_N^T$</td>
</tr>
<tr>
<td>$F_p$</td>
<td>pulling force measured with the force transducer</td>
</tr>
<tr>
<td>$F_{pre}^L, F_{pre}^T$</td>
<td>preload forces at leading and trailing edges</td>
</tr>
<tr>
<td>$I_g$</td>
<td>moment of inertia of friction material about the axis that passes through the supported edge, $I_g + m(0.5l)^2$</td>
</tr>
<tr>
<td>$I_o$</td>
<td>moment of inertia of friction material about its center of mass</td>
</tr>
<tr>
<td>$k$</td>
<td>effective contact stiffness</td>
</tr>
<tr>
<td>$k^x$</td>
<td>translational contact stiffness</td>
</tr>
<tr>
<td>$k^L, k^T$</td>
<td>stiffness of piston–hydraulic system</td>
</tr>
<tr>
<td>$k_f^L, k_f^T$</td>
<td>linear stiffness coefficients for contact model at leading/trailing edges</td>
</tr>
<tr>
<td>$K^0$</td>
<td>torsional contact stiffness</td>
</tr>
<tr>
<td>$l$</td>
<td>length of the friction material</td>
</tr>
<tr>
<td>$l^F, l^T$</td>
<td>distance between center of mass and leading/trailing edge actuators</td>
</tr>
<tr>
<td>$l_f^L, l_f^T$</td>
<td>distance between center of mass and leading/trailing contact locations</td>
</tr>
<tr>
<td>$m$</td>
<td>mass of friction material</td>
</tr>
<tr>
<td>$m_b$</td>
<td>total mass of translating components in experiment</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure at air cylinder</td>
</tr>
<tr>
<td>$p_N$</td>
<td>pressure at point actuators</td>
</tr>
<tr>
<td>$q$</td>
<td>order number</td>
</tr>
<tr>
<td>$r$</td>
<td>rotor radius</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
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**Nomenclature**

- $T$: brake torque
- $w$: width of the friction material
- $x$: translation of friction material in $e_s$
- $\dot{x}, \ddot{x}$: measured accelerations at leading and trailing edges of friction material
- $\dot{x}_a, \ddot{x}_a$: measured accelerations at leading and trailing edges of actuation system
- $\dot{x}_r, \ddot{x}_r$: relative accelerations between friction material and actuation system at leading and trailing edges
- $y$: displacement of friction material in $e_y$
- $y_b$: displacement of translating friction element in $e_y$
- $\dot{y}_b$: acceleration of translating friction element in $e_y$
- $\alpha^q, \alpha^q_r$: displacement excitation phase for qth order at leading and trailing edges
- $\gamma$: $\sqrt{l_g k - (0.5c)^2}$
- $\zeta$: $c/2\sqrt{l_g k}$
- $\theta$: rotation of friction material about $e_z$
- $\kappa$: specific coefficients
- $\alpha^L, \alpha^T$: acceleration response ratio from trailing edge to leading edge
- $\mu_g$: friction coefficient between the frictional guide and friction material
- $\mu_i$: friction coefficient at friction element/material interface
- $\xi^L, \xi^T$: displacement excitations acting on friction material due to geometric distortions of brake rotor at leading and trailing edges
- $\Xi_q$: displacement excitation amplitude for qth order
- $\rho$: distance between center of contact forces and supported edge of friction material
- $\rho^+$: $\rho$ for positive sliding velocity of friction material at supported edge
- $\rho^-$: $\rho$ for negative sliding velocity of friction material at supported edge
- $\rho_n$: normalized $\rho$ with $0.5l, \rho/0.5l$
- $\sigma$: regularizing factor for ‘sgn’ function
- $\psi$: pivoting angle of friction material around its supported edge
- $\omega$: excitation frequency
- $\omega_1$: $\sqrt{k/l_g}$

**Subscripts**

- $a$: alternating component
- $act$: denotes a property of actuation system
- $b$: translating friction element
- $e$: effective value
- $f$: property at friction material/element interface
- $g$: property at friction material/frictional guide interface
- $i$: interface
- $m$: mean component
- $N$: denotes external normal load
- $pre$: preload
- $pp$: peak-to-peak value
- $q$: relative value
- $r$: order number
The chief goal of this article is to develop a scientific problem that would conceptually examine the dynamic interactions between the brake pad and the caliper. In particular, the focus will be on the leading-to-trailing stiffness distribution effect within the disc–pad and pad–caliper interfaces. Such issues are recognized for high frequency brake squeal and groan [8–11], and it has recently been experimentally studied by the authors [12] from the brake judder point of view. Given a variety of caliper designs, it is extremely difficult to construct an experiment to evaluate alternate designs under controlled conditions. Therefore, a simplified translating friction element system, as conceptually illustrated in Fig. 1, is formulated, designed, and evaluated. In this simulated scientific problem, the rotating brake disc is replaced with a translating friction element that has minimal surface distortions. Further, the brake pad of a conventional brake assembly is substituted by a friction material in the simplified problem. This friction material is supported from either its leading or trailing edge with a frictional guide, which is the abutment bracket of a conventional brake assembly. By using this translating friction element problem, normal load application locations and shifts in the center of contact force location will be experimentally and mathematically studied. Since the proposed problem is motivated by the judder problem, only the low frequency dynamics are of interest, and thus lower dimensional models are adequate [1–5].

2. Problem formulation

2.1. Literature review and unresolved issues

The dynamic amplification issue of brake judder problem is investigated by Sen et al. [1,2] on a dynamic friction experiment, and different events in the operational speed range are identified. Jacobsson [4] used simplified vehicle models...
and predicted $T(t)$ along with its envelope function using analytical means. Duan and Singh [5] approached the problem with a source–path–receiver network model and related the dynamic amplifications of the steering wheel angular displacement to the rotor imperfections. Besides these papers that focus on the dynamic amplification issue with models excluding actuation systems, Leslie [6] developed a detailed lumped parameter model of a brake corner including caliper, abutment bracket, hydraulic pistons and brake pads and numerically predicted $T(t)$. Kang and Choi [7] numerically calculated $T(t)$ using a single degree of freedom model, and discussed the effects of key parameters on $T(t)$ response at non-resonant operating speeds. In another study, Kim et al. [13] used the multi-body dynamics approach to investigate the key parameters of the problem, which are mainly stiffness terms in their lumped parameter caliper model. The authors proposed applicable modifications on the brake corner to alter the stiffness of certain components and discussed the reduction on $T(t)$ response. Additionally, experimental studies done by the authors suggest that the frictional torque amplitudes at higher speeds (non-resonant speeds region) are influenced by the caliper dynamics, though the available judder models either exclude the actuation system in mathematical models [1–5] or focus on component stiffness issues for $T(t)$ reduction [6,7,13].

The interrelationship between a shift in the center of the contact force location and brake squeal noise generation was first observed by Spurr [14], who proposed the sprag–slip mechanism for high frequency noise problem. This problem has been subsequently studied with pin-on-disc models [15–17], which relate the system stability to the sprag angle, though other mechanisms of the brake squeal problem have been extensively investigated [18,19]. In particular, Fieldhouse et al. [9] measured the location of the center of pressure during a braking operation, and observed that a shift in the center of pressure towards the leading edge generates a high propensity to squeal noise. Further, Fieldhouse and Steel [10] and Fieldhouse et al. [11] extended the Spurr model [14] by adding another friction regime, which is located between the brake pad and the abutment (support) bracket. Fieldhouse et al. [11] calculated the distance ($\rho$) between the center of contact forces and this supported edge as follows from the perspective of squeal noise: $\rho = \mu_l w + 0.5(1 + \mu_g)l$, where $\mu_l$ and $\mu_g$ are the friction coefficients at the brake pad/rotor interface and the supported edge, and $w$ and $l$ are the pad thickness and length, respectively; the $\pm$ sign arises due to the directional behavior of the friction force at the supported edge. However, this expression is only valid for a pad that is supported at the trailing edge. Dreyer et al. [12] studied the effect of brake disc/pad contact geometry on the measured $T(t)$ response. Further, an effective dynamic stiffness expression has been proposed as it explains some experimental trends [12]. In the current article, this concept is fully investigated by focusing on the brake pad leading and trailing edge dynamics instead of brake pad–caliper geometric effects.

2.2. Translating friction element problem

Fig. 1 illustrates the proposed physical system in terms of a translating friction element with two different constraint location combinations. Two cases (designated as A and B) conceptually simulate the ‘push’ or ‘pull’ type calipers in automotive brake applications [20]. In the ‘push’ type caliper (simulated in Fig. 1(a)), the constraint is located at the trailing edge (superscript $T$), and the friction force ($F_f$) between the translating friction element (represents brake rotor) and friction material (brake pad) ‘pushes’ the friction material towards the frictional guide (abutment bracket). Therefore, another frictional regime emerges at the trailing edge. Conversely, in the ‘pull’ type caliper (simulated in Fig. 1(b)), the constraint is at the leading edge (superscript $L$); thus, the second frictional regime occurs at the leading edge of the friction material (brake pad) due to the friction interface between the translating friction element (brake rotor) and friction material (brake pad). The normal and contact forces at the leading and trailing edges are given with $F_{Nl}$, $F_{Nt}$, $F_{Tl}$, and $F_{Tt}$, respectively, in Fig. 1. In addition, the friction force at the friction material/frictional guide interface is described by $F_{fg}$ where the preceding $\pm$ sign represents the positive

<table>
<thead>
<tr>
<th>Normal load locations</th>
<th>Frictional guide at the trailing edge (A)</th>
<th>Frictional guide at the leading edge (B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point actuators closer to the outer edges (1)</td>
<td>A1</td>
<td>B1</td>
</tr>
<tr>
<td>Point actuators close to the center (2)</td>
<td>A2</td>
<td>B2</td>
</tr>
<tr>
<td>Point actuators at the leading edge (3)</td>
<td>A3</td>
<td>B3</td>
</tr>
<tr>
<td>Point actuators at the trailing edge (4)</td>
<td>A4</td>
<td>B4</td>
</tr>
</tbody>
</table>

Fig. 2. Illustration of all loading and constraint configurations as examined in this article. Letters (A and B) describe the constraint(s) at the trailing and leading edges, respectively. Numbers (from 1 to 4) designate the normal load configurations. Arrows show the forces acting on the body. Key: $F_{Nl}$ and $F_{Tl}$, $F_{Nt}$, $F_{Tt}$, $F_{fg}$.
and negative slipping velocity conditions at the supported edge. Besides different constraint locations (as given by cases A and B), the following possible normal load locations are shown in Fig. 2: close to the edges (1), center with a narrow gap (2), leading edge (3), and trailing edge (4) of the friction material. In addition, the existence of frictional guides creates scleronomic constraint with an equation $y = 0$, where $y$ is the friction material displacement in the $e_y$ direction; hence, no motion for the friction material in the $e_y$ direction is possible. Fig. 2 lists all possible force combinations in terms of the normal force (given by the solid arrows) and friction force (shown by the dashed and dash-dotted arrows for $F_f$ and $F_{fg}$, respectively) locations.

The main objectives of this article are as follows. (1) Develop simplified nonlinear and linear models of the translating friction element of Fig. 1, and solve them for the cases of Fig. 2. (2) Design a simple laboratory experiment to simulate the system of Figs. 1 and 2, conduct measurements to explore the effect for all loading and constraint conditions, and validate nonlinear and linear model predictions based on the rank ordering of all cases. (3) Calculate the center of contact forces location ($\rho$) for all cases based on the static equilibria, and explain the significance of $\rho$ using an equivalent nonlinear model. (4) Propose a simplified judder-like problem to study the cases of Fig. 2, demonstrate the role of the center of contact concept, and generate suggestions for the practicing engineers.

### 3. Nonlinear model of the translating friction element problem

A two degree of freedom nonlinear model of the translating friction element problem is shown in Fig. 3, where only the translation ($x$) in the $e_x$ direction and rotation ($\theta$) about the $e_z$ direction of the friction material are considered. The brake frictional regime, which is the contact interface between the friction material and the translating friction element, is

---

**Fig. 3.** Two degree of freedom nonlinear model for the translating friction element problem: (a) for the frictional guide at the trailing edge (cases A1 to A4); (b) for the frictional guide at the leading edge (cases from B1 to B4).
assumed to be given by point contacts at two different locations; they are described with linear springs \((k_L^I \text{ and } k_T^I)\) and viscous dampers \((c_L^I \text{ and } c_T^I)\). The governing equations for case A (Fig. 3(a)) are derived as

\[
m\ddot{x} + (c_L^I + c_T^I)x + (c_L^I - c_T^I)\dot{\theta} + (k_L^I + k_T^I)x + (k_L^I - k_T^I)\dot{\theta} + F_{fg}\text{sgn}(\dot{x} - 0.5\dot{\theta}) = F^L + F^T,
\]

\[
I_o \ddot{\theta} + (c_L^I - c_T^I)\dot{x} + (c_L^I l_L^I)\dot{\theta} + (c_T^I l_T^I)\dot{\theta} + (k_L^I l_L^I - k_T^I l_T^I)x + (k_T^I l_L^I + k_L^I l_T^I)\dot{\theta} - F_f w - 0.5F_{fg}\text{sgn}(\dot{x} - 0.5\dot{\theta}) = F_{fg}^L - F_{fg}^T,
\]

where \(m\) and \(I_o\) represent the mass and the inertia of the friction material. In addition, \(l_L, l_T, l_L^I, \text{ and } l_T^I\) are the distances between the center of mass and leading edge point actuator, trailing edge point actuator, leading edge contact location, and trailing edge contact location, respectively.

The friction force at the brake frictional regime \((F_f)\) in Eq. (2) is calculated as

\[
F_f = \mu[I^I(c_L^I(\dot{x} + l_L^I\dot{\theta}) + c_T^I(\dot{x} - l_T^I\dot{\theta}) + k_L^I(x + l_L^I\dot{\theta}) + k_T^I(x - l_T^I\dot{\theta})] / C_{138},
\]

where \(\mu\) is the friction coefficient at the friction material/translated friction element interface. Since \(F_f\) is transmitted to the frictional constraint as a normal force, the friction force at the support \((F_{fg})\) becomes:

\[
F_{fg} = \mu_v F_f,
\]

where \(\mu_v\) is the friction coefficient between the friction material and frictional guide. Since \(F_{fg}\) changes direction depending on the body velocity at the supported edge, the model becomes piecewise-linear due to the ‘sgn’ function as defined below.

\[
\text{sgn}(z) = \begin{cases} 
-1 & \text{if } z < 0 \\
0 & \text{if } z = 0 \\
1 & \text{if } z > 0 
\end{cases}
\]

\[
\begin{align*}
F_f [N] & \quad t [s] \\
95 & \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1
\end{align*}
\]

\[
\begin{align*}
\dot{x}^L [m] & \quad \dot{x}^T [m] \\
1.15 \times 10^{-7} & \quad 1.1 \times 10^{-7} \\
6.483 & \quad 6.482 \\
6.481 & \quad 6.481 \\
6.48 & \quad 6.48 \\
6.479 & \quad 6.479 \\
0.96 & \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1
\end{align*}
\]

Fig. 4. Numerical solutions for case A1 at 10 Hz modulation frequency using the two degree of freedom nonlinear model: (a) friction force \((F_f(t))\); (b) displacements of the leading \((\dot{x}^L = x + 0.5\dot{\theta})\) and trailing \((\dot{x}^T = x - 0.5\dot{\theta})\) edges. Key: \(\ldots \), \(\ldots \旦 \), \(\ldots \旦 \).
where \( z \) is the argument of the ‘sgn’ function. Here, the Coulomb friction formulation is utilized and the ‘sgn’ function represents the directional behavior of \( F_{fg} \). However, the ‘sgn’ function is approximated with a hyperbolic tangent function as 
\[
\text{sgn}(z) = \tanh(\sigma z)
\]
in numerical studies, where \( \sigma \) is a regularizing factor [1,2,21]. For the sake of completeness, governing equations for case B (Fig. 3(b)) are given with Eqs. (A.1) and (A.2) in Appendix A.

In order to simulate the judder-like behavior over the speed range of interest, the external forces acting on the body, \( F_{L} \) and \( F_{T} \), are described as a combination of mean (constant) loads (\( F_{Nm} \) and \( F_{Nm} \)) and an infinite sum of sinusoidal functions with a fundamental frequency of \( \omega \), and corresponding constant amplitudes of \( F_{Na} \) and \( F_{Na} \), as expressed below:
\[
F_{N} = F_{Nm} + F_{Na} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin((2n+1)\omega t); \tag{6a}
\]
\[
F_{T} = F_{Nm} + F_{Na} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin((2n+1)\omega t), \tag{6b}
\]

note that Eqs. (6a) and (6b) represent triangular waveforms with the constant DC terms (\( F_{Nm} \) and \( F_{Nm} \)); the reason for choosing this type of excitation will be explained in Section 5. The nonlinear model is numerically solved for all cases of Fig. 2 at fundamental excitation frequencies (\( \omega \)) of \( 10\pi \) and \( 20\pi \) rad/s (5 and 10 Hz) due to experimental limitations. Single point contacts are assumed to be located at the edges of the friction material; i.e. \( l_{L} = l_{T} = 0.5 \). The results of case A1 for \( \omega = 20\pi \) rad/s are displayed in Fig. 4 in terms of the friction force \( F_{f} \) and displacements at the leading (\( x+0.5\theta l \)) and trailing (\( x-0.5\theta l \)) edges. Simulations are run with constant \( \mu_{I} \) and \( \mu_{g} \) values that are assumed as 0.6 and 1.0, respectively, and results are shown only for the steady-state motion of the friction material in Fig. 4. As seen in Fig. 4(b), the leading edge motion of the pad has a peak-to-peak amplitude of 20 \( \mu \)m. However, the trailing edge is almost stationary (total displacement of 30 nm) relative to the leading edge due to the existence of a frictional guide. Such small displacement is a result of the approximation utilized for the ‘sgn’ function. This can be explained with the sticking condition as defined below.
\[
|F_{N}^{L} + F_{N}^{T} - F_{I}| \leq \mu_{g} \mu_{I} F_{I}, \tag{7}
\]
where \( F_{I} = F_{L}^{T} + F_{T}^{T} \) is the total contact force at the friction material/ translating friction element interface, and it is calculated by the expression inside the brackets of Eq. (3). Therefore, the left hand side of Eq. (7) represents the total force acting on the friction material in the \( e_{x} \) direction. The term \( \mu_{I} F_{I} \) is the total force acting on the friction material in the \( e_{y} \) direction, and thus

![Fig. 5. Simplified single degree of freedom linear model of the translating friction element system of Fig. 3: (a) for the frictional guide at the trailing edge (cases from A1 to A4); (b) for the frictional guide at the leading edge (cases from B1 to B4).](image-url)
the right hand side of Eq. (7) is the friction force generated at the friction material/frictional guide interface. Rewrite Eq. (7), after some algebraic manipulations, as

\[
\frac{F_{N}^{L} + F_{N}^{T}}{1 + \mu \mu_{g}} \leq F_{f} \leq \frac{F_{N}^{L} + F_{N}^{T}}{1 - \mu \mu_{g}},
\]

(8)
it is seen that the trailing edge sticks to the frictional guide only when \( F_{f} \) satisfies the condition of Eq. (8). Since this condition is always satisfied in numerical simulations (for both cases A and B, and fundamental frequencies 10\( \pi \) and 20\( \pi \) rad/s), the friction material should physically stick to the frictional guide surface at the supported edge. This also suggests that the pure sticking condition cannot be numerically obtained by approximating the ‘\( \text{sgn} \)’ function. Moreover, it is observed that the accuracy of numerical predictions depends on the value of \( \sigma \). The simulations with lower \( \sigma \) values lead to unrealistic results, as also reported before  [1].

4. Analytical solution of the translating friction element problem by using a linear model

Assume that the frictional guide creates another scleronomous constraint with equations \( x - 0.5l = 0 \) for case A and \( x + 0.5l = 0 \) for case B; this assumption is valid since the body sticks to the frictional guide in each case as mathematically shown in the previous section. Consequently, the two degree of freedom nonlinear model of Section 3 is simplified to a single degree of freedom linear model, in terms of the pivoting motion \( (\phi) \) of the friction material around the frictional guide. The corresponding single degree of freedom linear models are displayed in Fig. 5(a) and (b). The governing equation for case A is derived as

\[
I_{g}\ddot{\phi} + ((c_{l}^{T}(0.5l + l_{l})0.5l + l_{l} - \mu_{l}w)) + (c_{T}^{T}(0.5l - l_{l}))(0.5l - l_{l} - \mu_{l}w))\phi
\]

\[
+((k_{l}^{T}(0.5l + l_{l}))(0.5l + l_{l} - \mu_{l}w)) + (k_{T}^{T}(0.5l - l_{l}))(0.5l - l_{l} - \mu_{l}w))\phi,
\]

\[
= F_{N}^{L}(0.5l + l_{l}) + F_{N}^{T}(0.5l - l_{l})
\]

(9)

where \( I_{g} \) is now the effective moment of inertia of the body about the axis that passes through the frictional guide, and it is calculated by using the parallel axis theorem as \( I_{g} = I_{g0} + m(0.5l)^{2} \). Again refer to Appendix A for the case B equation as given by Eq. (A.3).

First consider the mean components of the normal force excitations \( (F_{Nm}^{L} \text{ and } F_{Nm}^{T}) \) and find the step response in the Laplace domain \( (s) \) with zero initial conditions for case A, as given by Eq. (9):

\[
\Phi_{m}(s) = \frac{F_{Nm}}{s(I_{g}S^{2} + cs + k)},
\]

(10)

where

\[
F_{Nm} = F_{Nm}^{L}(0.5l + l_{l}) + F_{Nm}^{T}(0.5l - l_{l}),
\]

(11a)

\[
c = c_{l}^{T}(0.5l + l_{l})(0.5l + l_{l} - \mu_{l}w) + c_{T}^{T}(0.5l - l_{l})(0.5l - l_{l} - \mu_{l}w),
\]

(11b)

\[
k = k_{l}^{T}(0.5l + l_{l})(0.5l + l_{l} - \mu_{l}w) + k_{T}^{T}(0.5l - l_{l})(0.5l - l_{l} - \mu_{l}w),
\]

(11c)

by taking the inverse Laplace transform of Eq. (10), \( \varphi_{m}(t) \) is calculated as

\[
\varphi_{m}(t) = \frac{F_{Nm}}{k} \cos \left( \frac{\gamma}{I_{g}} t \right) \exp(-\zeta\omega_{1}t) - \frac{F_{Nm}c}{2k\gamma} \sin \left( \frac{\gamma}{I_{g}} t \right) \exp(-\zeta\omega_{1}t),
\]

(12)

where \( \gamma = \sqrt{l_{g}k - (0.5c)^{2}} \), \( \omega_{1} = \sqrt{k/I_{g}} \), and \( \zeta = c/2\sqrt{k}I_{g} \). For the sinusoidal part of the excitations, \( \Phi_{a}(s) \) becomes

\[
\Phi_{a}(s) = \sum_{n=0}^{\infty} \frac{F_{Na}(n)\omega_{n}}{(I_{g}S^{2} + cs + k)(S^{2} + \omega_{n}^{2})},
\]

(13)

where

\[
F_{Na}(n) = F_{Na}^{l} \left( \frac{-1}{2n + 1} \right)^{2} \left( \frac{1}{2} + l_{l} \right) + F_{Na}^{T} \left( \frac{-1}{2n + 1} \right)^{2} \left( \frac{1}{2} - l_{l} \right),
\]

(14a)

\[
\omega_{n} = (2n + 1)\omega_{1},
\]

(14b)

again using the inverse Laplace transform, \( \varphi_{a}(t) \) is calculated as

\[
\varphi_{a}(t) = \sum_{n=0}^{\infty} \left\{ \frac{\kappa_{1}(n)\exp(-\zeta\omega_{1}t)\cos \left( \frac{\omega_{n}t}{I_{g}} \right) + 2\kappa_{2}(n)\omega_{n}\exp(-\zeta\omega_{1}t)\sin \left( \frac{\omega_{n}t}{I_{g}} \right)}{\omega_{n}^{2}} \right\},
\]

(15)
where

\[
\kappa_1(n) = \frac{F_{Na}(n)c\omega_n}{I_g^2\omega_n^4 + c^2\omega_n^2 - 2I_gk\omega_n^2 + k^2},
\]

(16a)

\[
\kappa_2(n) = \frac{F_{Na}(n)\omega_n(I_g^2\omega_n^2 + c^2 - kl_g)}{I_g^2\omega_n^4 + c^2\omega_n^2 - 2I_gk\omega_n^2 + k^2},
\]

(16b)

\[
\kappa_3(n) = -\frac{F_{Na}(n)c\omega_n}{I_g^2\omega_n^4 + c^2\omega_n^2 - 2I_gk\omega_n^2 + k^2},
\]

(16c)

\[
\kappa_4(n) = \frac{F_{Na}(n)\omega_n(k - l_g\omega_n^2)}{I_g^2\omega_n^4 + c^2\omega_n^2 - 2I_gk\omega_n^2 + k^2}.
\]

(16d)

Finally, the total response of the system is calculated from Eqs. (12) and (15):

\[
\varphi(t) = \varphi_m(t) + \varphi_a(t).
\]

(17)

Fig. 6. Comparison of measured and predicted normalized peak-to-peak friction forces \(\bar{F}_{fpp}\) for the translating friction element problem: (a) at 10 Hz modulation frequency; (b) at 5 Hz modulation frequency. Key: \(\square\), numerical predictions for cases A1–A4 using the nonlinear model; \(\square\), analytical predictions for cases A1–A4 using the linear model; \(\triangle\), numerical predictions for cases B1–B4 using the nonlinear model; \(\Delta\), analytical predictions for cases B1–B4 using the linear model.
hence, the friction force $F_f$ at the friction element/material interface is calculated with Eq. (18) for case A, as expressed below:

$$F_f = \mu_l [(c_1^A(0.5l + 0.5l – l_t^f) + c_1^B(0.5l – 0.5l – l_t^f))\omega + (k_1^A(0.5l + 0.5l – l_t^f) + k_1^B(0.5l – 0.5l – l_t^f))\omega].$$

(18)

it should be noted that expressions given with Eqs. (11), (14) and (18) are case specific (A or B), and the corresponding expressions of case B are given in Appendix A with Eq. (A.4).

As seen in Eq. (18), $\omega$ should be calculated in order to estimate $F_f$. This is done by taking the derivatives of Eqs. (12) and (15) with respect to $t$ as

$$\dot{\omega}(t) = \left(\frac{F_{nmC}}{l_c} + \frac{F_{nmC \cos n}}{2l_c} - \sum_{n=0}^{\infty} \left(\frac{k_1(n)\omega + (2k_2(n)l_c - c_k(n))\omega}{2l_c}\right) \sin \left(\frac{\omega t}{l_c}\right) \exp(-\omega t) \right) + \left(\frac{F_{nmC}}{l_c} - \sum_{n=0}^{\infty} \frac{k_1(n)\omega - (2k_2(n)l_c - c_k(n))}{2l_c} \right) \cos \left(\frac{\omega t}{l_c}\right) \exp(-\omega t) - \sum_{n=0}^{\infty} k_3(n)\omega_n \sin (\omega_n t) - k_4(n) \cos (\omega_n t).$$

(19)

The friction forces ($F_f$) are calculated for all cases at previously given frequencies $\omega = 10\pi$ and $20\pi$ rad/s, and the peak-to-peak values ($F_{pp}$) are compared in Fig. 6(a and b) for both linear (square and circle markers for cases A and B, respectively) and nonlinear (solid and dashed lines for cases A and B, respectively) model predictions (and with measurements as explained in the next section). Note that linear and nonlinear model predictions are given with the ordinates of Figs. 3 and 5. Symbols used to identify the sensors are explained in Section 5.

5. Translating friction element experiment

A translating friction element experiment is specifically designed and constructed as depicted in Fig. 7. In this experiment, a custom-built stationary friction material of length 82.55 mm (to simulate the brake pad) is pushed by two point actuators (to simulate the brake caliper) towards a translating friction element of length 300 mm; hence the element is compressed between two surfaces while translating. Teflon is preferred on the other side of the element due to its low friction coefficient (0.05–0.08), which is about an order of magnitude lower than the friction material side of the translating friction element [22]. Thus, the friction force generated on the Teflon side is negligible (about an order of magnitude less for the same normal load) compared to the friction element/material side. The actuation system consists of two steel L-brackets (stainless steel) with circular cross sections of 3 mm diameter. The friction element is pulled in only one direction by an air cylinder to ensure that the element does not bend. The following measurements are made during the motion of the element: force ($F(t)$) between the air cylinder and element using a load cell, pressures at the air inlets of point actuators ($P(t)$) and the air cylinder ($p(t)$) using pressure transducers, the translating friction element displacement ($y(t)$) using a linear potentiometer and accelerations on the element ($\ddot{y}(t)$), at the leading and trailing edges of custom built friction material ($x^L$ and $x^R$), and actuation system ($\dot{x}^L$ and $\dot{x}^R$) using uniaxial piezoelectric accelerometers.

This experiment is specifically designed to run all cases of Fig. 2. The design of the stationary friction material allows a controlled implementation of case A or B, i.e. friction interfaces between the stationary friction material and leading or trailing edge frictional guide.
trailing edge guides can be separately obtained. In addition, point actuators are placed in a slot such that they can slide parallel to the translating friction element motion direction; thus the experimental design permits different loading conditions. The normal forces applied by the point actuators are modulated by using an on–off solenoid valve. Hence a

![Graphs](image)

**Fig. 8.** Typical measurements of translating friction element motion at 10 Hz modulation frequency: (a) displacement $y_b(t)$; (b) velocity $\dot{y}_b(t)$; (c) acceleration $\ddot{y}_b(t)$. Key: —, case A1; —, case B1.
triangular wave shaped excitation is obtained as defined earlier. Experiments are performed at two different modulation frequencies, which are 5 and 10 Hz as given before. Due to the physical limitations of the solenoid valve, modulation frequencies higher than 10 Hz could not be used.

The typical displacement ($y_b$), velocity ($\dot{y}_b$) and acceleration ($\ddot{y}_b$) time histories of the translating friction element at 10 Hz modulation frequency are displayed in Fig. 8 for cases A1 and B1. Observe in Fig. 8(a) that $y_b$ decreases almost linearly, and thus, the velocity of the element is almost constant. Calculated translating friction element velocities (Fig. 8(b)) using a numerical differentiation method also show that the velocity oscillates around almost constant mean values ($-0.079$ m/s for case A1 and $-0.094$ m/s for case B1) with very small amplitudes (about 0.015 m/s for both cases). Here, it should be noted that an 8th order Butterworth low pass filter is applied to the calculated velocities, in order to eliminate high frequency noise that is introduced by the numerical differentiation operation. Further, a steady sliding motion is achieved, without any stick–slip behavior, as the velocity does not cross the zero velocity line. Since the translating friction element velocity is constant, the forces acting on the element in the $e_x$ direction should be equal due to very small beam acceleration in the $e_x$ direction; $\dot{y}_b$ oscillates around zero mean value with an amplitude of about 0.1g as seen in Fig. 8(c). Hence, the friction force $F_f$ should be equal to the measured pulling force $F_p$. Such measured friction force $F_f$ and normal force $F_N (F_N = F_F + F_{Y})$, acting on the translating friction element, are displayed again in Fig. 9 for both cases A1 and B1. Note that $F_N$ is estimated by taking a product of measured pressure ($p_N(t)$) at the point actuators and internal area of the actuators ($A^E$ and $A^T$). As seen in Fig. 9, modulation of pressure $p_N(t)$ leads to triangular wave shaped signals with almost constant mean. The mean value of $F_f$ is removed from the signal due to the nature of the dynamic sensor used to measure the pulling force $F_p$, and thus only the alternating component (in terms of peak-to-peak values) is reported. For 5 Hz modulation frequency, zero deceleration assumption is no longer valid. As depicted in Fig. 10, the translating motion has a significant alternating component. Even though the mean deceleration is still about zero as shown in Fig. 10(c), oscillation in the $\ddot{y}_b$ signal has an amplitude of about 0.5g that is 5 times more than that at 10 Hz frequency modulation. Further, the mean velocity is around $-0.21$ m/s, and it has a peak-to-peak amplitude of around 0.2 m/s, which is 7 times greater than that at 10 Hz frequency modulation. Nonetheless, there is still no stick–slip behavior in the translating friction element motion. Hence, $F_f$ is not equal to the

![Figure 9](https://example.com/fig9.png)

**Fig. 9.** Measured normal ($F_N(y)$) and tangential ($F_f(y)$) forces acting on the element at 10 Hz modulation frequency: (a) for case A1; (b) for case B1. Key: $F_N(y)$; $F_f(y)$. 
pulling force $F_p$, which is measured with the force transducer, and it is estimated from the force equilibrium of the element $m_b \ddot{y}_b = \sum F = F_p - F_f$, where $m_b$ is the total mass of the translating part of the experiment. Similarly, estimated $F_f$ along with $F_N$ are depicted in Fig. 11 again for cases A1 and B1. As seen in Fig. 11, the number of pulsations decreases due to low modulation frequency for the same amount of element displacement $y_b$. In addition, the mean value of $F_N$ decreased

![Fig. 10. Typical measurements of element motion at 5 Hz modulation frequency: (a) displacement $y_b(t)$; (b) velocity $\dot{y}_b(t)$; (c) acceleration $\ddot{y}_b(t)$. Key: ---, case A1; -----, case B1.](image)
although its peak-to-peak value increased. This is due to the modulation technique used during the experiment. As mentioned before, $F_N$ is modulated with an on–off solenoid valve; increasing the modulation period causes more air to be released. Hence, the pressure difference between the on and off states of solenoid valve is higher, and the mean value of $F_N$ is lower. This also explains the reason for having higher acceleration amplitudes with 5 Hz modulation frequency. Though a needle valve could be used to obtain similar $F_N$ levels for both modulation frequencies, in the current article, cases A and B are separately investigated in terms of rank ordering.

The measured peak-to-peak values of $F_f (F_{fpp})$ are extracted for all cases of Fig. 2 at two modulation frequencies and compared with predictions in Fig. 6. Measured $F_{fpp}$ values are given with the abscissa of Fig. 6, and they are again normalized with $F_{fpp}^{A1}$. Measurements also show the same rank orders at both modulation frequencies as calculated by the nonlinear and linear models where $F_{fpp}^{A1} < F_{fpp}^{A2} < F_{fpp}^{A4}$ for case A, and $F_{fpp}^{B3} < F_{fpp}^{B2} < F_{fpp}^{B4}$ for case B. This behavior suggests that the center of contact forces seem to affect the friction force amplitudes. Moreover, these trends show that system stiffness is case dependent, since the peak-to-peak variation in the response changes. In addition, Fig. 6(a) shows a reasonable match between measurements and predictions. A least square curve fit of the measured and predicted data (for all 8 cases) at 10 Hz modulation frequency is also shown in Fig. 6(b). The key for the figures is: $F_N(y)$: solid line, $F_f(y)$: dotted line.

### Table 1

<table>
<thead>
<tr>
<th>Normal load locations</th>
<th>Friction guide at trailing edge (A)</th>
<th>Friction guide at leading edge (B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0.18</td>
<td>7.49</td>
</tr>
<tr>
<td>(2)</td>
<td>0.23</td>
<td>7.81</td>
</tr>
<tr>
<td>(3)</td>
<td>0.19</td>
<td>6.32</td>
</tr>
<tr>
<td>(4)</td>
<td>0.06</td>
<td>5.39</td>
</tr>
</tbody>
</table>

Fig. 11. Measured normal ($F_N(y)$) and tangential ($F_f(y)$) forces acting on the element at 5 Hz modulation frequency: (a) for case A1; (b) for case B1. Key: $F_N(y)$: solid line, $F_f(y)$: dotted line.
and 5 Hz modulation frequencies with the equation $F_{\text{fpp, predicted}} \approx \beta_1 T_{\text{fpp, measured}} + \beta_2$ gives a slope $\beta_1$ of 1.08 and 2.69, and constant term $\beta_2$ of $-0.18$ and $-1.73$, respectively. The coefficient of determination of both fits is calculated as 0.89. The main reason for some discrepancies is due to the simplifying assumptions made in the models, such as constant friction coefficients and linear elastic characteristics.

In order to check the motion of the body at the supported edge, measured acceleration values are compared. As shown in Fig. 7, accelerations are measured on leading and trailing edges of both friction material and actuation system; hence relative accelerations at the edges of the friction material are calculated as

$$\ddot{x}^L_{\text{rel}}(t) = \ddot{x}^L(t) - \ddot{x}^L_{\text{act}}(t),$$

$$\ddot{x}^T_{\text{rel}}(t) = \ddot{x}^T(t) - \ddot{x}^T_{\text{act}}(t),$$

where subscript ‘act’ denotes the actuation system, and superscripts $L$ and $T$ represent leading and trailing edges, respectively. Further, $\ddot{x}^L_{\text{rel}}$ and $\ddot{x}^T_{\text{rel}}$ are transformed to the frequency domain, and the acceleration response ratio (from the trailing edge to the leading edge) is calculated as

$$\lambda_{LT}(\omega) = \frac{\ddot{x}^T(\omega)\ddot{x}^L(\omega)^*}{\ddot{x}^L(\omega)\ddot{x}^T(\omega)^*},$$

where * is the conjugate operator. The ratio $\lambda_{LT}$ is calculated in Table 1 at $\omega = 20\pi$ rad/s modulation frequency. Note that lower motion at the trailing edge is expected for case A, which is the supported edge. Hence, $\lambda_{LT} < 1$. However, for case B, the leading edge is supported, and thus, $\lambda_{LT} > 1$. As seen in Table 1, the acceleration values at the supported edge(s) are almost an order of magnitude lower than those at the unsupported edge(s).

### 6. Role of the center of contact forces

The distance ($\rho$) from the center of contact forces to the frictional guide constraint is calculated next for all cases. This is accomplished by combining $F^L_1$ and $F^T_1$ (of Fig. 1) into a single (total contact) force $F_1$ and solving for the force and moment equilibria. Fig. 12(a) and (b) represents the free body diagrams for cases A and B, respectively. Note that the equilibrium conditions must be solved for two sliding cases (with positive and negative velocities) as the direction of $F_{fg}$ is unknown.

![Fig. 12. Forces acting on the body assuming slipping conditions for the translating friction element problem: (a) for the frictional guide at the trailing edge (cases from A1 to A4); (b) for the frictional guide at the leading edge (cases from B1 to B4).](image-url)
Since $F_{Bg}$ is bounded by $\pm \mu \mu_s F_l$ for the stick case, i.e. $-\mu \mu_s F_l \leq F_{Bg} \leq \mu \mu_s F_l$, the center of contact forces for the stick case should lie between the two slipping (positive and negative velocity) conditions. First, the equations of force (Eq. (22a) and (22b)) and moment (Eq. (23)) equilibria (about point O in Fig. 12(a)) are given for case A.

\[
F_N^L + F_N^L - F_l - \mu \mu_s F_l = 0 \quad \text{for } \dot{x} > -0.5 \dot{l} \dot{\theta} > 0, \quad (22a)
\]

\[
F_N^L + F_N^L - F_l - \mu \mu_s F_l = 0 \quad \text{for } \dot{x} > -0.5 \dot{l} \dot{\theta} < 0, \quad (22b)
\]

\[
F_N^L (0.5 \dot{l} + \dot{\theta}) + F_N^L (0.5 \dot{l} - \dot{\theta}) - F_l \mu + \mu W F_l = 0, \quad (23)
\]

since there are two unknowns ($F_l$ and $\rho$), they are solved with Eqs. (22) and (23). and expressions for $\rho$ are obtained as

\[
\rho^+ = (1 + \mu \mu_s) \left[ \frac{l + \mu_l \mu_s}{1 + \mu \mu_s} + \frac{F_N^L - F_N^L}{F_N^L + F_N^L} \right] \quad \text{for } \dot{x} > -0.5 \dot{l} \dot{\theta} > 0, \quad (24a)
\]

\[
\rho^- = (1 - \mu \mu_s) \left[ \frac{l + \mu_l \mu_s}{1 - \mu \mu_s} + \frac{F_N^L - F_N^L}{F_N^L + F_N^L} \right] \quad \text{for } \dot{x} > -0.5 \dot{l} \dot{\theta} < 0, \quad (24b)
\]

note that $\rho$ is expressed with + or − superscript that correspond to positive or negative slip velocity. Interestingly, Eqs. (24a) and (24b) simplify to Fieldhouse et al.’s equation [11] when $F_N^L = F_N^T$. Again, expressions of $\rho$ for case B are given in Appendix A with Eq. (A.5).

The position of the center of contact forces relative to the geometric center of the body can be estimated based on different conditions from Eqs. (24a) and (24b) and (A.5); these are listed in Tables 2 and 3 for the cases of Fig. 2. Here, $\mu_s$ is the main parameter that affects $\rho^+$ and $\rho^-$. In addition, Eqs. (24a) and (24b) and (A.5) show $\rho^+ \approx \rho^-$ when $\mu \mu_s = 0$, and this is only possible when either $\mu \mu_s > 0$ or $\mu \mu_s < 0$. When $\mu_s = 0$, $F_l = F_{Bg} = 0$, which means that there is no friction force acting on the body and $\rho^+ = \rho^-$. When $\mu_s = 0$, $F_l \neq 0$, but $F_{Bg} = 0$, though the $\rho^+ = \rho^-$ condition is still valid. This concludes that the friction regime at the friction material/frictional guide interface has a significant effect on the shift in the center of contact force.

### Table 2

Summary of the slipping conditions and center of contact force locations ($\rho^+$) with positive velocity, based on Eqs. ((24a) and A.5a).

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Location of center of contact forces</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1: $(l^L-l^L=0)$</td>
<td>Always</td>
<td>$\rho^+ &gt; 0.5$</td>
</tr>
<tr>
<td>A2: $(l^L-l^L=0)$</td>
<td>Always</td>
<td>$\rho^+ &gt; 0.5$</td>
</tr>
<tr>
<td>A3: $(l^L-l^L=0)$</td>
<td>Always</td>
<td>$\rho^+ &gt; 0.5$</td>
</tr>
<tr>
<td>A4: $(l^L-l^L=0)$</td>
<td>$\mu_s &gt; 2 \mu l$</td>
<td>$\rho^+ &gt; 0.5$</td>
</tr>
<tr>
<td>B1: $(l^L-l^L=0)$</td>
<td>$\mu_s &lt; 2 \mu l$</td>
<td>$\rho^+ &lt; 0.5$</td>
</tr>
<tr>
<td>B2: $(l^L-l^L=0)$</td>
<td>$\mu_s &gt; 2 \mu l$</td>
<td>$\rho^+ &gt; 0.5$</td>
</tr>
<tr>
<td>B3: $(l^L-l^L=0)$</td>
<td>$\mu_s &lt; 2 \mu l$</td>
<td>$\rho^+ &lt; 0.5$</td>
</tr>
<tr>
<td>B4: $(l^L-l^L=0)$</td>
<td>$\mu_s &gt; 2 \mu l$</td>
<td>$\rho^+ &gt; 0.5$</td>
</tr>
</tbody>
</table>

### Table 3

Summary of the slipping conditions and center of contact force locations ($\rho^-$) with negative velocity, based on Eqs. ((24b) and A.5b).

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Location of center of contact forces</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1: $(l^L-l^L=0)$</td>
<td>$\mu_s &gt; 2 \mu l$</td>
<td>$\rho^- &lt; 0.5$</td>
</tr>
<tr>
<td>A2: $(l^L-l^L=0)$</td>
<td>$\mu_s &lt; 2 \mu l$</td>
<td>$\rho^- &gt; 0.5$</td>
</tr>
<tr>
<td>A3: $(l^L-l^L=0)$</td>
<td>$\mu_s &gt; 2 \mu l$</td>
<td>$\rho^- &gt; 0.5$</td>
</tr>
<tr>
<td>A4: $(l^L-l^L=0)$</td>
<td>$\mu_s &lt; 2 \mu l$</td>
<td>$\rho^- &lt; 0.5$</td>
</tr>
<tr>
<td>B1: $(l^L-l^L=0)$</td>
<td>Always</td>
<td>$\rho^- &gt; 0.5$</td>
</tr>
<tr>
<td>B2: $(l^L-l^L=0)$</td>
<td>Always</td>
<td>$\rho^- &gt; 0.5$</td>
</tr>
<tr>
<td>B3: $(l^L-l^L=0)$</td>
<td>Always</td>
<td>$\rho^- &gt; 0.5$</td>
</tr>
<tr>
<td>B4: $(l^L-l^L=0)$</td>
<td>$\mu_s &lt; -2 \mu l$</td>
<td>$\rho^- &lt; 0.5$</td>
</tr>
</tbody>
</table>
Calculated distances between the center of contact forces and frictional constraint location are listed in Table 4 for all cases. Note that each value is normalized with 0.5\(l\) as \(\rho = \frac{\rho}{0.5l}\). First, observe that the same rank ordering as in Fig. 6 is still valid, i.e., \(\rho_{A4} < \rho_{A1} < \rho_{A2} < \rho_{A3}\) for case A and \(\rho_{B3} < \rho_{B2} < \rho_{B1} < \rho_{B4}\) for case B. This concludes that as the center of pressure moves away from the frictional support, \(F_{fpp}\) increases for all cases. Second, Table 4 shows that \(\rho_{\pm} \) for all cases, which means that the motion of the body in the \(+e_x\) direction shifts \(\rho\) further away from the supporting guide than a motion in the \(-e_x\) direction. Third, \(\rho_{A4} > \rho_{B}\) for symmetrically loaded cases (A1, A2, B1, B2) due to the direction of element motion. This motion in the \(+e_y\) direction shifts the center of contact forces closer to the leading edge. Fourth, as the normal loads \(F_{N1}\) and \(F_{N4}\) move towards one edge, the center of contact forces also shifts towards the same edge. To better visualize the data,

![Fig. 13. Comparison of normalized peak-to-peak friction forces \((F_{fpp})\) and center of contact locations \((\rho)\) for the translating friction element problem: (a) at 10 Hz modulation frequency; (b) at 5 Hz modulation frequency. Key: \(\times\) \(\rho^+\) for cases A1–A4; \(\times\) \(\rho^-\) for cases A1–A4; \(\times\) \(\rho^\pm\) for cases B1–B4; \(\times\) \(\rho^\mp\) for cases B1–B4.](image-url)
measured $F_{fp}$ from Fig. 6 is plotted in Fig. 13 against the calculated $\rho^+$ and $\rho^-$ values of Table 4 again for both modulation frequencies. First, observe that the variation in $\rho^-$ is less than $\rho^+$ for both cases A and B. Moreover, there is an increasing trend for both $F_{fp}$ and $\rho$, i.e. $F_{fp}$ increases as $\rho$ increases. To observe the trend quantitatively, least square curve fitting is again applied to the data with equation $F_{fp} \approx F_0 + \beta_2 \rho^+ + \beta_3 \rho^-$, respectively. At 10 Hz modulation frequency, coefficients for $\rho^+$ and $\rho^-$ are calculated as $F_0 = 0.52, \beta_2 = 0.26$ and $F_0 = 1.28, \beta_3 = 0.57$, respectively. In addition, coefficients of correlation are obtained as 0.89 and 0.67 for $\rho^+$ and $\rho^-$, respectively. At 5 Hz modulation frequency, the same coefficients are obtained as $F_0 = 0.21, \beta_2 = 0.68$ and $F_0 = 0.53, \beta_3 = 0.80$ for $\rho^+$ and $\rho^-$, respectively. Similarly, coefficients of correlation are calculated as 0.92 and 0.71, respectively. Even though the least square curve fit is not perfect, there is almost a linear relationship between $F_{fp}$ and $\rho$. Further, Fig. 13 suggests that, $F_{fp}$ seems to vary less at low modulation frequency.

In order to better explain the effect of $\rho$, the total contact force $F_p$ of Fig. 12 is replaced with a translational spring ($k_x$) and damper ($c_x$). In addition, a fictitious torsional spring ($K_\theta$) and a fictitious damper ($C_\theta$) are connected at the center of the body to ensure some coupling between the $x$ and $\theta$ motions. These models are illustrated in Fig. 14 for cases A and B. The governing equations for case A are obtained as

$$m \ddot{x} + c_x \dot{x} - c_x (0.5l - \rho) \dot{\theta} + k_x x - k_x (0.5l - \rho) \theta + F_{fg} \text{sgn}(\dot{x} - 0.5l \dot{\theta}) = F^k_N + F^T_N,$$

by comparing Eqs. (25) and (26) with the nonlinear models (as given by Eqs. (1) and (2)), observe that Fig. 12 represents essentially the same physics. Thus, $c_x$, $C_\theta$, $k_x$ and $K_\theta$ are written as

$$c_x = c^1_x + c^1_T,$$

$$C_\theta = c^1_x ((l^1_x)^2 - (\rho - 0.5l)^2) + c^1_T ((l^1_T)^2 - (\rho - 0.5l)^2).$$

Fig. 14. Two degree of freedom model of the translating friction element problem with translational and torsional elastic and dissipative elements. This model is mathematically the same as the models of Fig. 3: (a) free body diagram for cases from A1 to A4; (b) free body diagram for cases from B1 to B4.
\begin{align*}
k^x &= k^x_L + k^x_T, \quad (27c) \\
K^\theta &= k^\theta_L((l^L)^2 - (\rho - 0.5l)^2) + k^\theta_T((l^T)^2 - (\rho - 0.5l)^2). \quad (27d)
\end{align*}

Equations (27a) and (27c) show that \( c^x \) and \( k^x \) essentially combine the damping and stiffness coefficients, respectively, corresponding to the leading and trailing edge contacts of Fig. 3. This is an expected result since the springs and dampers at the leading and trailing contacts are in parallel. In addition, Eqs. (27b) and (27d) show that the fictitious torsional damping (\( C^\theta \)) and torsional stiffness (\( K^\theta \)) values are functions of \( \rho \). Therefore, the torsional characteristics of the system would change as \( \rho \) changes. For the sake of simplicity, assume \( k^L_I = k^T_I = k_I \), \( c^L_I = c^T_I = c_I \) and \( l^L_I = l^T_I = 0.5l \). These assumptions are reasonable, and they are already utilized for the nonlinear and linear models. Hence, Eqs. (27b) and (27d) simplify to
\begin{align*}
C^\theta &= 2c_I(1 - \rho), \quad (28a) \\
K^\theta &= 2k_I(1 - \rho). \quad (28b)
\end{align*}

As explained before, \( \rho \) is the distance between the center of contact forces and the edge where the frictional constraint locates; thus, \( (1 - \rho) \) is the distance from the center of contact forces to the other edge. Obviously, \( K^\theta \) is maximum when \( \rho = 0.5l \). Hence, as the system shows more resistance to rotation and normal force variations, \( F_{fpp} \) would decrease. As \( \rho \) moves further away from the geometric center of the body, \( K^\theta \) would decrease; thus, the system tends to rotate more, and \( F_{fpp} \) would increase. The validity of these simplifications is evident from Figs. 6 and 13, and Table 4.

7. A simplified brake vibration (judder-like) model

A simplified brake judder model with motion excitation (from the geometric distortions of a rotor) is developed next. The two degree of freedom nonlinear model of the translating friction element (Fig. 3) is modified as follows: (i) apply the periodic displacement (\( \xi^L \) and \( \xi^T \)) and velocity (\( \dot{\xi}^L \) and \( \dot{\xi}^T \)) excitations to contact springs and dampers and (ii) replace the external loads \( F_{L}N \) and \( F_{T}N \) with elastic and dissipative elements that now represent the piston and hydraulic system.

Fig. 15. Two degree of freedom nonlinear brake judder-like model: (a) for the frictional guide at the trailing edge (cases from A1 to A4); (b) for the frictional guide at the leading edge (cases from B1 to B4).
stiffness and damping characteristics of a brake system as displayed in Fig. 15. The governing nonlinear equations for case A are now:

\[
m\ddot{\mathbf{x}} + (c^L + c^T + c^f + c^g + c^T_f - c^L_f - c^T_f - c^L_f) \dot{\mathbf{x}} + \left(k^L + k^T + k^f + k^g + k^T_f - k^L_f - k^T_f - k^L_f\right) \mathbf{x} + F_{pg} \text{sgn}(\dot{x} - 0.5\dot{\theta}),
\]

\[
= F_{pre}^L + F_{pre}^T + c^L_x z + c^T_x z + k^L z^2 + k^T z^2 \tag{29}
\]

\[
I_\alpha \dot{\theta} + (c^L \dot{\theta} - c^T \dot{\theta} + c^f \dot{\theta} - c^g \dot{\theta} + c^T_f \dot{\theta} - c^L_f \dot{\theta} + c^T_f \dot{\theta} - c^L_f \dot{\theta}) \dot{\theta} + \left(k^L \dot{\theta} - k^T \dot{\theta} + k^f \dot{\theta} - k^g \dot{\theta} + k^T_f \dot{\theta} - k^L_f \dot{\theta} - k^T_f \dot{\theta} - k^L_f \dot{\theta}\right) \dot{\theta} + F_{Tf} \omega \cdot 0.5F_{pg} \text{sgn}(\dot{x} - 0.5\dot{\theta}) = F_{pre}^L \omega - 5F_{pg} \text{sgn}(\dot{x} - 0.5\dot{\theta}) + c^L_x z^2 - c^T_x z^2 \tag{30}
\]

where \( c^L \), \( c^T \), \( k^L \) and \( k^T \) represent the dissipative and elastic elements of the piston–hydraulic system at the leading and trailing edges, respectively. External loads \( F_{pre}^L \) and \( F_{pre}^T \) in Eqs. (29) and (30) are the preloads applied to \( k^L \) and \( k^T \) to simulate the mean actuation pressure. Periodic displacement excitation \( z^L \) and \( z^T \) represent the geometric distortions of the brake rotor at the leading and trailing edges, respectively, and they are defined as

\[
z^L = \sum_{q=1}^{Q} \Xi_q \sin(\omega t - \alpha^L_q), \tag{31a}
\]

\[
z^T = \sum_{q=1}^{Q} \Xi_q \sin(\omega t + \alpha^T_q), \tag{31b}
\]

where \( \Xi \) and \( q \) are the amplitude of the surface distortion and speed order index, respectively; \( \omega \) (rad/s) is the rotor speed. Phases \( \alpha^L_q \) and \( \alpha^T_q \) of Eqs. (31a) and (31b) consist of two different phases: (1) phase between different orders \( q \); and (2) phase due to the kinematics between the leading and trailing edges. The latter of these two phasing relationships can be calculated as \( \dot{\theta} / \tau \) and \( \dot{\theta} / \tau \) for leading and trailing edges, respectively, where \( \tau \) is the radius of the brake rotor.

Unlike Eq. (3), the friction force \( F_I \) in Eq. (30) for the brake judder models is calculated as

\[
F_I = \mu_I \left[c^T_f (x + \frac{\dot{L} \dot{\theta}}{\pi} - \dot{z}^L) + c^L_f (x - \frac{\dot{L} \dot{\theta}}{\pi} - \dot{z}^T) + k^L_f (x + \frac{L \dot{\theta}}{\pi} + \dot{z}^L) + k^T_f (x - \frac{L \dot{\theta}}{\pi} - \dot{z}^T)\right], \tag{32}
\]

though Eq. (4) still holds for the calculation of \( F_{Tf} \).

This problem is numerically solved at the same excitation frequencies (\( \omega = 10\pi \) and \( \omega = 20\pi \text{ rad/s} \)) for all the cases of Fig. 2; the same approximation for the ‘sgn’ function is utilized. In addition, a first order \( (q=1) \) rotor profile is assumed for the sake of simplicity, hence phasing between \( z^L \) and \( z^T \) is only due to the kinematics. Simulations are carried out for two different sets of stiffness values for \( k^L \) and \( k^T \) \( (k^L = k^T = k^L_1 = k^T_1 \text{ and } 100k^L = 100k^T = k^L_2 = k^T_2) \) and the calculated \( F_{pp}^{L} = F_{pp}^{T} / F_{pp}^{A} \) results are compared in Table 5 for the wheel angular speeds of \( \omega = 10\pi \) and \( \omega = 20\pi \text{ rad/s} \), respectively.

Observe that Table 5 yield the same rank orders \( F_{pp}^{L} < F_{pp}^{A} < F_{pp}^{G} < F_{pp}^{A} \) for case A, and \( F_{pp}^{L} < F_{pp}^{T} < F_{pp}^{A} < F_{pp}^{A} \) for case B) for lower stiffness values \( (k^L = k^T = k^L_2 = k^T_2) \). This concludes that the locations of the external loads and frictional guides indeed affect the judder response of a brake system. Nevertheless, the variation in \( F_{pp} \) from case to case is not as prominent as the translating friction element solutions. In order to enhance such variations, the values of two key parameters \( (k^L \text{ and } k^T) \) should be increased since their locations are case-specific. Now larger differences are obtained in Table 5 for the higher stiffness case \( (k^L = k^T = k^L_1 = k^T_1) \) though the rank order for case A is slightly changed. Note that the difference is only between cases A1 and A2 where the external load locations are at the edges and center of the pad with a narrow gap, respectively.

**Table 5**

<table>
<thead>
<tr>
<th>Normal load locations</th>
<th>Peak-to-peak friction forces ( F_{pp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Friction guide at trailing edge (A)</td>
</tr>
<tr>
<td></td>
<td>100( k^L ) = 100( k^T )</td>
</tr>
<tr>
<td></td>
<td>( k^L = k^T )</td>
</tr>
<tr>
<td>5 Hz</td>
<td>1.000</td>
</tr>
<tr>
<td>10 Hz</td>
<td>1.003</td>
</tr>
<tr>
<td></td>
<td>1.020</td>
</tr>
<tr>
<td></td>
<td>0.985</td>
</tr>
</tbody>
</table>

Forces predicted by the nonlinear brake judder-like system model at wheel speeds of 5 Hz and 10 Hz. All forces are normalized by using the force prediction of case A1.
Recall from Section 6 where it is observed that the torsional stiffness of the translating friction element system should be higher for case A1 due to the center of contact location; hence, the system should have more resistance to rotation compared to case A2. The same result is obtained for the current brake judder problem, i.e. \( \theta_{A1} < \theta_{A2} \); thus, the torsional stiffness of case A1 is still higher than A2. Therefore, the reason for having \( \mathbf{F}_{ip}^{A2} < \mathbf{F}_{ip}^{A1} \) is a result of the phasing difference between the displacement excitation and motion response of the system. For case A1, the locations of the elastic and dissipative elements (\( c_1^*, c_1^*, k_1^*, k_1^* \)) that are excited with displacement excitations are the same as the elastic and dissipative elements (\( c_1, c_1, k_1, k_1 \)) that support the pad. Hence, the motion response of the pad is always in phase with the displacement excitations. However, there is a phase delay between the excitation and response for case A2, and this lowers the value of \( \mathbf{F}_{ip} \). As another proof, it is seen that the rank order for case B for the brake judder problem is exactly the same as found in the translating friction element. This again suggests that changes in the external load or frictional guide location for both brake judder and translating friction element models are similar.

8. Conclusions

The chief contribution of this article is the proposed translating friction element problem (with 8 cases of different loading and constraint conditions) that conceptualizes the dynamic interactions between the brake pad and the caliper. Two degree of freedom nonlinear model successfully predicts trends and the simplified single degree of freedom linear model, yields analytical tractable solutions. Further, an analogous laboratory experiment is successfully constructed, and mathematical models are validated with measurements. Results show that friction force amplitude increases as the center of contact force location moves away from the supported edge of the friction material. Further, the effect of center of contact force location on the torsional characteristics of the system is shown with two degree of freedom models. As the center of contact forces approach the geometric center of the friction material, the effective torsional stiffness of the system increases. This creates more resistance to the rotational motion of the friction material, and consequently, the amplitudes of normal friction force oscillations decrease. Finally the center of contact concept is extended to a brake judder-like problem via two degree of freedom nonlinear models with displacement excitations (due to rotor surface imperfections), and the same rank order and boundary conditions are obtained. Unlike the existing literature that recommends a decrease in the stiffness of the system (in the normal direction of the rotor surface) as a means of suppressing judder vibration amplitudes, the current article successfully links the torsional stiffness of the system to the center of contact force location. For instance, the torsional stiffness of the system could be increased without changing the stiffness in normal direction by keeping the center of contact force location in the vicinity of the geometric center of the pad. Experiments on a real-life vehicle brake system are still required to interpret and extend the results of this article.

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Appendix A. Equations for case B (translating friction element with supported at the leading edge)

Governing equations of the two degree of freedom nonlinear mathematical model as given with Fig. 3(b) for case B are obtained as follows. Note that, Eqs. (3) and (4) still apply for the solution of case B.

\[
\begin{align*}
mx + (c_1^* + c_1^*)x + (c_1^* l^5_t - c_1^* l^5_t)\dot{\theta} + (k_1^* + k_1^*)x + (k_1^* l^5_t - k_1^* l^5_t)\dot{\theta} &= F_{lg} \text{sgn}(x + 0.5l_0) = F_N + F_{F}^N, \\
I_{\theta} \ddot{\theta} + (c_1^* l^5_t - c_1^* l^5_{\theta})x + (c_1^* l^5_t)^2 + c_1^* (l^5_{\theta})^2 \dot{\theta} + (k_1^* l^5_t - k_1^* l^5_t)\dot{x} &= (c_1^*(0.5l_0 - l^5_\theta + \mu_l w))w + (c_1^*(0.5l_0 - l^5_\theta + \mu_l w))\dot{w} + (k_1^*(0.5l_0 - l^5_\theta + \mu_l w) + (k_1^*(0.5l_0 - l^5_\theta + \mu_l w)))\dot{\theta} = F_{N}^x(0.5l_0 - l^5_\theta) + F_{N}^w(0.5l_0 + l^5_\theta). 
\end{align*}
\]

The governing equation of the single degree of freedom linear model for case B is derived as follows:

\[
I_{\theta} \ddot{\varphi} + ((c_1^*(0.5l_0 - l^5_\theta)(0.5l_0 - l^5_\theta + \mu_l w)) + (c_1^*(0.5l_0 - l^5_\theta)(0.5l_0 - l^5_\theta + \mu_l w)))\dot{\varphi} + (k_1^*(0.5l_0 - l^5_\theta)(0.5l_0 - l^5_\theta + \mu_l w) + (k_1^*(0.5l_0 - l^5_\theta)(0.5l_0 - l^5_\theta + \mu_l w)))\dot{\varphi} = F_{N}^x(0.5l_0 - l^5_\theta) + F_{N}^w(0.5l_0 + l^5_\theta). 
\]

Case specific expressions of case B for the analytical solution of the single degree of freedom linear model are given below.

\[
F_{Nm} = F_{N}^m(0.5l_0 - l^5_\theta) + F_{N}^w(0.5l_0 + l^5_\theta), 
\]

(A.4a)
Expressions derived for the calculation of $\rho$ for case B are given below. Again, $\rho$ is derived for both positive and negative slip velocity at the supported (leading) edge.

$$\rho^+ = (1 + \mu \mu_g) \left[ \frac{l}{2} - \frac{\mu_W}{1 + \mu \mu_g} \frac{F_{f \rho} - F_{f \rho}^* F_{f \rho}}{F_N^* + F_N} \right] \text{ for } \dot{x} + 0.5 \dot{\theta} > 0.$$  \hspace{1cm} (A.5a)

$$\rho^- = (1 - \mu \mu_g) \left[ \frac{l}{2} - \frac{\mu_W}{1 - \mu \mu_g} \frac{F_{f \rho} - F_{f \rho}^* F_{f \rho}}{F_N^* + F_N} \right] \text{ for } \dot{x} + 0.5 \dot{\theta} < 0.$$  \hspace{1cm} (A.5b)

Governing equations for the two degree of freedom nonlinear model of Fig. 12(b) are given below for case B. Again, Eqs. (A.6) and (A.7) also represent the same physics with Eqs. (A.1) and (A.2).

$$m \ddot{x} + c x - c'(\rho - 0.5) \dot{x} - k x - k'(\rho - 0.5) \dot{x} + F_{f \rho} \text{sgn}(x + 0.5 \dot{\theta}) = F_{f \rho}^* + F_{f \rho}^*,$$  \hspace{1cm} (A.6)

$$l_0 \ddot{\theta} - c(\rho - 0.5)l_0 \dot{x} + (c_2(\rho - 0.5)^2) + c(\rho - 0.5) l_0 \dot{x} - k(\rho - 0.5) l_0 \dot{\theta} + (k(\rho - 0.5)^2 + k l_0) \dot{\theta} - F_{f \rho} \text{sgn}(x + 0.5 \dot{\theta}) = F_{f \rho}^* - F_{f \rho}^*.$$

Governing equations of the two degree of freedom nonlinear brake judder model of Fig. 13(b) are derived as below for case B.

$$m \ddot{x} + (c_l + c_l^T + c_l^T - c_l^T + c_l^T - c_l^T) \dot{x} + (c_l^T - c_l^T + c_l^T - c_l^T + c_l^T - c_l^T) \dot{x} + (k_l + k_l^T + k_l^T + k_l^T + k_l^T + k_l^T) \dot{x} + F_{f \rho} \text{sgn}(x + 0.5 \dot{\theta}),$$  \hspace{1cm} (A.8)

$$l_0 \ddot{\theta} + (c_l^T - c_l^T + c_l^T - c_l^T + c_l^T - c_l^T) \dot{x} + (c_l^T - c_l^T + c_l^T - c_l^T + c_l^T - c_l^T) \dot{x} + (k_l + k_l^T + k_l^T + k_l^T + k_l^T + k_l^T) \dot{x} + F_{f \rho} \text{sgn}(x + 0.5 \dot{\theta}),$$

$$-F_{f \rho} l_0 + 0.5 F_{f \rho} \text{sgn}(x + 0.5 \dot{\theta}) = F_{f \rho}^* - F_{f \rho}^* + c_l^T - c_l^T + c_l^T - c_l^T + c_l^T - c_l^T,$$  \hspace{1cm} (A.9)

## References


