Modal analysis of thin cylindrical shells with cardboard liners and estimation of loss factors

Hasan Koruk\textsuperscript{a,b}, Jason T. Dreyer\textsuperscript{a}, Rajendra Singh\textsuperscript{a,*}

\textsuperscript{a}Acoustics and Dynamics Laboratory, Department of Mechanical and Aerospace Engineering, The Ohio State University, Columbus, OH 43210, USA
\textsuperscript{b}Istanbul Technical University, Mechanical Engineering Department, 34437 Istanbul, Turkey

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\textbf{A B S T R A C T}

Cardboard liners are often installed within automotive drive shafts to reduce radiated noise over a certain frequency range. However, the precise mechanisms that yield noise attenuation are not well understood. To overcome this void, a thin shell (under free boundaries) with different cardboard liner thicknesses is examined using analytical, computational and experimental methods. First, an experimental procedure is introduced to determine the modal behavior of a cylindrical shell with a cardboard liner. Then, acoustic and vibration frequency response functions are measured in acoustic free field, and natural frequencies and the loss factors of structures are determined. The adverse effects caused by closely spaced modes during the identification of modal loss factors are minimized, and variations in measured natural frequencies and loss factors are explored. Material properties of a cardboard liner are also determined using an elastic plate treated with a thin liner. Finally, the natural frequencies and modal loss factors of a cylindrical shell with cardboard liners are estimated using analytical and computational methods, and the sources of damping mechanisms are identified. The proposed procedure can be effectively used to model a damped cylindrical shell (with a cardboard liner) to predict its vibro-acoustic response.

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1. Introduction

Some vehicles include cardboard liners within the drive shafts to control the structureborne and radiated noise over the mid-frequency range. Although there are numerous articles on homogenous and undamped cylindrical shells [1–4], no prior study has analytically examined the vibro-acoustics of cylindrical shells with cardboard liners. Although a few accelerance frequency response functions (with and without cardboard liners) have been measured [5–8], the modal parameters of such structures have not been quantified in a systematic way. Some attempts have been made to simulate the cardboard liner as a distributed mass element in large scale computational models [9]. Overall, the effect of cardboard thickness has not been studied, and their damping mechanisms have not been explored in the literature [6–9]. This paper attempts to fill this void with a controlled experimental study and by analytically or computationally examining certain shell vibration modes and associated damping mechanisms.

\*Corresponding author. Tel.: +1 614 292 9044.
E-mail addresses: koruk@itu.edu.tr (H. Koruk), dreyer.24@osu.edu (J.T. Dreyer), singh.3@osu.edu (R. Singh).

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### Nomenclature

- $a$: acceleration  
- $a_{mn}^{Bij}$: modal constant of $(m,n)$ mode for response point $i$ and excitation point $j$  
- $D$: bending stiffness of the shell  
- $d$: outer diameter of cylinder  
- $E$: Young's modulus  
- $E^*$: complex Young's modulus  
- $F$: excitation force  
- $h$: thickness of the cylinder  
- $h_1$: thickness of base shell  
- $h_2$: thickness of cardboard liner  
- $I$: second moment of inertia  
- $j$: square root of $-1$  
- $k$: structural wave number  
- $k_{n1}k_{n2}$: wave numbers in circumferential and axial directions  
- $K$: global stiffness matrix of a cylindrical shell with a cardboard liner  
- $K_c$: stiffness matrix of a composite element  
- $K_{c1}$: stiffness matrix of layer $l$ of a composite element computed as if the layer is on the neutral plane  
- $K_{c2}$: stiffness matrix of layer $l$ expressed with respect to the neutral plane of a composite element  
- $l$: length of cylinder  
- $m$: axial modal index  
- $n$: circumferential modal index  
- $m,n$: indices for shell modes  
- $M$: global mass matrix of a cylindrical shell with a cardboard liner  
- $M_c$: mass matrix of a composite element  
- $M_{c1}$: mass matrix of layer $l$ of a composite element computed as if the layer is on the neutral plane  
- $M_{c2}$: mass matrix of layer $l$ expressed with respect to the neutral plane of a composite element  
- $p$: sound pressure  
- $s,q$: indices for a mode shape of a plate (the numbers of half-waves in a mode shape along the long and short edges)  
- $R$: radius of the middle surface of a cylindrical shell  
- $T$: transformation matrix  
- $m,n^{eij}$: residual term of $(m,n)$ mode for response point $i$ and excitation point $j$  
- $\tilde{R}_{ij}$: compliance function for response point $i$ and excitation point $j$  
- $\tilde{R}^{(w)}_{ij}$: difference between the actual compliance and the value of the compliance at the fixed frequency $\omega$  
- $\omega$: fixed frequency near natural frequency  
- $Z$: inverse compliance parameter  
- $\eta$: modal loss factor  
- $\eta_{mn}$: modal loss factor for $(m,n)$ mode  
- $\eta_r$: modal loss factor of $r$th mode  
- $\eta_M$: loss factor due to material damping  
- $\eta^Q$: loss factor due to dry friction  
- $\eta^*_{M}^{T}$: total (or effective) loss factor  
- $\mu_a$: average loss factor  
- $\mu$: Poisson's ratio  
- $\rho$: mass density  
- $\sigma_0$: standard deviation of the loss factor  
- $\sigma$: standard deviation of the loss factor  
- $\phi$: circumferential modal indices  
- $\phi_r$: circumferential direction  
- $\mu_e$: neutral plane of an individual layer of a composite element  
- $\omega_{mn}$: natural frequency of shell mode $(m,n)$  
- $\omega$: natural frequency of $r$th mode  
- $\phi_a,\phi_b$: two angles around natural frequency used in circle-fit method  

### Subscripts

- $1$: index for base tube  
- $2$: index for cardboard liner  
- $c$: composite element  
- $i$: index integer  
- $ij$: response and excitation point pair  
- $l$: layer index  
- $m,n$: indices for shell modes  
- $n$: circumferential modal index  
- $r$: mode number  

### Superscripts

- $e$: neutral plane of an individual layer of a composite element  
- $o$: neutral plane of a composite element  
- $M$: material  
- $Q$: frictional  
- $S$: total (or effective)  
- $T$: transpose of a matrix  
- $\sim$: complex valued  

### Operators

- $| |$: determinant

### Abbreviations

- DOFs: degrees of freedom  
- FE(M): finite element (method)

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2. Problem formulation

Fig. 1 illustrates the example case of a cylindrical shell with a cardboard liner; sensors used for the experimental study are also shown. The boundary conditions at \( z = 0 \) and \( l \) are assumed to be free. The following approximate solutions for the natural frequencies of a homogenous, thin cylindrical shell are first utilized to design the experiment and select the nominal dimensions such as the length \( l \), thickness \( h \) and radius \( R \). The natural frequency, \( \omega_{mn} \) (rad/s) of a homogeneous cylindrical shell for mode \((m,n)\) is approximated from the following frequency equation that is based on the classical Love’s equation [1] and an infinite shell model [2]:

\[
K \left( k_{zm}^2 + \frac{1}{h^2} k_{zm}^2 \right) - \rho h \omega_{mn}^2 \frac{k_{zm} \left( k_{zm} + \frac{D}{K} k_{zm} \right)}{k_{zm} \left( k_{zm} + \frac{D}{K} k_{zm} \right)} = 0
\]

Here, \( m \) and \( n \) are axial and circumferential modal indices, respectively; \( R \) is the radius of the middle surface and \( h \) is the thickness of a cylindrical shell; \( k_{zm} = n/R \) is the wave number in the circumferential direction, \( k_{zm} \) is the modal number in the axial direction and \( k_{zm} = \sqrt{k_{zm}^2 + k_{zm}^2} \) is the structural wave number; \( K = E \mu/(1-\mu) \) is the membrane stiffness of the shell, and \( D = Eh^3/[12(1-\mu^2)] \) is the bending stiffness of the shell where \( \mu \) is Poisson’s ratio, \( E \) is Young’s modulus and \( \rho \) is the mass density. The modal wave number in the axial \((z)\) direction under free-free boundary conditions is given using the beam functions \( k_{zm} = (m-0.5)\pi/l \) for \( m \geq 2 \). Note that the beam functions cannot provide a solution for \( m = 0 \) and \( m = 1 \). Accordingly, analytical expressions for \( \omega_{0n} \) and \( \omega_{1n} \) (in rad/s) for \( m = 0 \) and \( m = 1 \) modes are determined using

\[
\omega_{0n}^2 = \frac{E h^2}{12R^4} \frac{n^2(n^2-1)^2}{n^2 + 1} \\
\omega_{1n}^2 = \omega_{0n}^2 \frac{n^2 + 24(1-\mu)R^2}{12R^2 + n^2(n^2 + 1)}
\]

Both acceleration to force \( \tilde{A}_j(\omega) = \hat{a}_j(\omega) / \hat{F}_j(\omega) \) and acoustic pressure to force \( \tilde{G}_j(\omega) = \hat{p}_j(\omega) / \hat{F}_j(\omega) \) transfer functions are measured on shell samples with different cardboard liners in acoustic free field where “a” is the acceleration, “p” is the sound pressure and “F” is the excitation force; \( i \) and \( j \) are the response and excitation point indices, respectively; \( \omega \) is the excitation frequency; and, superscript \( \sim \) implies a complex valued quantity. Preliminary experiments performed on shells with typical cardboard liners showed that it is difficult to determine their modal parameters (especially the modal loss factors) when the structure is highly damped. Accordingly, several cardboard samples of increasing thickness \((h_2)\) are prepared. Furthermore, results with a different thickness \((h_2)\) should provide a better understanding of the modes and sources of damping. Specifically, the shell thickness \((h_1)\), outer diameter \((d)\) and length \((l)\) are determined to be 2.1, 152.4 and 457.2 mm, respectively, while three cardboard liners are prepared with \( h_2 = 0.31, 0.62 \) and 0.98 mm. The outer diameter of the cardboard is equal to the inner diameter of the base (aluminum) cylinder, and thus an interference fit describes the assembled structure. The mass density \((\rho)\) of the aluminum tube is 2704 kg/m\(^3\) while the cardboard density is determined to be 649 kg/m\(^3\). Young’s modulus \((E)\) and Poisson’s ratio \((\mu)\) of the base aluminum cylinder are 71.0 GPa and 0.33, respectively. The experiments include the same cardboard material and the same base shell (and thus the surface properties do not vary as a function of liner thickness). Therefore, the results are expected to be independent of surface properties.

Overall, the main objectives of this study are as follows: (i) design a controlled vibro-acoustic experiment to measure \( \tilde{A}_j(\omega) \) and \( \tilde{G}_j(\omega) \) functions on a cylindrical shell with several cardboard liners; (ii) determine the first ten natural frequencies and modes of a cylindrical shell without and with the cardboard liners (up to 1950 Hz), and evaluate the accuracy of the measured natural frequencies; (iii) extract the loss factors of the first five modes of a cylindrical shell without and with the cardboard liners using several frequency response based methods and a number of measured \( \tilde{A}_j(\omega) \) functions, and explore the uncertainty levels in the measured loss factors; (iv) determine material properties of a cardboard liner using an elastic plate treated with a thin liner; (v) estimate the loss factors of a cylindrical shell with the cardboard liners using an analytical
approach; (vi) estimate the natural frequencies and modal loss factors of those structures using a computational method; and (vii) identify possible sources of damping.

3. Vibro-acoustic experiments

Before conducting detailed $\tilde{A}_{ij}(\omega)$ and $\tilde{G}_{ij}(\omega)$ measurements, the appropriate locations of force excitation, accelerometer and microphone, as well as the shell suspension method [10,11], are identified using a computational (finite element) model [12] of an untreated cylindrical shell. For this, an analysis of the base structure is conducted to determine modal deflections for each mode. It is noted that the degrees of freedom (DOF) with lower displacement response levels (suggesting negligible interactions between structure and suspension cord) should provide good suspension locations while the DOFs with higher velocity response levels (which may induce double hit impacts) as well as the nodal points must be avoided when determining suitable impulse hammer excitation locations. The acceleration and sound pressure measurements are simulated via the highest acceleration and surface velocity levels, respectively, over the frequency range of interest. Overall, the best positions are determined by averaging the corresponding parameters for the modes of interest [10,11]. Sample results for measurement locations considering the first 12 modes of a cylinder are displayed in Fig. 2. The results for accelerometer and microphone simulations are very similar, as expected.

Preliminary $\tilde{A}_{ij}(\omega)$ and $\tilde{G}_{ij}(\omega)$ measurements are conducted to select the best signal processing parameters. For instance, $\tilde{A}_{ij}(\omega)$ measurements are obtained with a frequency resolution of 0.078, 0.125 and 0.250 Hz, while $\tilde{G}_{ij}(\omega)$ measurements are performed with 0.063 and 0.125 Hz resolution. Measured data with a finer frequency resolution is used to identify the damping at the lower modes. The autospectra of the force signal are examined to ensure that the structure is properly excited over the frequency range of interest. Each power spectrum is found using five averages.

Since the mass loading by an accelerometer would shift the frequencies, the natural frequencies of the shell structure are determined by the acoustic functions $\tilde{G}_{ij}(\omega)$. Nevertheless, the modal loss factors are estimated using structural accelerances $\tilde{A}_{ij}(\omega)$; half-power, line-fit and circle-fit methods [10,13,14] are utilized. Also, a very fine frequency resolution is employed for better accuracy and for a clear recognition of closely spaced modes. The measurement period is selected to be sufficiently long to ensure that the signal approaches zero at the end of its minimal period. Consequently, there is no need to apply a particular windowing function; hence, there should be minimal uncertainty in the identified modal loss factors. Also, experiments are performed only under free–free boundary conditions to eliminate any uncertainty due to boundary damping and stiffness. Overall, the first ten shell modes of structures are successfully identified.

The assembled structure consists of the main shell and a cardboard liner. A nonlinear interface with a dry friction element is expected between the base tube and cardboard liner. Therefore, the variations in the measured $\tilde{A}_{ij}(\omega)$ and $\tilde{G}_{ij}(\omega)$ should be identified before determining the modal parameters from the measured $\tilde{A}_{ij}(\omega)$ and $\tilde{G}_{ij}(\omega)$ database. For this purpose, some precautions are taken. First, in order to verify the repeatability of the measurement system, including the behavior of the assembly, three $\tilde{A}_{ij}(\omega)$ functions were measured on the treated cylindrical shell with $h_2=0.62$ mm for the same response and excitation points; these are overlaid in Fig. 3. Here, the response and excitation locations are on the same line on the outer surface of the cylinder, i.e., $i$ is given by $r=d/2$, $\theta=0$, $z=200$ mm and $j$ is given by $r=d/2$, $\theta=0$, $z=150$ mm. It is seen that the $\tilde{A}_{ij}(\omega)$ functions are nearly the same even at the anti-resonances, though there are some deviations beyond...
1950 Hz. Also, the reciprocity experiment is performed, i.e., the measurements are repeated by interchanging the excitation \((i)\) and response \((j)\) coordinates as given above. The corresponding \(\tilde{A}_{ij}(\omega)\) functions measured on the treated cylindrical shell (with \(h_2 = 0.62\) mm) are compared in Fig. 4. Observe that the \(\tilde{A}_{ij}(\omega)\) functions are nearly the same below 1200 Hz though there are negligible amplitude deviations between 1200 Hz and 1950 Hz. However, higher deviations are seen beyond 1950 Hz. Such variations at higher frequencies as determined via repeatability and reciprocity experiments are attributed to the nonlinear behavior of the assembly of the base tube and cardboard liner due to friction, the inability to properly excite the structure, and the effect of damping. It is noted that higher increased damping is expected at higher frequencies due to the enhanced effects of friction and radiation damping. Therefore, the lower modes are more important from the vehicle vibration and sound perspective. Accordingly, in this paper, the natural frequencies of the first ten modes (below 1950 Hz) and the damping levels of only the first five modes are identified. Overall, it is seen that \(\tilde{A}_{ij}(\omega)\) and \(\tilde{G}_{ij}(\omega)\) measurements are very repeatable, and the coherence function is at least 0.999 at and around the natural frequencies. It is noted that the frequency domain based methods utilize data at and around natural frequencies to identify modal parameters.

4. Extraction of modal parameters

Since it is virtually impossible to have a perfectly symmetrical shell, closely spaced peaks are found at each mode in measured accelerances \(\tilde{A}_{ij}(\omega)\) as previously mentioned. Also, some resonant peaks may contain more than one mode.
The adverse effects of such closely spaced modes encountered during the identification process are minimized via the circle-fit [13] and line-fit [14] type methods that take into account both accelerance $A_{ij}(\omega)$ amplitude and phase data when distinct modes are well excited; these methods are used in addition to the conventional half-power method [10].

**Fig. 5.** Effect of cardboard liner thickness ($h_2$) on measured accelerances. Key: $•••• h_2=0$, $•••• h_2=0.31$ mm, $••• h_2=0.62$ mm and $••• h_2=0.98$ mm.

**Fig. 6.** Effect of cardboard liner thickness ($h_2$) on measured acoustic frequency response functions. Key: $•••• h_2=0$ mm, $•••• h_2=0.31$ mm, $••• h_2=0.62$ mm and $••• h_2=0.98$ mm.

**Fig. 7.** Effect of cardboard liner thickness ($h_2$) on measured accelerances showing (0,2) and (1,2) type mode pairs. Key: $•••• h_2=0$ mm, $•••• h_2=0.31$ mm, $••• h_2=0.62$ mm and $••• h_2=0.98$ mm.

The adverse effects of such closely spaced modes encountered during the identification process are minimized via the circle-fit [13] and line-fit [14] type methods that take into account both accelerance $A_{ij}(\omega)$ amplitude and phase data when distinct modes are well excited; these methods are used in addition to the conventional half-power method [10].

In the half-power method [10], the loss factor ($\eta_{mn}$) for mode $(m,n)$ is determined by the following:

$$\eta_{mn} = 2\zeta_{mn} = \frac{\omega_{mn,2}^2 - \omega_{mn,1}^2}{2\omega_{mn}^2}$$

where $\zeta_{mn}$ is the corresponding viscous damping ratio and $(\omega_{mn,1}, \omega_{mn,2})$ are the frequencies corresponding to half power points around $\omega_{mn}$. In the circle-fit method [13], the modal loss factor is determined by the following:

$$\eta_{mn} = \frac{\omega_{mn,b}^2 - \omega_{mn,a}^2}{\omega_{mn}^2 (\tan(\varphi_{mn,a}/2) + \tan(\varphi_{mn,b}/2))}$$

where two frequencies $(\omega_{mn,a}, \omega_{mn,b})$ correspond to the angles $(\varphi_{mn,a}, \varphi_{mn,b})$ around $\omega_{mn}$ when the dynamic compliance function is plotted using the Nyquist diagram [10]. Note that the expression in Eq. (5) is equal to the expression in Eq. (4) when the angles $(\varphi_{mn,a}, \varphi_{mn,b})$ are selected as $\pi/2$. The line-fit method [14] is shown to be quite effective for the determination of modal loss factors [15]. In this method, the compliance function $\tilde{R}_{ij}$ for response point $i$ and excitation point $j$ can be written as follows:

$$\tilde{R}_{ij}(\omega) \approx \omega_{mn} \approx \eta_{mn} = \frac{\omega_{mn,b}^2 - \omega_{mn,a}^2}{\omega_{mn}^2 (\tan(\varphi_{mn,a}/2) + \tan(\varphi_{mn,b}/2))} + mn\varepsilon_{ij}$$

where $mnB_{ij}$ is the modal constant and $mn\varepsilon_{ij}$ is the residual term for the mode $(m,n)$ corresponding to $i$ and $j$. A new form of compliance function, $\tilde{R}_{ij}(\omega)$, is the difference between the actual compliance function and the value of the compliance function at one fixed frequency, $\omega$, in the range of interest is defined as follows:

$$\tilde{R}_{ij}(\omega) = \tilde{R}_{ij}(\omega) - \tilde{R}_{ij}(\omega)$$

(7)

to cancel the residual term in the compliance function. After that, an inverse parameter is defined using $\tilde{R}_{ij}(\omega)$ for modal analysis [10]; it is given in Eq. (8). The expression in Eq. (8) can also be expressed as in Eq. (9):

$$Z(\omega) = \frac{\omega^2}{\omega_{mn}}$$

(8)

$$Z(\omega) = (\omega_{mn}^2 - \omega^2 + \varepsilon_{ij})$$

(9)

The inverse parameter in Eq. (9) is separated into real and imaginary parts which are supposed to fit onto a line when plotted against $\omega^2$. The modal parameters are then estimated by determining the best lines that provide the best fits for the measured data. See Refs. [10,13,15] for more details.

5. Comparison of natural frequencies

The natural frequencies of the untreated shell are first calculated using the approximate analytical formulas given by Eqs. (1)–(3). Then, the homogeneous shell elements (such as S4R in [16] and 3SHL04 in [17]) are utilized for computational (finite element) modal analysis. The mesh convergence is checked and the finite element models of the cylindrical shell are developed using 2500 (3SHL04) and 8554 (S4R) elements.

Before presenting the natural frequencies of cylinders, variations in natural frequencies need to be addressed. When the measured acoustic $G_{ij}(\omega)$ functions are analyzed for each test sample, small variations in natural frequencies are found, though such variations increase as the cardboard thickness $h_2$ increases. For example, the natural frequencies for the degenerate mode pairs of the first mode of the treated sample with $h_2 = 0.31$ mm are 243.16, 243.22, 243.12, 243.05, 243.14 and 243.25 Hz for the first peak and 244.15, 244.25, 244.48, 244.50, 244.54 and 244.52 Hz for the second peak when six measured $G_{ij}(\omega)$ spectra for various excitation and response points are analyzed. Similarly, the natural frequencies are between 238.52 and 238.74 Hz for the first peak and from 239.75 to 240.12 Hz for the second peak for the degenerate mode pairs of the first mode of the treated shell with $h_2 = 0.62$ mm. Overall, when all $G_{ij}(\omega)$ functions are analyzed for each mode of each test sample, it is seen that the variations in the natural frequencies are less than $\pm 0.03\%$, $\pm 0.05\%$, $\pm 0.10\%$ and $\pm 0.25\%$ with $h_2 = 0, 0.31, 0.62$ and 0.98 mm, respectively. Such variations are mainly due to finite frequency resolution, the effect of excitation position, the nonlinear effect of the cardboard liner and dry friction. However, they are deemed negligible.

Table 1 compares the experimental, analytical and computational natural frequencies of a homogeneous cylindrical shell. The degenerate mode pairs are also identified for both untreated and treated samples with $h_2 = 0.31$ and 0.62 mm liners, though the degenerate modes disappear when a thicker ($h_2 = 0.98$ mm) liner is employed due to higher damping. Observe that the error is less than 1% for the computational method, but higher errors (say up to 6%) are seen for the analytical method. The averaged errors are about 0.4 and 4.4% for the computational and approximate analytical methods, respectively. Error in the analytical method is due to the simplifying assumptions made in the shell vibration formulation [2], especially for the $(0,n)$ and $(1,n)$ modes.
Table 1
Measured and calculated natural frequencies of a homogeneous thin cylinder with free boundaries.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experimental</th>
<th>Approximate analytical solution</th>
<th>Finite element model (FEM)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>FEM-1 [16]</td>
</tr>
<tr>
<td>Index (m,n)</td>
<td>$\omega_{mn}$ (Hz)</td>
<td>$\omega_{ave}$ (Hz)</td>
<td>$\omega_{ave}$ (Hz)</td>
</tr>
<tr>
<td>1</td>
<td>(0.2)</td>
<td>247.0, 248.9</td>
<td>247.7</td>
</tr>
<tr>
<td>2</td>
<td>(1.2)</td>
<td>258.6, 259.9</td>
<td>259.3</td>
</tr>
<tr>
<td>3</td>
<td>(0.3)</td>
<td>699.4, 701.9</td>
<td>700.6</td>
</tr>
<tr>
<td>4</td>
<td>(1.3)</td>
<td>714.3, 716.0</td>
<td>715.1</td>
</tr>
<tr>
<td>5</td>
<td>(2.3)</td>
<td>1023.4, 1025.8</td>
<td>1024.6</td>
</tr>
<tr>
<td>6</td>
<td>(2.2)</td>
<td>1339.3, 1343.8</td>
<td>1341.5</td>
</tr>
<tr>
<td>7</td>
<td>(0.4)</td>
<td>1353.9, 1358.3</td>
<td>1356.1</td>
</tr>
<tr>
<td>8</td>
<td>(1.4)</td>
<td>1365.6, 1372.1</td>
<td>1368.8</td>
</tr>
<tr>
<td>9</td>
<td>(2.4)</td>
<td>1472.4, 1476.0</td>
<td>1474.2</td>
</tr>
<tr>
<td>10</td>
<td>(3.4)</td>
<td>1834.6, 1841.8</td>
<td>1838.2</td>
</tr>
</tbody>
</table>

The measured natural frequencies of the treated shells (with different $h_2$ values) are given in Table 2. Also, selected mode shapes of the cylindrical shells are displayed in Fig. 8. It is noted that the mode shapes of the cylindrical shell with a cardboard liner are expected to be the same as that of an untreated structure. Observe that the natural frequencies of a treated shell decrease at all modes as $h_2$ is increased. The overall reductions are about 1.6%, 3.2% and 4.2% for the treated cylinders with $h_2=0.31, 0.62$ and 0.98 mm.

6. Extraction of modal loss factors

Although most sources of uncertainties are minimized as much as possible using well-established experimental methodology, damping levels still contain some uncertainty mainly due to the nonlinear nature of damping, the degenerate mode pairs of cylindrical shells and the dry friction between the main shell and a cardboard liner. Therefore, the issue about uncertainty in the loss factor should be addressed first. For this purpose, a number of $A_{ij}(\omega)$ functions are measured, and the degenerate loss factors are determined for well-excited modes. For example, the loss factors for the first mode of the treated sample with $h_2=0.62$ mm are determined to be $0.00654, 0.00768, 0.00607, 0.00600$ and $0.00626$ using the circle-fit, $0.00586, 0.00593, 0.00569, 0.00579$ and $0.00565$ using the line-fit and $0.00632, 0.00659, 0.00622, 0.00619$ and $0.00644$ using the half-power when five accelerance $A_{ij}(\omega)$ functions are analyzed. Similarly, the loss factors for the second mode of this structure are determined to be $0.00696, 0.00740, 0.00697, 0.00740$ and $0.00653$ using the line-fit, $0.00737, 0.00795, 0.00712, 0.00730$ and $0.00653$ using the half-power. Typical loss factor ranges, averages ($\mu_\eta$) and standard deviations ($\sigma_\eta$) of the treated cylindrical shell with $h_2=0.62$ mm are listed in Table 3. It is seen that three methods predict similar loss factors and their standard deviations are about 10% or less. This value is quite reasonable considering the nature of damping and closely spaced modes. Also, the averaging process reduces random errors, and thus the results are believed to be quite accurate.

The averaged modal loss factors ($\eta_{mn}$) of untreated and treated shells are compared in Table 4. It is seen that the average loss factor of the aluminum cylinder is about 0.0009; it is comparable to values found in the literature. The modal loss factors of the treated cylindrical shells increase as cardboard thickness ($h_2$) increases as expected. The loss factors are about...
0.004, 0.007 and 0.012 with $h^2 = 0.31, 0.62$ and 0.98 mm, respectively. Note that the degenerate modes at a higher frequency with $h^2 = 0.98$ mm could not be separated, and thus no results are reported for such modes.

7. Determination of material properties of cardboard liners

The material properties of the cardboard liner used in this paper are determined using a lightly damped elastic plate treated with a cardboard liner. The dimensions of the base plate are selected so that this plate would have no symmetrical modes, and the modes would also not be closely spaced. Two steel plates (300 mm × 201 mm × 1 mm) are treated with cardboard liners with $h^2 = 0.31$ and 0.62 mm. The density of the base steel plate is 7664 kg/m³ while Poisson’s ratio is 0.3. The cardboard is glued to the base plate using an adhesive whose elastic modulus is much lower than the liner. Furthermore, the thickness of the adhesive is negligible as the thickness of the treated plate is nearly equal to the thickness of the base plate plus the thickness of cardboard. Therefore, the effect of the adhesive is assumed to be negligible. Also, each liner with different thickness used here can be considered as a single layer which is nearly isotropic although cardboard materials are expected to be anisotropic [18].

A preliminary test on the plates showed that the accelerometer provided additional damping. Therefore, modal parameters of the plates are again determined using acoustic $C_{ij}(\omega)$ functions measured in the acoustic free field and half-power method. Both untreated and treated plates are modeled using homogeneous and composite finite element (FE) formulations, which have already been found to predict the modal parameters of structures with high accuracy [17]; this FE formulation will be explained in Section 8.2. Overall, the plates are modeled using a quite fine mesh (2400 elements).

First, Young’s modulus and loss factor of the untreated plate (steel) are determined by comparing experimental and theoretical modal parameters. Then, the material properties of the cardboard liner are extracted by comparing experimental and theoretical modal parameters of the treated plates and by the material properties of the base steel plate.

![Fig. 8. First nine mode shapes of a thin cylindrical shell with free boundaries.](image)

Table 3

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<thead>
<tr>
<th>Mode</th>
<th>Circle-fit</th>
<th>Line-fit</th>
<th>Half-power</th>
<th>$\eta$ (%)</th>
<th>$\sigma$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>$(m,n)$</td>
<td>$\eta_{mn}$ (%)</td>
<td>$\eta_{mn}$ (%)</td>
<td>$\eta_{mn}$ (%)</td>
<td>$\eta_{mn}$ (%)</td>
</tr>
<tr>
<td>1</td>
<td>(0,2)</td>
<td>0.600–0.768</td>
<td>0.565–0.743</td>
<td>0.619–0.762</td>
<td>0.651</td>
</tr>
<tr>
<td>2</td>
<td>(1,2)</td>
<td>0.673–0.740</td>
<td>0.599–0.710</td>
<td>0.653–0.795</td>
<td>0.693</td>
</tr>
<tr>
<td>3</td>
<td>(0,3)</td>
<td>0.647–0.705</td>
<td>0.830–0.705</td>
<td>0.647–0.717</td>
<td>0.668</td>
</tr>
<tr>
<td>4</td>
<td>(1,3)</td>
<td>0.651–0.789</td>
<td>0.624–0.783</td>
<td>0.687–0.759</td>
<td>0.716</td>
</tr>
<tr>
<td>5</td>
<td>(2,3)</td>
<td>0.577–0.656</td>
<td>0.554–0.711</td>
<td>0.568–0.711</td>
<td>0.638</td>
</tr>
</tbody>
</table>

0.004, 0.007 and 0.012 with $h^2 = 0.31, 0.62$ and 0.98 mm, respectively. Note that the degenerate modes at a higher frequency with $h^2 = 0.98$ mm could not be separated, and thus no results are reported for such modes.
are determined to be as follows: the two-layer system can be written as follows:

\[ I = \text{the two-layer system} \]

where \( EcIc \) is the second moment of area, \( EcIc \) is the loss factor of a two-layer structure and \( h \) refers to the base structure, coated (cardboard) layer and composite structure, respectively. It should be stated that as the damping of the base structure could be significant, and consequently it should be taken into account, the base structure damping is usually ignored in the base structure and the cardboard layer. It is seen that the natural frequencies match well as the average differences between the experimental and theoretical natural frequencies are 0.53, 0.55 and 0.88% for plates with 

8.1. Analytical approach

The Ross–Kerwin–Ungar (RKU) equations [19] are used to predict the equivalent elastic and damping properties of a composite structure assuming that the cardboard liner acts as a free-layer damping treatment. The flexural rigidity, \( EIc \), of the two-layer system can be written as follows:

\[ \frac{EIc}{EI1} = 1 + \frac{Eh^3}{1 + Eh} + 3(1 + \frac{Eh}{1 + Eh})^2 \]  \( \text{(10)} \)

where \( I \) is the second moment of area, \( E = E2/E1 \) and \( h = h2/h1 \), and subscripts 1, 2 and c refer to the base structure, coated (cardboard) layer and composite structure, respectively. It should be stated that as the damping of the base structure could be significant, and consequently it should be taken into account, the base structure damping is usually ignored in the literature [19]. When the loss factors of the base structure and the cardboard layer are \( \eta_1 \) and \( \eta_2 \), respectively, rewrite Eq. (10) as follows:

\[ \frac{EIc(1+jh2c)}{EI1(1+jh1c)} = 1 + \frac{Eh^3}{1 + Eh} (1 + jh2) + 3(1 + \frac{Eh}{1 + Eh})^2 \left( \frac{Eh(1+jh2)}{1+Eh(1+jh2)} \right) \]  \( \text{(11)} \)

where \( \eta_c \) is the loss factor of a two-layer structure and \( j = \sqrt{-1} \). Here, it is assumed that the cardboard liner follows the elastic deformation of the base tube and the interfacial damping mechanism is negligible. As a result, the averaged loss factors of the treated cylindrical shells with \( h2 = 0.31, 0.62 \) and 0.98 mm are found to be 0.0023, 0.0041 and 0.0068, respectively. These values are, however, smaller than the measured results of Section 6.

8.2. Computational approach

A recently developed finite element formulation for composite structures [17] is used to model a cylindrical shell with a cardboard liner. This formulation is based on a complex eigenvalue method, and it represents multilayer composite structures with an equivalent layer. In this formulation, the damping capability is added to the element by defining the complex Young’s modulus of the material as \( E = E(1+j\eta) \) where \( E \) and \( \eta \) are the storage Young’s modulus and the loss factor, respectively. The stiffness and mass matrices for individual layers of the composite element are expressed with respect to

### Table 4

Effect of cardboard liner thickness on measured loss factors. See Tables 1 and 2 for natural frequencies.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( h_2 ) (mm)</th>
<th>( h_2=0 ) mm</th>
<th>( h_2=0.31 ) mm</th>
<th>( h_2=0.62 ) mm</th>
<th>( h_2=0.98 ) mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.2)</td>
<td>0.130</td>
<td>0.514</td>
<td>0.651</td>
<td>1.192</td>
</tr>
<tr>
<td>2</td>
<td>(1.2)</td>
<td>0.114</td>
<td>0.434</td>
<td>0.693</td>
<td>1.093</td>
</tr>
<tr>
<td>3</td>
<td>(0.3)</td>
<td>0.068</td>
<td>0.364</td>
<td>0.668</td>
<td>1.252</td>
</tr>
<tr>
<td>4</td>
<td>(1.3)</td>
<td>0.055</td>
<td>0.392</td>
<td>0.716</td>
<td>a</td>
</tr>
<tr>
<td>5</td>
<td>(2.3)</td>
<td>0.073</td>
<td>0.357</td>
<td>0.638</td>
<td>a</td>
</tr>
</tbody>
</table>

\( \eta \) could not be identified.

the neutral plane of the composite element are defined as follows:

\[
\begin{align*}
K_l^c &= \begin{bmatrix}
T & 0 & 0 & 0 & T & 0 \\
0 & T & 0 & 0 & T & 0 \\
0 & 0 & T & 0 & T & 0 \\
\end{bmatrix}^T \\
M_l^c &= \begin{bmatrix}
T & 0 & 0 & 0 & T & 0 \\
0 & T & 0 & 0 & T & 0 \\
0 & 0 & T & 0 & T & 0 \\
\end{bmatrix}^T \\
\end{align*}
\]

where \( K_l^c \) and \( M_l^c \) are the stiffness and mass matrices of each layer of the composite element which are computed as if each layer is on the neutral plane, \( T \) is a transformation matrix of dimension \( 6 \times 6 \), \( 0 \) is a \( 6 \times 6 \) null matrix, and subscript \( l \) refers to the individual layer of a composite element [17]. Although any shell element can be used to develop the composite element, the specific shell element utilized in this formulation includes the “physical drilling degrees of freedom” in the normal direction to the element [17]. The stiffness and mass matrices for the composite element are obtained by adding the contributions of individual layers via a summation process as follows:

\[
\begin{align*}
K_c &= \sum_{l=1}^{12} K_l^c, \\
M_c &= \sum_{l=1}^{12} M_l^c,
\end{align*}
\]

where the subscripts \( c \) refers to a composite shell. It should be noted that the summation above represents the matrix building process rather than a simple matrix summation. After the elemental stiffness and mass matrices are obtained with respect to the neutral plane of the composite element, these matrices are transformed to a common global coordinate system and the natural frequencies and modal damping values for a damped system can be obtained by solving the conventional eigenvalue problem:

\[
(K - \lambda^2 M)\Psi = 0
\]

where \( K \) and \( M \) are the global stiffness and mass matrices of a cylindrical shell with the cardboard liners, respectively, \( \lambda^2 \) is the eigenvalue and \( \Psi \) is the eigenvector. The solution of the eigenvalue problem above results in complex eigenvalues (\( \lambda^2 \)) and complex eigenvectors (\( \Psi \)). The natural frequencies (\( \omega_r \)) and loss factors (\( \eta_r \)) are determined by defining \( \lambda^2 = \omega_r^2(1 + j\eta_r) \) and by the following:

\[
\omega_r^2 = \text{Re}(\lambda^2)
\]
\[
\eta_r = \text{Im}(\lambda_r^2)/\text{Re}(\lambda_r^2)
\]  

(18)

As a result, the loss factors of the treated cylindrical shells are determined using the FE formulation presented here, and the results are listed in Table 6. This method yields averaged loss factors to be about 0.0021, 0.0037 and 0.0059% for \( h_2 = 0.31, 0.62 \) mm and 0.98 mm, respectively. Observe that these values are close to the results predicted via the analytical method.

The predicted natural frequencies and corresponding errors are given in Table 7a. It is seen that the predicted natural frequencies are higher than the measured ones. This is expected because the layers are assumed to be fully joined in the computational method. In Section 6, it was shown that the natural frequencies decrease as \( h_2 \) increases. It is clear that the cardboard liner has mostly the mass effect. Therefore, the cardboard liner is modeled as a distributed mass and its stiffness effect is ignored. The natural frequencies of the treated cylinders for this case are given in Table 7b. It is seen that the predicted natural frequencies are very close to the measured ones, i.e., the error is less than 1% even for \( h_2 = 0.98 \) mm.

Although the strain field within the cardboard liner would yield some damping, experimental results suggest that the elastic effect of the liner can be ignored when calculating the natural frequencies of a cylinder with a liner. The effects of surface properties of the test samples, frictional sliding between the base shell and cardboard liner, etc. is the subject of a future work.

9. Identification of damping mechanisms

The averaged loss factors (for the first 5 modes) of a cylindrical shell with \( h_2 = 0.31, 0.62 \) and 0.98 mm are determined to be \( \eta_r = 0.0023, 0.0041 \) and 0.0068 via the analytical approach and \( \eta_r = 0.0021, 0.0037 \) and 0.0059 via the computational

Table 6

Predicted modal loss factors of the treated cylinders using a computational method.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( h_2=0.31 ) mm</th>
<th>( h_2=0.62 ) mm</th>
<th>( h_2=0.98 ) mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>((m,n))</td>
<td>( \eta_{mn} ) (%)</td>
<td>( \eta_{mn} ) (%)</td>
</tr>
<tr>
<td>1</td>
<td>(0,2)</td>
<td>0.219</td>
<td>0.385</td>
</tr>
<tr>
<td>2</td>
<td>(1,2)</td>
<td>0.219</td>
<td>0.385</td>
</tr>
<tr>
<td>3</td>
<td>(0,3)</td>
<td>0.218</td>
<td>0.383</td>
</tr>
<tr>
<td>4</td>
<td>(1,3)</td>
<td>0.218</td>
<td>0.383</td>
</tr>
<tr>
<td>5</td>
<td>(2,3)</td>
<td>0.183</td>
<td>0.294</td>
</tr>
</tbody>
</table>

Table 7

Predicted natural frequencies of the treated cylinders determined via a computational method where the elastic effect of the liner is included in (a) and ignored in (b).

<table>
<thead>
<tr>
<th>Mode</th>
<th>( h_2=0.31 ) mm</th>
<th>( h_2=0.62 ) mm</th>
<th>( h_2=0.98 ) mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>((m,n))</td>
<td>( \omega_{mn} ) (Hz)</td>
<td>Error (%)</td>
</tr>
<tr>
<td>(a) Elastic effect of the liner is included</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(0,2)</td>
<td>249.3</td>
<td>2.24</td>
</tr>
<tr>
<td>2</td>
<td>(1,2)</td>
<td>259.5</td>
<td>1.77</td>
</tr>
<tr>
<td>3</td>
<td>(0,3)</td>
<td>705.3</td>
<td>2.28</td>
</tr>
<tr>
<td>4</td>
<td>(1,3)</td>
<td>718.6</td>
<td>2.13</td>
</tr>
<tr>
<td>5</td>
<td>(2,3)</td>
<td>1012.5</td>
<td>0.35</td>
</tr>
<tr>
<td>6</td>
<td>(0,4)</td>
<td>1311.6</td>
<td>0.88</td>
</tr>
<tr>
<td>7</td>
<td>(1,4)</td>
<td>1352.8</td>
<td>1.40</td>
</tr>
<tr>
<td>8</td>
<td>(2,4)</td>
<td>1366.7</td>
<td>1.33</td>
</tr>
<tr>
<td>9</td>
<td>(3,4)</td>
<td>1478.6</td>
<td>1.86</td>
</tr>
<tr>
<td>10</td>
<td>(4,4)</td>
<td>1830.3</td>
<td>1.10</td>
</tr>
<tr>
<td>Average absolute error (%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.53</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| (b) Elastic effect of the liner is ignored | | | | | | | |
| 1    | (0,2) | 244.7 | 0.34 | 240.7 | 0.56 | 236.3 | 0.00 |
| 2    | (1,2) | 254.7 | -0.12 | 250.5 | 0.14 | 246.0 | -0.34 |
| 3    | (0,3) | 692.3 | 0.39 | 681.1 | 0.62 | 668.8 | -0.60 |
| 4    | (1,3) | 705.4 | 0.25 | 693.9 | 0.43 | 681.3 | -0.75 |
| 5    | (2,3) | 999.6 | -0.93 | 983.3 | -1.01 | 965.4 | -1.87 |
| 6    | (0,4) | 1324.2 | 0.32 | 1302.4 | 0.29 | 1278.4 | -0.65 |
| 7    | (1,4) | 1328.1 | -0.46 | 1306.5 | -0.57 | 1282.9 | -1.43 |
| 8    | (2,4) | 1341.7 | -0.52 | 1319.9 | -0.80 | 1296.0 | -1.37 |
| 9    | (3,4) | 1453.0 | 0.09 | 1429.3 | 0.44 | 1403.4 | -0.75 |
| 10   | (4,4) | 1804.3 | -0.33 | 1774.9 | -0.27 | 1742.5 | 1.13 |
| Average absolute error (%) | | | | | | | |
| 0.37 | | | | | | | |

method, respectively. The corresponding measured loss factors are 0.0041, 0.0067 and 0.0118 for the cylindrical shell with \( h_2 = 0.31, 0.62 \) and 0.98, respectively. Since both analytical and computational methods consider only the material damping, the difference between measured (real) and computational or analytical results (based on simplifying formulation) could be attributed to the dry friction damping. The loss factors due to friction should increase as the cardboard thickness (and thus the normal load) is increased. Indeed, the values of frictional (the difference between experimental and computational) loss factors are 0.0200, 0.0031 and 0.0059 for the treated cylinders with \( h_2 = 0.31, 0.62 \) and 0.98 mm, respectively.

The total (equivalent) loss factor (\( \eta^p \)) of a composite cylindrical shell with a cardboard liner can be approximated by \( \eta^p = \eta^M + \eta^Q \) where \( \eta^M \) and \( \eta^Q \) are the loss factors due to material damping and friction. The \( \eta^M \) can be predicted using analytical or computational methods once the material properties of a cardboard liner are known. Interestingly, the results of this paper show that there is a relationship between frictional and material (or total) loss factors. The ratios between the \( \eta^Q \) and \( \eta^p \) are 0.49, 0.46 and 0.50 for the treated cylinders with \( h_2 = 0.31, 0.62 \) and 0.98 mm, respectively. In fact, the following relationship \( \eta^Q = c_0 \eta^M \) is seen where \( c_0 \approx 0.85 \) – 1.00 for the cases presented here. Therefore, the effective damping of a cylindrical shell with a cardboard liner can be simply approximated by \( \eta^p = c_1 \eta^M \) where \( c_1 \approx 1.85 \) – 2.00 for the cases presented here.

10. Conclusion

This paper investigates the effect of cardboard liners on the resonant vibro-acoustic responses of cylindrical shells. Specific contributions of this paper include the following. A controlled experiment is first designed to measure acoustic and vibration frequency response functions with different cardboard thicknesses in acoustic free field. By analyzing the experimental data, the difficulties in the identification of modal parameters of cylindrical shells with cardboard liners are explored, the natural frequencies and loss factors of structures are determined using several frequency domain methods and measured data, and the variations of natural frequencies and uncertainty levels in loss factors are explored. The material properties of a cardboard liner are also determined by utilizing an elastic plate treated with a cardboard liner. The modal loss factors of a cylindrical shell with cardboard liners are then estimated using analytical and computational approaches while the natural frequencies of cylinders are predicted using a computational method. Finally, potential sources of damping mechanisms are identified by comparing experimental, analytical and computational damping levels. This suggests that the dry friction contributes to about half of the overall damping.

The results show that an equivalent loss factor for a cylindrical shell with a cardboard liner can be calculated. Once the equivalent values of loss factor and flexural rigidity are available, analytical methods for a homogeneous shell can be used to predict the modal behavior of a cylindrical shell with a cardboard liner. Similarly, computational methods like the one given in Section 8.2 can be used to predict the vibro-acoustic behavior of a cylindrical shell with a cardboard liner. Subsequently, one could calculate the modal radiation efficiency of a cylindrical shell using the methodology of this paper. The effect of surface properties and the role of sliding friction between base shell and cardboard liner should be pursued in future.

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References


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