Estimation of coefficient of friction for a mechanical system with combined rolling–sliding contact using vibration measurements

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ABSTRACT

A new dynamic experiment is proposed to estimate the coefficient of friction for a mechanical system with a combined rolling–sliding contact under a mixed lubrication regime. The experiment is designed and instrumented based on an analogous contact mechanics model, taking into consideration the constraints to ensure no impact and no sliding velocity reversal. The system consists of a cam (rotating with a constant speed) having a point contact with a follower that oscillates about a frictionless pivot, while maintaining contact with the cam with the help of a well-designed translational spring. The viscous damping elements for contact are identified for two different lubricants from an impulse test using the half-power bandwidth method. Dynamic responses (with the cam providing an input to the system) are measured in terms of the follower acceleration and the reaction forces at the follower pivot. A frequency domain based signal processing technique is proposed to estimate the coefficient of friction using the complex-valued Fourier amplitudes of the measured forces and acceleration. The coefficient of friction is estimated for the mechanical system with different surface roughnesses using two lubricants; these are also compared with similar values for both dry and lubricated cases as reported in the literature. An empirical relationship for the coefficient of friction is suggested based on a prior model under a mixed lubrication regime. Possible sources of errors in the estimation procedure are identified and quantified.

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1. Introduction

Friction plays a significant role in the dynamics of mechanical systems under sliding contacts [1–7]. The friction force is often modeled using the Coulomb formulation, though the analyst must judiciously select the value of the coefficient of friction (μ). In many prior experimental studies as summarized by Persson [8], μ is found from a simple and pure translational sliding contact (without rolling) system. For instance, Espinosa et al. [9] used a modified Kolsky bar apparatus, while Hoskins et al. [10] used a sliding block of rocks to estimate the normal and friction forces. Furthermore, translational sliding experiments were employed by Worden et al. [11] to estimate the dependence of friction forces on displacement and velocity, and then by Schwingshackl et al. [12] to model the non-linear friction interface. Several investigators have also
conducted friction experiments on rotating systems such as a pin-disk apparatus [13,14], two rotating circular plates [15], and a radially loaded disk-roller system [16,17]. Also, Kang and Kim [18] determined the Coulomb friction insight stabilization equipment using torque and angular displacement characteristics, while Povey and Paniagua [19] estimated the bearing friction for a turbo machinery application. Such pure sliding contact experiments cannot be employed to estimate $\mu$ for a system with combined rolling–sliding contact since the kinematics is different. Radzimovsky et al. [20] conducted experiments on gears to determine the instantaneous $\mu$ over a mesh cycle. However, none of the prior combined rolling–sliding contact experiments rely on vibration measurements. Furthermore, the conventional or direct approaches given in the literature [9–19] focus on estimating the $\mu$ for only a pure sliding contact systems (without rolling). Since the kinematics of such systems is generally complicated, new effort is needed to estimate the $\mu$ from vibration measurements (measured forces and acceleration) under certain conditions.

Some researchers have experimentally studied cam–follower mechanisms [21,22] from the stability and bifurcation perspective under impacting conditions. In contrast, a cam–follower mechanism with rotational sliding contact (with no impacts) is used to experimentally determine $\mu$ in this study. Since $\mu$ cannot be directly measured from vibration experiments, an analogous contact mechanics model [23] is developed to aid the process. The goal is to vary the surface roughness, lubrication film thickness, contact pressure and velocities at contact (sliding and entrainment). The proposed system could then be utilized to simulate the contact conditions seen in drum brakes and geared systems.

2. Problem formulation

Fig. 1 shows the mechanical system with an elliptic cam (with semi-major and minor axes as $a$ and $b$, respectively). Though the kinematics of combined rolling–sliding contact systems are complex compared to systems with pure sliding contact, this is one of the simplest systems which one can devise to measure the coefficient of friction in such systems which would allow controlled measurements of the reaction forces and system acceleration. The cam is pivoted at $E$ along its major axis with a radial run out, $e$, from its centroid ($G_c$, with subscript $c$ denoting cam). The angle made by the end point of the major axis ($A$) with the horizontal axis ($\hat{e}_x$) is $\Theta(t)$, which is an excitation to the system (where $t$ represents the time). The equation of the elliptic cam is given by the following, where $r$ is the radial distance from $G_c$ to any point on the circumference of the cam, and $\Delta$ is the polar angle of that point,

$$ r(\Delta) = \frac{ab}{\sqrt{[a \sin(\Delta)]^2 + [b \cos(\Delta)]^2}}. \quad (1) $$

The cam is in a point contact (at $O_c$) with the follower (at $O_b$, with subscript $b$ denoting follower), which consists of a thin cylindrical dowel pin (of radius $r_d$) attached to a bar (of length $l_b$) of square cross-section (of width $w_b$). The center of gravity of the follower lies at $G_b$ at a distance of $l_g$ from the pivot point $P$ (using roller bearings) which is at $d_y$ distance above the ground. The follower is supported by a linear spring ($k_s$) along the vertical direction ($\hat{e}_y$), which is at a distance of $d_x$ from $P$ as shown in Fig. 1. The angular motion of the follower is given by $\alpha(t)$ in the clockwise direction from the $\hat{e}_x$ axis; it is also the only dynamic degree-of-freedom of the system. The contact mechanics at $O$ between the cam and the follower is represented by non-linear contact stiffness ($k_\lambda$) and viscous damping ($c_\lambda$) elements. Viscous damping is valid in this study since the system is designed to not lose the contact at any point of time, and the indentation velocity is low (no impacts), and hence the contact damping force is insignificant (compared to contact stiffness force) regardless the damping

![Fig. 1. Example case: a mechanical system with an elliptic cam and follower supported by a lumped spring ($k_s$).](image-url)
mechanisms involved. Damping forces and mechanisms become significant only when the system undergoes impacts, hence the viscous damping assumption is valid for this system under non-impact conditions. The choice of contact damping mechanism will not affect the results of the study due to insignificance of the contact damping force. Hence a simple viscous damping model is chosen over a non-linear amplitude dependent damping model. A coordinate system \((\hat{i}, \hat{j})\) attached to the follower is defined with its origin at \(Q\) where \(\hat{i}\) is orthogonal to the follower. The angle subtended by \(e_\text{C}\) from \(\hat{e}_x\) is given by \(\varphi(t)\), which is used in the following equation to calculate the \(\Delta_\text{C}(t)\) for the contact point \(O_\text{C}\) as

\[
\Delta_\text{C}(t) = \text{mod}(\varphi(t) - \Theta(t), 2\pi).
\]

Here, “mod” is the modulus function defined as \(\text{mod}(x, y) = x - y \cdot \text{floor}(x/y)\), if \(y \neq 0\). The vector \(\vec{O}_\text{C}Q\) is represented in the \((\hat{i}, \hat{j})\) coordinate system by \(\nu_j = \nu_j + \nu_j^\text{C}\). When the instantaneous value of \(\varphi(t)\) is negative, that would ensure that the cam and the follower are in contact.

The scope of the current study is restricted to an estimation of \(\mu\) under a mixed lubrication regime. The key assumptions in the proposed system are as follows: (i) The bearings at the follower pivot are frictionless and rigid; (ii) the surfaces of the cam and follower have no other irregularities with the exception of random surface roughness; (iii) the sliding friction between cam and the follower can be described by the Coulomb friction model; (iv) the contact force is represented by the Hertzian point contact model [23], since it is widely used for point contact in machine elements; and (v) the bending moment of the follower is negligible. The Coulomb friction model is used in the study only to model the variation in \(\mu\) with the relative velocity like, Benedict and Kelley model [17]. With the variation in friction with relative velocity being low in the system under study, it can be assumed to be almost constant. Furthermore, to capture the variation in the amplitude of \(\mu\) with surface roughness, contact pressure and dynamic viscosity, an empirical model based on the Benedict and Kelley model [17] will be used. Hence this also takes care of applicability of \(\mu\) under boundary and mixed lubrication conditions. The specific objectives of this study are (1) Develop a contact mechanics model for a mechanical system with a combined rolling-sliding contact to design a suitable experiment and to predict the dynamic response; (2) Design a controlled laboratory experiment for the cam–follower system to measure dynamic forces and acceleration; (3) Propose a signal processing technique to estimate \(\mu\) using the Fourier amplitudes of measurements and obtain an empirical formula for \(\mu\); and (4) Identify the potential sources of errors in the proposed technique.

### 3. Contact mechanics model

The 0-state of the system (represented by superscript 0) is defined as the state when \(\vec{O}_\text{C}Q = \vec{0}\) and the major axis is parallel to the follower \((\alpha^0 = -\Theta^0)\). In the 0-state \(Q^0\), \(O_\text{C}^0\) and \(O_\text{C}\) are coincident. From the geometry of the system, \(\alpha\) and the magnitude of \(\vec{P}E\) along \(\hat{j}(\chi)\) are calculated in the 0-state as

\[
\alpha^0 = \cos^{-1} \left( \frac{\sqrt{|\vec{P}E|_x^2 + |\vec{P}E|_y^2} - |\vec{P}E|_x (0.5w_b + 2r_d + b)^2 + |\vec{P}E|_y (0.5w_b + 2r_d + b) \cos \phi}{|\vec{P}E|_x^2 + |\vec{P}E|_y^2} \right),
\]

\[
\chi^0 = |\vec{P}E|_x \cos (\alpha^0) - |\vec{P}E|_y \sin (\alpha^0) - e.
\]

Here, \(|\vec{P}E|_x\) and \(|\vec{P}E|_y\) represent the magnitudes of \(\vec{P}E\) along \(\hat{e}_x\) and \(\hat{e}_y\), respectively. The instantaneous values of the moving coordinates \(\psi_i(t)\) and \(\psi_j(t)\) are determined from \(\alpha(t)\), \(\Theta(t)\) and the system geometry using the following vector equation:

\[
\vec{P}E = \vec{P}O_b + \vec{O}_bO_\text{C} + \vec{O}_\text{C}E.
\]

Employing the vector polygon procedure discussed by Sundar et al. [24], the equations for \(\psi_i(t)\) and \(\psi_j(t)\) are obtained as

\[
\psi_i(t) = (\chi^0 + e) \sin \left( \alpha(t) - \alpha^0 \right) + (r(\Delta_\alpha(t)) + 0.5w_b + 2r_d) \cos \left( \alpha(t) - \alpha^0 \right)
+ (\Delta_\alpha(t)) \sin \left( \alpha(t) + \varphi(t) \right) - \sin \left( \alpha(t) + \Theta(t) - 0.5w_b + 2r_d \right).
\]

\[
\psi_j(t) = (\chi^0 + e) \cos \left( \alpha(t) - \alpha^0 \right) + (r(\Delta_\alpha(t)) - 0.5w_b + 2r_d) \sin \left( \alpha(t) - \alpha^0 \right)
- r(\Delta_\alpha(t)) \cos \left( \alpha(t) + \Theta(t) \right) + e \cos \left( \alpha(t) + \Theta(t) \right).
\]

Differentiating Eqs. (6) and (7) with respect to time, \(\dot{\psi}_i(t)\) and \(\dot{\psi}_j(t)\) are obtained as follows:

\[
\dot{\psi}_i(t) = (\chi^0 + e) \cos \left( \alpha(t) - \alpha^0 \right) \dot{\alpha}(t) - (r(\Delta_\alpha(t)) + 0.5w_b + 2r_d) \sin \left( \alpha(t) - \alpha^0 \right) \dot{\alpha}(t)
+ r(\Delta_\alpha(t)) \cos \left( \alpha(t) + \varphi(t) \right) \dot{\alpha}(t) + \dot{\varphi}(t) + r(\Delta_\alpha(t)) \sin \left( \alpha(t) + \varphi(t) \right)
- e \cos \left( \alpha(t) + \Theta(t) \right) \dot{\alpha}(t) + \dot{\Theta}(t).
\]

\[
\dot{\psi}_j(t) = (\chi^0 + e) \sin \left( \alpha(t) - \alpha^0 \right) \dot{\alpha}(t) + (r(\Delta_\alpha(t)) + 0.5w_b + 2r_d) \cos \left( \alpha(t) - \alpha^0 \right) \dot{\alpha}(t)
+ r(\Delta_\alpha(t)) \sin \left( \varphi(t) + \Theta(t) \right) \dot{\varphi}(t) + \dot{\alpha}(t) - r(\Delta_\alpha(t)) \cos \left( \varphi(t) + \alpha(t) \right) - \sin \left( \alpha(t) + \Theta(t) \right) \dot{\alpha}(t) + \dot{\Theta}(t).
\]
The non-linear contact stiffness is defined for a point contact based on the Hertzian contact theory \[23\] as given by

\[
\chi = \frac{0.5ab\left(b^2 - a^2\right) \sin \left(2\Delta_0(t)\right)}{\left[(a \sin \left(\Delta_0(t)\right))^2 + (b \cos \left(\Delta_0(t)\right))^2\right]^{1.5}}
\]  

(10)

The angle \(\theta(t)\) corresponding to the contact point \(O_c\) is determined at every instant for a given \(\alpha(t)\) and \(\Theta(t)\) by locating the point on the elliptic profile of the cam which is tangential to the follower. Hence the slope of the follower, \(s_0'(t) = \tan(-\alpha(t))\), should be equal to the slope of the cam at \(O_c\) \((s_0(t))\) which is calculated as follows:

\[
s_0'(t) = \tan \left(\theta(t) + \tan^{-1}\left(-\frac{b^2}{a^2 \tan (\theta(t) - \Theta(t))}\right)\right)
\]  

(11)

Equating \(s_0'(t)\) and \(s_0(t)\) and rearranging, \(\alpha(t)\) is calculated by the following:

\[
\alpha(t) = \theta(t) - \tan^{-1}\left(\frac{b^2}{a^2 \tan (\theta(t) - \Theta(t))}\right)
\]  

(12)

The equation of motion of the follower when it is in contact with the cam is derived by balancing the moments (from Fig. 2) about \(P\) as

\[
I_P^b \ddot{\alpha}(t) = m_b g l_k \cos (\alpha(t)) - F_s(t) d_x + F_n(t) \chi(t) - \Theta(t)(0.5w_b + 2r_d).
\]  

(13)

Here, \(I_P^b\) is the moment of inertia of the follower about \(P\), \(m_b\) is the mass of the follower, \(g\) is the acceleration due to gravity, and \(\chi(t)\) is the moment arm of the contact force about the pivot \(P\). The elastic force from the spring, \(F_s(t)\), is given by the following, where \(L^b_s\) is the original length of the follower spring:

\[
F_s(t) = k_s \left[l^b_s - d_y + d_x \tan (\alpha(t)) + 0.5w_b \sec (\alpha(t))\right].
\]  

(14)

The normal force \((F_n(t))\) arising from the point contact with the cam is given by

\[
F_n(t) = -k_s (\dot{\gamma}(t)) \dot{\psi}_f(t) - c_2 \ddot{\psi}_f(t).
\]  

(15)

The non-linear contact stiffness is defined for a point contact based on the Hertzian contact theory \[23\] as

\[
k_s (\dot{\gamma}(t)) = \left(4/3\right) Y^e (\rho^e(t)) \psi_f(t))^{0.5}.
\]  

(16)

Here, \(Y\) is Young's modulus (with superscript \(e\) denoting equivalent) in accordance with the Hertzian contact theory given by the following, where \(\nu\) is Poisson's ratio,

\[
Y^e = \left[1 - \nu^2 + \frac{1 - \nu_b^2}{Y_b}\right]^{-1}
\]  

(17)

The equivalent radius of curvature at the contact \((\rho^e(t))\) and the radius of curvature of the elliptical cam at \(O_c\) \((\rho_c(\Delta_0(t)))\) are given by

\[
\rho^e(t) = \left[\rho_c(\Delta_0(t)) \right]^{-1} + (r_d)^{-1}
\]  

(18)

\[
\rho_c(\Delta_0(t)) = \left[\frac{a \sin (\gamma(\Delta_0(t)))^2 + b \cos (\gamma(\Delta_0(t)))^2}{ab}\right]^{1.5}
\]  

(19)

Fig. 2. Free-body diagram of the follower; refer to Fig. 1 for the two coordinate systems.
The contact damping is modeled as linear viscous damping as the system is designed to not undergo any impact. The viscous contact damping is given by the following expression, where $\zeta$ is the modal damping ratio which will be experimentally found under lubricated conditions (as explained later in section 4) and $\Theta$ is the linearized natural frequency of the system, and $\chi^*$ is static equilibrium value of $\chi(t)$ (discussed later in this section).

$$c_i = \frac{2\zeta\Theta r_i^0}{(\chi^*)^2}$$

(20)

The friction force is given as

$$F_f(t) = \mu F_n(t) \text{sgn}(v_r(t)).$$

(21)

The Coulomb friction is used to describe a variation in coefficient of friction with the relative velocity along the lines of the Benedict and Kelley model [17] and Stribeck formulation [32]. When the variation of $\mu$ with relative velocity is low, it can be assumed to be almost constant, which is found to be the case with the following system. Here the relative sliding velocity, $v_r(t)$, is given by

$$v_r(t) = \dot{\psi}_f(t) - \left[r(D(t))\sin(\varphi(t) + \alpha(t)) + e \sin(\alpha(t) + \Theta(t))\right]\left(\dot{\alpha}(t) + \dot{\Theta}(t)\right).$$

(22)

The static equilibrium point is used as the initial condition while numerically solving Eq. (13). In Eqs. (6), (7) and (13), $\alpha(t)$, $\psi_f(t)$, and $\psi_i(t)$ are replaced with their corresponding values at the static equilibrium point (with superscript $*$), and all time-derivative terms are set to zero and solved. Using the method of Jacobian matrix as discussed by Sundar et al. [24], $\Theta$ is then calculated at the static equilibrium point.

4. Experiment for the determination of $\mu$

Since the measured time domain signals are bound to have significant noise, a frequency domain based signal processing technique is preferred for the estimation of $\mu$. Accordingly, measured forces and acceleration must not be affected by discontinuities and system resonances. Design criteria for the experimental system can be given by the following. First, the follower must always be in contact with the cam, as a loss in contact would generate impulses in force and acceleration discontinuities and system resonances. Second, $v_r(t)$ must be in contact with the cam, as a loss in contact would generate impulses in force and acceleration discontinuities and system resonances. Third, the cam should rotate with a constant speed $\omega_c$ in order to accurately measure the spectral contents of forces and acceleration. Fourth, at least the first five harmonics of $\omega_c$ should lie in the stiffness controlled regime. Fifth, the experiment should permit a mixed lubrication regime. Finally, a variation in the slide-to-roll ratio should be possible in the experiment.

Fig. 3 shows the schematic of a cam–follower experiment having a hollow cylindrical cam of outer radius, $a$, driven by the output shaft of an electric motor. The radial runout between the center of rotation (axis of the shaft) and the centroid of the cam can be easily varied. A point contact is obtained, as the cam and the dowel pin have cylindrical surfaces with their axes oriented orthogonal to each other. The contact is continuously lubricated using either a heavy gear oil (AGMA 4EP) [25,26] or a light hydraulic oil (ISO 32) [25,26]. The follower is hinged at one of its ends with two frictionless rolling element bearings and is supported by a coil spring. A tri-axial force transducer (PCB 260A01 [27]) located at the follower hinge measures the reaction forces, $N_x(t)$ and $N_y(t)$, along $\xi$ and $\eta$, respectively. An accelerometer (PCB 356A15 [28,29]) located at the end of the follower measures its tangential acceleration. These are dynamic transducers with a very high frequency bandwidth [27,29]. Both force and acceleration signals are simultaneously sampled.

Fig. 3. Mechanical system experiment used to determine the coefficient of friction ($\mu$) at the cam–follower interface.
5. Identification of system parameters

5.1. Identification of geometrical parameters

The following parameters for the cam–follower system are carefully chosen to satisfy the design constraints stated in Section 4: \(m_b=0.21\) kg, \(a=17.5\) mm, \(b=2020\) kg mm\(^2\), \(l_b=179\) mm, \(l_s=89\) mm, \(w_0=12.7\) mm, \(r_d=3.2\) mm, \(k_s=2954\) N/m, \(l^*_s=57\) mm, \(d_s=40\) mm and \(d_r=61\) mm. The relative positions of the pivot points of the cam and the follower are given by \(PE = 86\) mm \(\hat{e}_x + 24\) mm \(\hat{e}_y\). The averaged surface roughness \(R_s\) and root-mean-square roughness \(R_{\text{rms}}\) of the cam and follower surfaces are measured using an optical profilometer. For the precision ground surfaces used in the experiment \(R_s=0.29\) \(\mu\)m and \(R_b=0.25\) \(\mu\)m, while for sand-blasted surfaces \(R_s=0.36\) \(\mu\)m and \(R_b=0.89\) \(\mu\)m. The key parameters that dictate a loss of contact between the follower and the cam and the sign reversal in \(v_i(t)\) are \(e\) and \(\Omega_e\). Inverse kinematics [30] is employed, as explained below, to predict a range of values for these two parameters over which the system neither has a loss of contact nor a sign reversal in \(v_i(t)\). For a given value of \(e\) and \(\Omega_e\), the angle of the follower (assuming it is just in contact) with the cam (\(\alpha^k(t)\)) is kinematically calculated for different values of \(\Theta(t)\) in the range \([0, 2\pi]\) (superscript \(k\) represents values calculated using the inverse kinematics).

By setting \(y_k=0\) in Eq. (6), \(\alpha^k(t)\) is calculated using the following equation:

\[
\begin{align*}
(\rho^0+e) \sin (\alpha^k(t)-\alpha^0) + (r(\Delta_o(t))+0.5w_b+2r_d) \cos (\alpha^k(t)-\alpha^0) + r^k(\Delta_o(t)) \sin \left(\alpha^k(t)+q^k(t)\right) - e \sin \left(\alpha^k(t)+\Theta(t)\right) - (0.5w_b+2r_d) &= 0.
\end{align*}
\]

Here, \(r^k(\Delta_o(t))\) is obtained using Eqs. (1) and (2) as

\[
r^k(\Delta_o(t)) = \frac{ab}{\sqrt{[a \sin (q^k(t)-\alpha^k(t))]^2 + [b \cos (q^k(t)-\alpha^k(t))]^2}}
\]

Eqs. (23) and (24) are solved along with Eq. (12) after replacing \(\alpha(t)\) with \(\alpha^k(t)\), to get \(r^k(\Delta_o(t)), \alpha^k(t)\) and \(q^k(t)\). Then, differentiating \(\alpha^k(t)\) with respect to \(t\), \(\alpha^k(t)\) and \(\Delta^k(t)\) are obtained. The normal force is estimated (as stated below) from the moment balance about \(P\) and by neglecting the moment due to \(F_f(t)\) in comparison with the moment due to \(F_n(t)\) because of system geometry.

\[
F^k_n(t) = \frac{\rho^0 \cos (\alpha^k(t)-\alpha^0) - (b+0.5w_b+2r_d) \sin (\alpha^k(t)-\alpha^0) + r^k(\Delta_o(t)) \cos (\alpha^k(t)-q^k(t))}{\rho^0 \cos (\alpha^k(t)-\alpha^0) + F^k_f(t) \Delta_x - m_b \dot{g}_b \cos (\alpha^k(t))}.
\]

Here, \(F^k_n(t)\) is calculated from Eq. (14) corresponding to \(\alpha^k(t)\). If the minimum value of \(F^k_n(t)\) calculated from Eq. (25) is negative, it would indicate that the follower would lose contact with the cam during the steady-state operation. Similarly, the relative velocity \(v^k_i(t)\) is kinematically calculated to check for any sign reversal from Eqs. (9) and (22) by replacing \(\alpha(t)\) and \(q(t)\) with \(\alpha^k(t)\) and \(q^k(t)\), respectively. The procedure mentioned above is repeated for different values of \(e\) and \(\Omega_e\) to calculate the \(\Omega_e\)–\(e/\alpha\) map as shown in Fig. 4; the regimes with and without loss of contact and reversal in the sliding velocity direction are clearly marked. All experiments are conducted in the \(e/\alpha\) range from 0.05 to 0.15, and \(\Omega_e\) is varied only between 10.1 Hz and 11.7 Hz; thus the system is well within the contact regime (as shown) with a constant sign \(\text{sign} (v_i(t)) = -1\). With these parameters, the linearized natural frequency of the system is found to be 1040 Hz for a steel cam and a steel follower \((E=Y_b=200\ GPa; \nu_c=\nu_b=0.3)\); thus the first five harmonics of \(\Omega_e\) lie in the stiffness controlled regime. Also, the lubrication regime is identified based on the “lambda ratio” \((\Lambda)\), which is the ratio

![Fig. 4. Classification of response regimes of the mechanical system with a circular cam in terms of \(\Omega_e\) vs. \(e/\alpha\) map with the parameters of Section 5.](image)

Key: Operational range of the experiment.
of minimum lubrication film thickness [31] to the composite surface roughness $R_{rms,c}^2 + R_{rms,b}^2$). With AGMA 4EP oil [25,26] (with dynamic viscosity, $\eta = 0.034$ kg m$^{-1}$ s$^{-1}$, pressure viscosity coefficient $= 20 \times 10^{-9}$ m$^2$/N at 60°C) 2 < $\Lambda$ < 5, the system should lie in the mixed lubrication regime. Also, lower values of $\Lambda$ are achieved (0.7–1.5) with ISO 32 oil [25,26] (with $\eta = 0.012$ kg m$^{-1}$ s$^{-1}$, pressure viscosity coefficient $= 18 \times 10^{-9}$ m$^2$/N at 60°C). Since the temperature at the contact is higher than the ambient (due to continuous sliding), it is assumed that the interfacial oil operates at 60°C. Furthermore, the slide-to-roll ratio which is given by the ratio of $v_r(t)$ (as given in Eq. (22)) and entrainment velocity ($\nu_e(t)$ as defined below), varies between 0.75 and 1.25 for $e/a=0.116$ and $\Omega_c=11.55$ Hz as shown in Fig. 5. Here,

$$v_e(t) = \psi(t) + [r(\Delta(t)) \sin (\phi(t)+\alpha(t))+e \sin (\alpha(t)+\Theta(t))](\alpha(t)+\Theta(t)).$$ (26)

The slide-to-roll ratio ($|v_r(t)/v_e(t)|$) could be easily changed by altering the geometry, such as $PE$, $a$, $b$ and $e$.

5.2. Identification of the modal damping ratio

The modal damping ratio under lubrication depends on the oil viscosity and the materials in contact; hence it is determined experimentally using the half-power bandwidth method with both lubricants. The experimental setup consists of two masses ($m_1 = 1.4$ kg and $m_2 = 1.8$ kg) connected by three identical point contacts which are lubricated as shown in Fig. 6. These point contacts are obtained by placing three dowel pins ($r_d = 3.2$ mm) attached to $m_1$ in one direction and two more dowel pins attached to $m_2$ in the orthogonal direction, as shown. The system is placed on a compliant base (foam), and two accelerometers are attached to each mass. An impulse excitation is imparted to the system in the vertical direction with

![Fig. 5.](image1.png) Slide-to-roll ratio for the cam–follower system with $e/a = 0.12$ and $\Omega_c = 11.55$ Hz and other parameters of Section 5.

![Fig. 6.](image2.png) Impulse experiment to determine the viscous damping ratio associated with the lubricated contact regime. (a) Experimental setup; (b) top view of the dowel pin arrangement showing the three point contacts. Key: $\bullet$, contact point.
an impact hammer. The response accelerance spectrum of each mass along the vertical direction is then found by averaging signals from two accelerometers. Impact tests are conducted with two lubricants. Fig. 7 shows the relative accelerance spectra (between $m_1$ and $m_2$), focusing on the system resonance ($\sim 1000$ Hz). As observed, there is a reduction in the amplitude and the natural frequency with lubrication. For a single point contact, the damping ratio ($\zeta$) with unlubricated, ISO 32 oil and AGMA 4EP oil conditions are found to be 1.8%, 1.9% and 4.1%, respectively. Note that the damping for ISO 32 oil is very close to the dry case. It is assumed that these values of $\zeta$ are also valid for the running cam–follower experiment.

6. Signal processing technique to estimate $\mu$

The $\mu$ is estimated from measured reaction forces ($N_x(t)$ and $N_y(t)$) along the $\hat{e}_x$ and $\hat{e}_y$ directions, respectively, and the tangential acceleration ($b_0 \dot{a}(t)$) of the follower at its free end. By dividing the measured tangential acceleration by $b_0$, $\dot{a}(t)$ is obtained, and then numerically integrating it twice w.r.t. time, the time-varying component of $\alpha(t)$ is computed, while the integration constant ($\alpha^0$) is obtained from the time-averaged value of $\alpha(t)$. The instantaneous elastic force $F_s(t)$ is calculated from $\alpha(t)$ using Eq. (14). From Fig. 2, $N_x(t)$ and $N_y(t)$ are evaluated as follows:

\[ N_x(t) = F_n(t) \sin(\alpha(t)) + F_f(t) \cos(\alpha(t)) - m_b \ddot{b} \dot{a}(t) \sin(\alpha(t)), \tag{27} \]

\[ N_y(t) = F_n(t) \cos(\alpha(t)) - F_f(t) \sin(\alpha(t)) + m_b \ddot{b} \cos(\alpha(t)) - F_n. \tag{28} \]

Rearrange Eqs. (27) and (28) to yield the friction and normal forces as

\[ F_f(t) = m_b \ddot{b} \dot{a}(t) + N_x(t) \sin(\alpha(t)) + [N_y(t) + F_n - m_b \ddot{b}] \cos(\alpha(t)). \tag{29} \]

\[ F_n(t) = m_b \ddot{b} \cos(\alpha(t)) + N_x(t) \cos(\alpha(t)) \tag{30} \]

Since the dynamic force transducer used does not measure the DC component, a technique to estimate $\mu$ is proposed that utilizes complex-valued Fourier amplitudes while maintaining the phase relationship among the measured signals. First, the measured $N_x(t)$, $N_y(t)$ and $\dot{a}(t)$ are converted to the frequency domain using the fast Fourier transform (FFT) algorithm. Then, the harmonic reaction forces are reconstructed (with superscript $r$) using only their DC components (with superscript $d$) and the fundamental harmonic component of $\Omega_c$ (with superscript 1) as

\[ \widetilde{N}_x(t) = \widetilde{N}_x^d + \widetilde{N}_x^i \cos(\Omega_c t), \quad \widetilde{N}_y(t) = \widetilde{N}_y^d + \widetilde{N}_y^i \cos(\Omega_c t). \tag{31a, b} \]

In the above equation, $\widetilde{N}_x^d$ and $\widetilde{N}_y^i$ (where $\sim$ represents a complex-valued signal) are known from measurements while $\widetilde{N}_x^i$ and $\widetilde{N}_y^d$ are unknown. Similarly, the following harmonic signals have also been reconstructed as the following where $\zeta_c(t) = \sin(\alpha(t))$ and $\zeta_c(t) = \cos(\alpha(t))$:

\[ \widetilde{\zeta}_x^d(t) = \zeta_x^d + \zeta_x^i \cos(\Omega_c t), \quad \widetilde{\zeta}_x^i(t) = \zeta_x^d + \zeta_x^i \cos(\Omega_c t), \]

\[ \widetilde{F}_x^d(t) = F_x^d + F_x^i \cos(\Omega_c t), \quad \widetilde{F}_x^i(t) = F_x^d + F_x^i \cos(\Omega_c t), \]

\[ \widetilde{F}_y^d(t) = F_y^d + F_y^i \cos(\Omega_c t). \tag{32a–e} \]

Since $\dot{a}(t)$ does not have a DC component, it is written as

\[ \widetilde{\dot{a}}(t) = \widetilde{\dot{a}} \cos(\Omega_c t). \tag{33} \]

---

**Fig. 7.** Relative accelerance spectra in the vicinity of the system resonance. Key: ◼ dry (unlubricated); ○ lubricated with AGMA 4EP oil; + lubricated with ISO 32 oil.
Substituting these reconstructed harmonic signals in Eqs. (29) and (30) and rearranging, the following DC components and first harmonic components of $F_j(t)$ and $F_n(t)$ are found as

\[
F_j = N_{x,j}^{ed} - N_{y,j}^{ed} + \left\{ -F_{j,s}^{ed} + m_b g_{j,s}^{ed} + 0.5 \left| N_{x,s}^{c1} - N_{y,s}^{c1} - F_{j,s}^{c1} \right| \right\},
\]

\[
F_n^{ed} = N_{x,n}^{ed} + N_{y,n}^{ed} + \left\{ 0.5 m_b \alpha \ddot{\alpha} + F_{n,s}^{ed} - m_b g_{n,s}^{ed} + 0.5 \left| N_{x,s}^{c1} + N_{y,s}^{c1} + F_{n,s}^{c1} \right| \right\},
\]

\[
F_n^{c1} = N_{x,n}^{c1} + N_{y,n}^{c1} + \left\{ N_{x,s}^{c1} + N_{y,s}^{c1} - m_b g_{n,s}^{c1} + F_{n,s}^{c1} + 0.5 m_b \alpha \dot{\alpha} \right\}.
\]

From Eq. (21) and since \( sgn (v_i(t)) = -1 \) is a constant, the following relationships can be derived:

\[
F_j = -\mu F_n,
\]

\[
F_j = -\mu F_n.
\]

Since $F_j(t)$ and $F_n(t)$ are exactly out-of-phase,

\[\angle F_j = -\angle F_n.\]

Substituting Eqs. (34)–(37) into Eq. (38) to (40), three non-linear equations with three unknowns ($\mu, N_j^{ed}$ and $N_n^{ed}$) are obtained as shown below.

\[
N_{x,j}^{ed} = N_{y,j}^{ed} + \left\{ -F_{j,s}^{ed} + m_b g_{j,s}^{ed} + 0.5 \left| N_{x,s}^{c1} - N_{y,s}^{c1} - F_{j,s}^{c1} \right| \right\}
\]

\[
= -\mu \left\{ N_{x,s}^{ed} + N_{y,s}^{ed} + \left\{ 0.5 m_b \alpha \ddot{\alpha} + F_{n,s}^{ed} - m_b g_{n,s}^{ed} + 0.5 \left| N_{x,s}^{c1} + N_{y,s}^{c1} + F_{n,s}^{c1} \right| \right\} \right\}.
\]

\[
N_{x,j}^{ed} = N_{y,j}^{ed} + \left\{ N_{x,s}^{ed} + N_{y,s}^{ed} + \left\{ m_b g_{j,s}^{ed} + F_{n,s}^{ed} + 0.5 m_b \alpha \dot{\alpha} \right\} \right\}
\]

\[= \mu \left\{ N_{x,s}^{ed} + N_{y,s}^{ed} + \left\{ m_b g_{n,s}^{ed} + F_{n,s}^{ed} + 0.5 m_b \alpha \dot{\alpha} \right\} \right\}.
\]

These equations are numerically solved to estimate $\mu, N_j^{ed}$ and $N_n^{ed}$. In order to computationally validate this technique, predicted forces and acceleration from the contact mechanics model with $e/a=0.3, \Omega_e=11.55$ Hz and a known $\mu=0.3$ are used. The signal processing technique (with 9460 Hz sampling frequency and frequency resolution of 1.15 Hz) yields an estimate of $\mu$ as 0.302, which is about 99.3% accurate. This method also accurately estimates $N_j^{ed}$ and $N_n^{ed}$ as $-2.26$ N and $-12.88$ N, respectively, compared to the known values of $-2.26$ N and $-12.7$ N, respectively.

7. Experimental results and friction model

Spectral tests are conducted under lubricated conditions with different surface roughness levels at the contact. Care is taken during the experiments to record the steady state force and acceleration measurements only after the initial transients have sufficiently decayed. Using the measured data and the signal processing technique discussed in Section 6, the $\tilde{\mu}$ (estimated value of $\mu$) is identified for various values of mean surface roughness $R_m = 0.5 \sigma(R_e + R_b)$ as shown in Fig. 8. This $\tilde{\mu}$ is compared with the values reported in the literature [13] for dry friction contact. A higher range is observed in the case of a pure dry friction regime in comparison with $\tilde{\mu}$ for the mixed lubrication regime. Also, observe that $\tilde{\mu}$ with ISO 32 lubricant (with a low $A$ value) is similar to the dry friction contact case [13].

He et al. [32] used the Benedict–Kelley friction model [17] to develop an empirical relationship between $\mu$ and $R_m$, but this was specific to a line contact in gears. Hence that relationship is generalized for both point and line contacts as the following, where $\langle \gamma \rangle$ is the time-average operator,

\[
\mu = \frac{C_1}{C_2 - R_m} \log_{10} \left( \frac{p_1}{\eta'(v_i(t))_t (v_e(t))_t} \right).
\]
Here, $C_1$ and $C_2$ are the arbitrary constants and $p_\lambda$ is the time-averaged Hertzian contact pressure given by

$$p_\lambda = \frac{3}{2\pi} \frac{F_n(t)}{\mu^2(t) \eta(t)}\left(\frac{v(t)}{p(t)}\right)^{-1}.$$  \hspace{1cm} (45)

With a non-linear curve-fitting technique, the constants of Eq. (44) are found from the experimental results for each lubricant: $C_1 = 0.0288 \mu m$ and $C_2 = 2.03 \mu m$ for AGMA 4EP oil, and $C_1 = 0.0509 \mu m$ and $C_2 = 1.6512 \mu m$ for ISO 32 oil.

The measured force and acceleration spectra are compared with the contact mechanics model (with estimated $\mu$) in Table 1 for a typical case with $e/a = 0.116$, $\Omega_c = 11.55$ Hz and $\bar{\mu} = 0.51$. The contact mechanics model successfully predicts the forces and acceleration at the first three harmonics of $\Omega_c$, which are dominant compared with the higher harmonics.

The normalized coefficient of friction ($\mu$) for the empirical model of Eq. (44) is defined as

$$\mu = \log_{10}\left(\frac{p_\lambda}{\eta v(t) \mu^2(t)}\right).$$  \hspace{1cm} (46)

From Fig. 9 it is observed that $\mu$ monotonically increases with $R_m$. Also $\mu$ is lower with AGMA 4EP (with higher $\Lambda$) as compared to ISO 32 oil (with lower $\Lambda$). Fig. 9 compares some results of prior friction experiments [16,33–35] in terms of selected $\mu$ values which are calculated based on certain assumptions given a lack of pertinent data. For instance, Shon et al. [16] and Xu and Kahraman [33] conducted experiments under the Elastohydrodynamic lubrication (EHL) regime, and hence their $\mu$ values are very low. Conversely, Grunberg and Campbell [34] and Furey [35] conducted experiments under poorly lubricated conditions (mixed lubrication regime). It can be easily inferred that $\mu$ decreases as $\Lambda$ increases. There are some differences in the $\mu$ values from (Eq. (44)) and the ones reported in the literature; these may be attributed to different lubrication regimes as well as potential sources of error in the $\mu$ estimation process which is discussed next.

8. Potential sources of error in the estimation of $\mu$

Some of the common measurement errors which are difficult to minimize include the following. First, a variation in the frictional load torque on the cam causes small variations in $\Omega_c$ during the experiment. This in turn introduces inaccuracy in the harmonic contents of the measured forces and acceleration, thereby affecting the estimated $\mu$. Second, a small error in the angular alignment ($\kappa$) of a force transducer could measure $N_x(t) \cos(\kappa)$ instead of the actual $N_x(t)$. From the static analysis it is found that for $\kappa = 5^\circ$, the $\mu$ estimate has only a 0.5% error. Third, if the follower spring is oriented at an angle of $\sigma$ (from
Based on the static force balance, the error in the estimation of \( \mu \) decreases (as observed from Table 3) with an increase in \( e / a \); this is because the amplitude of \( \hat{\alpha} \) at the second harmonic of \( \Omega_c \) becomes significant compared with that of the first. Next the error is calculated for different \( e / a \) values [38]. For instance, bias errors [37] might be caused in the computation of the spectral contents of forces and acceleration due to a coarse frequency resolution (constrained by the length of the measured time domain signal and the usage of the Hanning window). Furthermore, Eqs. (41)–(43) are solved using the Levenberg–Marquardt algorithm which has limited accuracy as dictated by its relative and absolute tolerance values [38].

The error in \( \mu \) is simulated for the system with a circular cam for different values of \( e / a \) under a constant \( \Omega_c = 11.55 \text{ Hz} \). Using the predicted force and acceleration responses from the contact mechanics model (with known \( \mu = 0.3 \)) in the signal processing technique, \( \hat{\mu} \) is calculated, and the results are given in Table 2. For a very low value of \( e / a \), the error in \( \hat{\mu} \) is high because the amplitude of \( \hat{\alpha} \) (and the reaction forces) at the first harmonic of \( \Omega_c \) is very small. As the amplitude of \( \hat{\alpha} \) at the fundamental harmonic of \( \Omega_c \) increases, the error reduces and reaches a minimum at \( e / a = 0.3 \) (error = 0.67%). Beyond \( e / a = 0.3 \), the error again starts increasing since the amplitude of \( \hat{\alpha} \) at the second harmonic of \( \Omega_c \) becomes significant compared with that of the first. Next the error is calculated for different \( \Omega_c \) with a constant \( e \). The error monotonically decreases (as observed from Table 3) with an increase in \( \Omega_c \); this is because the amplitude of \( \hat{\alpha} \) (and the reaction forces) at the first harmonic of \( \Omega_c \) increases, while the amplitude ratio of the second harmonic to the fundamental harmonic is a constant.

### Table 2

<table>
<thead>
<tr>
<th>( e / a )</th>
<th>( \hat{\mu} ) (rad/s²)</th>
<th>( \hat{\mu} ) at the first harmonic of ( \Omega_c )</th>
<th>( \hat{\mu} ) at the second harmonic of ( \Omega_c )</th>
<th>Error (%) = ( \frac{| \text{known} - \text{predicted} |}{| \text{known} |} \times 100 )</th>
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<tbody>
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<td>472.7</td>
<td>81.5</td>
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</table>
A similar analysis is done for the system with an elliptic cam given $e/a = 0.1$, for different values of eccentricity $\varepsilon = \sqrt{1 - \left(\frac{b^2}{a^2}\right)}$ with known $\mu = 0.3$ (other parameters remaining the same as in Section 5). Fig. 10 gives a map of $\Omega_c - b/a$ showing different regimes that are obtained using the inverse kinematics procedure of Section 5. Comparison of Fig. 10 with Fig. 4 suggests that $\varepsilon$ for an elliptic cam provides a similar motion input as $e$ does for a circular cam. Care is taken so that the system lies in the regime without a loss of contact and no direction reversal of the $v(t)$. Table 4 shows the $\bar{\mu}$ values for an elliptic cam for different $\varepsilon$ at $\Omega_c = 8.33$ Hz. Only a small variation in the error is observed. However, an increase in the $\varepsilon$ increases the acceleration amplitude at the second harmonic of $\Omega_c$ due to a change in the type of motion input to the system. Overall, it is inferred that $\mu$ can be satisfactorily estimated even for a system with an elliptic cam.

### 9. Conclusion

The main goal of this article is to estimate $\mu$ using both model and experimental measurement and not to validate the contact mechanics model using the experiment. The major contributions of these analytical and experimental studies are as follows. First, a new vibration experiment has been designed to estimate $\mu$ for a mechanical system with combined rolling-sliding contact under a mixed lubrication regime. This experiment permits the contact pressure, “lambda ratio”, contact

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Error in the estimation of $\mu$ for the mechanical system with circular cam for different cam speeds with $e/a = 0.1$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_c$ [Hz]</td>
<td>$\dot{a}$ [rad/s$^2$]</td>
</tr>
<tr>
<td>At the first harmonic of $\Omega_c$</td>
<td>At the second harmonic of $\Omega_c$</td>
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<td>233.8</td>
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<tr>
<td>21</td>
<td>356.7</td>
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</tbody>
</table>

Table 4

| $\varepsilon$ | $\dot{a}$ [rad/s$^2$] | $\bar{\mu}$ | Error (%) = $\frac{|\bar{\mu} - \mu|}{\mu} \times 100$ |
|----------------|------------------|------------|---------------------------------|
| At the first harmonic of $\Omega_c$ | At the second harmonic of $\Omega_c$ | 0.283 | 5.7 |
| 0 | 52.2 | 2.18 | 0.283 | 5.7 |
| 0.31 | 52.1 | 52.7 | 0.281 | 6.2 |
| 0.44 | 52.1 | 104.9 | 0.281 | 6.2 |
| 0.53 | 52.05 | 156.8 | 0.284 | 5.5 |
| 0.6 | 52.0 | 208.4 | 0.288 | 4.2 |
| 0.66 | 51.95 | 259.7 | 0.294 | 2.2 |

A similar analysis is done for the system with an elliptic cam given $e/a = 0.1$, for different values of eccentricity $(e = \sqrt{1 - \left(\frac{b^2}{a^2}\right)}^{0.5})$ with known $\mu = 0.3$ (other parameters remaining the same as in Section 5). Fig. 10 gives a map of $\Omega_c - b/a$, showing different regimes that are obtained using the inverse kinematics procedure of Section 5. Comparison of Fig. 10 with Fig. 4 suggests that $\varepsilon$ for an elliptic cam provides a similar motion input as $e$ does for a circular cam. Care is taken so that the system lies in the regime without a loss of contact and no direction reversal of the $v(t)$. Table 4 shows the $\bar{\mu}$ values for an elliptic cam for different $\varepsilon$ at $\Omega_c = 8.33$ Hz. Only a small variation in the error is observed. However, an increase in the $\varepsilon$ increases the acceleration amplitude at the second harmonic of $\Omega_c$ due to a change in the type of motion input to the system. Overall, it is inferred that $\mu$ can be satisfactorily estimated even for a system with an elliptic cam.
velocity (sliding and entrainment), lubrication regime and surface roughness to be changed while satisfying the design constraints. Thus, the same experiment can be used to estimate $\mu$ for similar rolling-sliding contact systems such as gears and drum brakes. Second, a refined contact mechanics model for a cam–follower system with an elliptic cam is formulated that successfully predicts the system responses, as theory and experiment match well. This mathematical model yields a better understanding of the system dynamics as well as the accuracy of the $\mu$ estimation procedure. Third, a new signal processing technique is proposed to calculate $\mu$ using the complex-valued Fourier amplitudes (without DC component) of measured forces and acceleration. The DC components of the measured signals are also estimated by this method (along with $\mu$) by numerically solving a set of nonlinear equations. The validity of the assumptions made is proved by the existence of a solution to this problem, since a direct solution technique is employed to estimate the coefficient of friction without using any kind of approximation or residue minimization techniques. The chief limitation of this study is related to the angular alignment of the follower spring. Also the error in $\bar{\mu}$ is controlled by the choice of system geometry and cam speed; in particular the speed should be fairly low in order to avoid impacting conditions.

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