Analysis of start-up transient for a powertrain system with a nonlinear clutch damper

Laihang Li, Rajendra Singh *

Acoustics and Dynamics Laboratory, Ssmart Vehicle Concept Center, Department of Mechanical and Aerospace Engineering, The Ohio State University, Columbus, OH 43210, USA

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A B S T R A C T
The transient vibration phenomenon in a vehicle powertrain system during the start-up (or shut-down) process is studied with a focus on the nonlinear characteristics of a multi-staged clutch damper. First, a four-degree-of-freedom torsional model with multiple discontinuous nonlinearities under flywheel motion input is developed, and the powertrain transient event is validated with a vehicle start-up experiment. Second, the role of the nonlinear torsional path on the transient event is investigated in the time and time – frequency domains; interactions between the clutch damper and the transmission transients are estimated by using two metrics. Third, the harmonic balance method is applied to examine the nonlinear characteristics of clutch damper during a slowly varying non-stationary process in a simplified and validated single-degree-of-freedom powertrain system model. Finally, analytical formulas are successfully developed and verified to approximate the nonlinear amplification level for a rapidly varying process.

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1. Introduction

The engine start-up (or shut-down) process is receiving attention recently for both conventional internal combustion (IC) engines [1–4] and hybrid electric vehicles [5–9].Transient torsional vibration phenomena are more noticeable for reasons related to the enhancement of fuel consumption efficiency. First, downsizing of the IC engine, including fewer cylinders and lightweighting, has led to higher torque fluctuations. Second, the IC engine in some modern vehicles frequently starts up and shuts down [5–9]. The engine instantaneous firing frequency passes through the natural frequencies of the powertrain torsional modes, creating a vibration amplification and thus human discomfort. Such instantaneous frequency driven problems have been studied primarily for a linear single-degree-of-freedom (SDOF) mechanical system [10–14]. However, the transient vibration in a multi-degree-of-freedom powertrain system is yet be analyzed because of multiple discontinuous nonlinearities in the clutch damper and the transmission [15,16]. Therefore, this article intends to develop powertrain system models and examine the non-stationary process using numerical, experimental, and semi-analytical methods.

The effect of a multi-staged clutch damper on the stationary periodic vibro-impacts processes (such as gear rattle) has been studied [16–20]. In addition, the discontinuous nonlinearities, such as piecewise linear stiffness, hysteresis, and preload, have been studied in the frequency and time domains under harmonic excitation with a time-invariant excitation frequency [21–27]. For the non-stationary process with an instantaneous excitation frequency, the transient envelope of the

* Corresponding author. Tel.: +1 614 292 9044.
E-mail address: singh.3@osu.edu (R. Singh).

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Vibratory response is normally used to evaluate the severity of the transient amplification. For instance, Sen et al. [13,14] utilized the Hilbert transform [28] to find the transient envelope of a nonlinear SDOF system with a clearance; Markert and Seidler [29] and Hok [30] analytically found the transient envelope in the closed form of a linear SDOF system. However, the effects of the discontinuous nonlinearities on the transient envelope have not been adequately studied in depth.

2. Problem formulation

Based on the void seen in previous studies, a simplified powertrain system of a conventional vehicle with a clutch damper and manual transmission is considered as shown in Fig. 1(a). A four-degree-of-freedom (4DOF) nonlinear torsional model with instantaneous motion input $\dot{\theta}_0(t)$, $\ddot{\theta}_0(t)$ will be developed where a bar over a symbol indicates a dimensional parameter. Specific objectives are as follows: 1. Develop a new nonlinear 4DOF powertrain model excited by the flywheel motion and validate it by using a vehicle start-up experiment; 2. Estimate the role of the multi-staged clutch damper on $\theta_0(t)$ (where $\theta_0(t) = \theta_0(t) - \theta_1(t)$ is the relative torsional displacement between the flywheel and the clutch hub) and study its interaction with transient gear rattles in the...
transmission (given by $\widetilde{\delta}_{33}(t)$ and $\widetilde{\delta}_{34}(t)$) during the engine start-up process; 3. Examine the nonlinear characteristics of the multi-staged clutch damper and estimate the transient envelope by using the harmonic balance method (HBM) and approximation formulas.

3. Nonlinear model

The proposed 4DOF nonlinear model is developed under the assumption that the flywheel motion should not be affected by the motion of the clutch hub and other elements due to its massive inertia, and therefore it can be used as an input to the system. The downstream driveline components including the propeller shaft, axles, and the differential are decoupled from the transmission during the start-up process. Here, $I_i$, $C_{ij}$, and $K_{ij}$ indicate the torsional inertia, viscous damping, and stiffness of the lumped element, respectively. Two gear backlashs with the backlash size $b$ (mm) between the headset gear pair and the unloaded gear pair are present. An asymmetric multi-staged clutch damper is located between the flywheel and the clutch hub; its characteristic curve is shown in Fig. 1(b) where $T_{01}$ is the torque transmitted through the clutch damper. The nonlinear function of $T_{01}(\theta_{01})$ is given as below, where $\Delta(\varphi)$ is a unit function which is normally replaced by $\tan(\varphi)+1)/2$, and $\Delta(\varphi)$ is a sign function which is replaced by $\tan(\varphi)$; $\sigma$ (say of the order of $10^6$–$10^7$) is a regularizing factor that is used to smoothen the discontinuities to facilitate numerical calculation [31]:

$$ T_{01}(\theta_{01}) = T_i(\theta_{01}) + T_H(\theta_{01} - \theta_{01}), \quad (1a) $$

$$ T_i(\theta_{01}) = T_1(\theta_{01}) + T_2(\theta_{01}) + T_3(\theta_{01}) + T_4(\theta_{01}) + T_5(\theta_{01}), \quad (1b) $$

$$ T_1(\theta_{01}) = K_{III}(\theta_{01} + \varphi_{II,d} - \varphi_{II,c}) - K_{III}\varphi_{II,c} - K_{III}\varphi_{II,d} - T_p, \quad (1c) $$

$$ T_2(\theta_{01}) = u(\theta_{01} + \varphi_{II,d} + \varphi_{II,c}) \left\{ (K_{III} - K_{III})(\theta_{01} + \varphi_{II,d} + \varphi_{II,c}) \right\}, \quad (1d) $$

$$ T_3(\theta_{01}) = u(\theta_{01} + \varphi_{II,c}) \left\{ (K_{III} - K_{III})(\theta_{01} + \varphi_{II,c}) + T_p \right\}, \quad (1e) $$

$$ T_4(\theta_{01}) = u(\theta_{01} - \varphi_{II,d} + \varphi_{II,c}) \left\{ (K_{III} - K_{III})(\theta_{01} - \varphi_{II,d} + \varphi_{II,c}) \right\}, \quad (1f) $$

$$ T_5(\theta_{01}) = u(\theta_{01} - \varphi_{II,d} - \varphi_{II,c}) \left\{ (K_{III} - K_{III})(\theta_{01} - \varphi_{II,d} - \varphi_{II,c}) \right\}, \quad (1g) $$

$$ T_H(\theta_{01} - \theta_{01}) = \left\{ \frac{H_{II,c}}{2} + u(\theta_{01} + \varphi_{II,c}) \frac{H_{II,c} - H_{II,d}}{2} + u(\theta_{01} - \varphi_{II,d}) \frac{H_{II,d} - H_{III}}{2} \right\} \Delta(\theta_{01}). \quad (1h) $$

Here, $T_i(\theta_{01})$ represents the torque from the multi-staged and asymmetric springs with a preload; the stiffness $K_i$ is in the pre-damper stage $I$, $K_{III}$ and $K_{III}$ correspond to the main spring (in stage II), and $K_{III}$ refers to the stopper (in stage III); preload $T_p$ is between stages I and II, and subscripts $c$ and $d$ imply coast and driving sides, respectively; also, the asymmetric length of each stage is given in $\varphi_i$. $T_H(\theta_{01} - \theta_{01})$ describes the multi-staged and asymmetric hysteresis caused by the dry friction between the spring and clutch shell [15,16].

The governing equations under the flywheel motion input $\theta_{0}(t)$, $\dot{\theta}_{0}(t)$ are given as follows:

$$ I_1\ddot{\theta}_1(t) + C_{12}(\ddot{\theta}_1(t) - \theta_{01}(t)) - T_{01}(\theta_{01}(t), \dot{\theta}_{01}(t)) + C_{12}(\ddot{\theta}_1(t) - \ddot{\theta}_2(t)) + K_{12}(\ddot{\theta}_1(t) - \ddot{\theta}_2(t)) = 0, \quad (2) $$

$$ I_2\ddot{\theta}_2(t) + C_{12}(\ddot{\theta}_2(t) - \ddot{\theta}_1(t)) + K_{12}(\ddot{\theta}_2(t) - \theta_{01}(t)) + F_{\text{head}}(\ddot{\theta}_{23}(t), \dot{\theta}_2) \tau_{n,i} + C_{23}\ddot{\theta}_{23}(t) r_{n,i} = T_{D3}. \quad (3) $$

$$ I_3\ddot{\theta}_3(t) + F_{\text{head}}(\ddot{\theta}_{23}(t), \dot{\theta}_3) \tau_{n,o} + C_{23}\ddot{\theta}_{23}(t) r_{n,o} + F_{\text{unload}}(\ddot{\theta}_{34}(t), \dot{\theta}_3) \tau_{u,i} + C_{34}\ddot{\theta}_{34}(t) r_{u,i} = T_{D2}. \quad (4) $$

$$ I_4\ddot{\theta}_4(t) + F_{\text{unload}}(\ddot{\theta}_{34}(t), \dot{\theta}_4) \tau_{u,o} + C_{34}\ddot{\theta}_{34}(t) r_{u,o} = T_{D1}. \quad (5) $$

Here, $T_{D}(t)$ ($i = 1, 2, 3$) are the drag torques on the gears and shafts in the transmission which are caused by the lubricant and bearings and assumed to be constants; $F_{\text{head}}(\ddot{\theta}_{23}(t), \dot{\theta}_2)$ and $F_{\text{unload}}(\ddot{\theta}_{34}(t), \dot{\theta}_3)$ represent the nonlinear gear mesh forces from the two gear pairs with backlashes. The nonlinear functions are given below, where $K_g = K_{23} = K_{34}$ is the translational gear mesh stiffness, $b = 0.1$ mm is the net backlash ($\pm 0.5b$), $r_{n,i}, r_{n,o}, r_{u,i}, r_{u,o}$ are the gear radii for the headset gear pair and the unloaded gear pair, respectively, and $\ddot{\theta}_{23}(t) = \ddot{\theta}_2(t) r_{n,i} + \ddot{\theta}_3(t) r_{n,o}$, $\ddot{\theta}_{34}(t) = \ddot{\theta}_3(t) r_{u,i} + \ddot{\theta}_4(t) r_{u,o}$:

$$ F_{\text{head}}(\ddot{\theta}_{23}(t), \dot{\theta}_2) = K_g \frac{\Delta(\ddot{\theta}_{23}(t) - \ddot{\theta}_2) \Delta(\ddot{\theta}_{23}(t) - \ddot{\theta}_2) - \Delta(\ddot{\theta}_{23}(t) + \ddot{\theta}_2) \Delta(\ddot{\theta}_{23}(t) + \ddot{\theta}_2)}{2}, \quad (6) $$
\begin{align*}
F_{\mathrm{unload}}(\delta_{34}(t),\bar{b}) &= R_{\delta} \left( \frac{\text{sgn}(\delta_{34}(t) - (\bar{b}/2)) (\delta_{34}(t) - (\bar{b}/2)) - \text{sgn}(\delta_{34}(t) + (\bar{b}/2)) (\delta_{34}(t) + (\bar{b}/2))}{2} \right),
\end{align*}

(7)

4. Experimental validation of 4DOF nonlinear model

A vehicle start-up experiment is conducted on a medium-duty pickup truck with a 4-cylinder gasoline engine and a 6-speed manual transmission to validate the nonlinear model. The schematic of the test rig is given in Fig. 2; note that other downstream driveline components including the propeller shaft, axles, and the differential are not shown as they are decoupled during the engine start-up. The absolute torsional velocities of the flywheel (\(\dot{\theta}_0(t)\)) and clutch hub (\(\dot{\theta}_1(t)\)) are measured by two magnetic sensors, and the key system parameters are estimated from the physical dimensions or vehicle measurements [16,19]. First, the nonlinear model is linearized by assuming that \(\bar{b} = 0\) and only one stage of the clutch damper is activated. The eigensolutions are calculated as reported in Table 1. Second, the measured flywheel motion (\(\dot{\theta}_0(t)\)) is applied to the nonlinear 4DOF model as an input. Third, the relative velocity through the clutch damper (\(\dot{\theta}_{01}(t) = \dot{\theta}_0(t) - \dot{\theta}_1(t)\)) is calculated in both time and time–frequency domains. The comparisons of Fig. 3a and b and Table 2 suggest that the 4DOF nonlinear model yields accurate predictions, especially for \(\theta_{01,\max}, \theta_{01,\min}, \theta_{01,pp},\) and \(\tau_{\max}\). In addition, the time–frequency domain results, by a short-time Fourier transform (STFT) algorithm, in Fig. 3c and d indicate that the 4DOF nonlinear model accurately captures the first two dominant orders of the instantaneous frequency and the instant (around 0.4 s) when the maximum amplification (\(\dot{\theta}_{01,\max}\) or \(\dot{\theta}_{01,\min}\)) occurs (dashed circle). Also, Fig. 3c and d shows that the clutch mode with stage I stiffness (12 Hz) is excited leading to an amplification. The discrepancies in the time or time–frequency domains may be due to other damping mechanisms in the vehicle experiment which are beyond the scope of this article.

5. Effect of multi-staged clutch damper on amplification

The role of the multi-staged clutch damper and its interaction with the transmission are found in both the time and time–frequency domains. Fig. 4b–d shows the transient responses calculated by the validated nonlinear 4DOF model with

![Fig. 2.](image-url)

**Table 1**

Eigensolutions of the linearized 4DOF powertrain system model.

<table>
<thead>
<tr>
<th>Mode (Stage I activated)</th>
<th>Mode (Stage II(_d) activated)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural frequency (Hz)</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>446</td>
<td>2172</td>
</tr>
<tr>
<td>2172</td>
<td>2515</td>
</tr>
<tr>
<td>2515</td>
<td></td>
</tr>
<tr>
<td>Flywheel</td>
<td></td>
</tr>
<tr>
<td>Eigenvector</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
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<tr>
<td>0.00</td>
<td>0.00</td>
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<tr>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Clutch hub</td>
<td></td>
</tr>
<tr>
<td>Eigenvector</td>
<td></td>
</tr>
<tr>
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<td>1.00</td>
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<tr>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>0.00</td>
<td>0.93</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>Counter shaft and “welded” gears</td>
<td></td>
</tr>
<tr>
<td>Eigenvector</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>-0.22</td>
</tr>
<tr>
<td>1.00</td>
<td>-0.19</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.18</td>
</tr>
<tr>
<td>0.33</td>
<td>0.93</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.20</td>
</tr>
<tr>
<td>0.37</td>
<td>1.00</td>
</tr>
<tr>
<td>Output gear</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.93</td>
<td>-0.20</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.37</td>
</tr>
</tbody>
</table>
the parameters of Section 4 and with the measured flywheel motion input $\tilde{\theta}_0(\tilde{t})$ that is displayed in Fig. 4 (a). The time domain predictions indicate that $\tilde{\theta}_0(\tilde{t})$ rapidly increases up to 1600 rpm within the first 1.0 s and starts to decrease from 1.5 s to the idle speed (in the neutral status). Accordingly, $\tilde{\theta}_0(\tilde{t})$ exhibits significant vibration between 0.3 and 0.5 s which induces vibro-impacts at the transition between stages I and II on both the driving and the coast sides. Similarly, $\delta_{23}(\tilde{t})$ and $\delta_{34}(\tilde{t})$ show severe transient gear rattle phenomenon (single- and double-sided impacts) over the same duration. Short time Fourier transform is utilized to conduct the time–frequency analysis for a clear explanation. In Fig. 5 (a), a spectrogram of $\tilde{\theta}_0(\tilde{t})$ shows three dominant firing orders. The acceleration of the instantaneous firing frequency at each order is significantly high before 1.0 s. Observe the following from the spectrograms of $\tilde{\theta}_{01}(\tilde{t})$, $\tilde{\delta}_{23}(\tilde{t})$ and $\tilde{\delta}_{34}(\tilde{t})$ in Fig. 5b–e: First, Mode 1 (clutch mode in Table 1) is excited by the instantaneous firing frequencies (at 2nd, 4th, and 6th orders) through the natural frequencies of the clutch modes around 0.5 s (for 12 Hz) and 3.0 s (for 52 Hz) though the 12 Hz mode is more

<table>
<thead>
<tr>
<th>Symbol (Units)</th>
<th>Experiment</th>
<th>4DOF nonlinear model</th>
<th>Prediction error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\theta}_{01_\text{max}}$ (rpm)</td>
<td>299</td>
<td>298</td>
<td>0.3%</td>
</tr>
<tr>
<td>$\tilde{\theta}_{01_\text{min}}$ (rpm)</td>
<td>-215</td>
<td>-232</td>
<td>8.2%</td>
</tr>
<tr>
<td>$\tilde{\theta}_{01_pp}$ (rpm)</td>
<td>513</td>
<td>531</td>
<td>3.3%</td>
</tr>
<tr>
<td>$\tilde{t}_{\text{max}}$ (s)</td>
<td>0.44</td>
<td>0.44</td>
<td>0%</td>
</tr>
</tbody>
</table>

Fig. 3. Experimental validation of the nonlinear powertrain model with an instantaneous flywheel motion input. (a) $\tilde{\theta}_0(\tilde{t})$ from the experiment; (b) $\tilde{\theta}_0(\tilde{t})$ from the 4DOF model; (c) Spectrogram of $\tilde{\theta}_0(\tilde{t})$ from the experiment where the magnitude has a reference value of 1.0 rad/s; (d) Spectrogram of $\tilde{\theta}_0(\tilde{t})$ from the 4DOF model where the magnitude has a reference value of 1.0 rad/s.

Table 2
Validation of the 4DOF nonlinear powertrain system model given measured flywheel motion input $\tilde{\theta}_0(\tilde{t})$. 

dominant since most of the vibration occurs within stage I (Fig. 5(a)); Second, the resonant amplification within the clutch damper induces significant vibro-impacts near 0.5 s and 3.0 s both within the clutch damper and across the gear backlashes, and these vibro-impacts excite higher modes (say 460 Hz, 2170 Hz, and 2500 Hz). Therefore, the clutch damper has a decisive effect on the transient events.

Evaluation metrics are proposed next to correlate the transient vibration within the clutch damper and transmission from the numerical results. Within the start-up time window (say from 0.0 s to 1.5 s), discrete response vectors of dimension \( n \) are defined as \( \delta_{23}(j), \delta_{34}(j) \), and \( \delta_{23,\text{loss}}(j) \), \( 1 \leq j \leq n \) given the time vector \( t = [t_1, t_2, \ldots, t_n] \). In addition, two new response vectors \( \delta_{23,\text{loss}}(k) \) and \( \delta_{34,\text{loss}}(l) \) are defined when the gear pairs lose contact over \( 1 \leq k \leq n_{23,\text{loss}} \leq n, 1 \leq l \leq n_{34,\text{loss}} \leq n \) respectively; these are numerically extracted from \( \delta_{23}(j) \) and \( \delta_{34}(j) \). The first metric is the peak to peak value \( \tilde{\delta}_{01,\text{PP}} = \max(\delta_{01}(j)) - \min(\delta_{01}(j)) \), which can be also experimentally measured (as shown in Fig. 3). Then two new metrics \( \delta_{23,\text{rattle}}(\%) \) and \( \delta_{34,\text{rattle}}(\%) \) are developed as

\[
\delta_{23,\text{rattle}}(\%) = \frac{\sqrt{\sum_{k=1}^{n_{23,\text{loss}}} \left( \frac{\delta_{23,\text{loss}}(k) - (\text{B}/2)^2}{n_{23,\text{loss}}} \right)^2}}{\text{B}} \times 100\%.
\]

\[
\delta_{34,\text{rattle}}(\%) = \frac{\sqrt{\sum_{l=1}^{n_{34,\text{loss}}} \left( \frac{\delta_{34,\text{loss}}(l) + (\text{B}/2)^2}{n_{34,\text{loss}}} \right)^2}}{\text{B}} \times 100\%.
\]

The stiffness of stage I (\( K_{\text{r}} \)) is defined as \( \beta_{01}^e \text{K}_{\text{r}} \), where \( \text{K}_{\text{r}} \) is the reference value of stage I stiffness estimated from the vehicle experiment, and \( \beta_{01}^e \) denotes the variation. Two metrics \( \delta_{01,\text{PP}} \) and \( \delta_{23,\text{rattle}} \) are calculated when \( \beta_{01}^e \) is varied from 0 to 2 as an example. Similar trends between \( \delta_{01,\text{PP}} \) and \( \delta_{23,\text{rattle}} \) in Fig. 6 suggest that these two metrics successfully correlate the transient vibration events within two components. Given that it is difficult to measure the relative motion in the transmission, this correlation would be helpful in evaluating the transient gear rattle severity based on measured transient amplification \( (\delta_{01,\text{PP}}) \) in the clutch damper. Further, the nonlinear 4DOF model may be reduced to allow a focus on the clutch damper characteristics. A semi-definite 2DOF torsional model with flywheel motion input is developed by lumping the clutch hub and the gears together, and eliminating the gear backlash elements. Eqs. (2)–(5) are simplified as follows, where \( T_D = T_{D1} - T_{D2}(\tau_{h,j}/\tau_{h,u}) + T_{D1}(\tau_{h,j}/\tau_{h,u}) \), \( \tau = I_1 + I_2 + I_3(\tau_{h,j}/\tau_{h,u})^2 + I_4(\tau_{h,j}/\tau_{h,u})^2 \),

\[
\tilde{\tau}_{D1}(\tau) + \tau_{01}(\tilde{\tau}_{D1}(\tau) - \tilde{\theta}_0(\tau)) - \tau_{01}(\tilde{\theta}_{01}(\tau)) = T_D.
\]

Comparisons between the semi-definite 2DOF (or definite SDOF) nonlinear model and the vehicle start-up experiment in both the time and time–frequency domains in Fig. 7 demonstrate the accuracy and utility of the reduced order nonlinear model.
Fig. 5. Predicted spectrograms of the nonlinear 4DOF system given measured motion input $\tilde{\theta}_0(t)$. (a) Spectrogram of $\tilde{\theta}_0(t)$ with a reference value of 1.0 rad/s$^2$; (b) Spectrogram of $\tilde{\theta}_0(t)$ with a reference value of 1.0 rad/s; (c) enlarged view of (b); (d) Spectrogram of $\tilde{\delta}_{23}(t)$ with a reference value of 1.0 m/s; (e) Spectrogram of $\tilde{\delta}_{34}(t)$ with a reference value of 1.0 m/s.

Fig. 6. Correlation between the transient events at the clutch damper and the gearbox, as a function of $\beta$ using two metrics $\theta_{01,pp}$ and $\delta_{23,stable}$.
Although the start-up operation is of primary interest in this article, the proposed 4DOF and SDOF nonlinear models with instantaneous motion input are also applicable to other stationary and non-stationary vibration issues that have been discussed in the literature [5–9, 17–20]; these include engine shut-down and idle gear rattle problems. For such torsional vibration issues, the flywheel speed should be measured and utilized as an excitation as successfully illustrated in this article.

6. Nonlinear characteristics of multi-staged clutch damper

Eq. (10) is further simplified from a semi-definite 2DOF system to a definite SDOF system with normalized unity torsional inertia as given below where the effect from the drag torque is not considered:

\[ \ddot{\theta}(t) + 2\zeta \omega_1 \dot{\theta}(t) + T(\dot{\theta}(t), \ddot{\theta}(t)) = T_e(t). \]  

(11)

Here, \( \omega_1 \) is the natural frequency of the clutch mode with stage II\(_a\) activated, and \( \zeta \) is the associated damping ratio. An equivalent torque \( T_e(t) \) (corresponding to the flywheel motion input) is applied, and it consists of a mean part \( T_m(t) \) and an alternating part \( T_a(t) \) [32–34] which are given as

\[ T_e(t) = T_m(t) + T_a(t), \]  

(12a)

\[ T_m(t) = \bar{m}_1^2 T_m, \]  

(12b)

\[ T_a(t) = \bar{m}_1^2 T_a \sin \left( \left[ \omega_0 + \frac{1}{2} \bar{m}_1^2 t \right] + \varphi \right). \]  

(12c)

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Fig. 7. Experimental validation of the nonlinear SDOF powertrain model with an instantaneous flywheel motion input. (a) \( \bar{\theta}_{01}(t) \) from the experiment; (b) \( \bar{\theta}_{01}(t) \) from the SDOF model; (c) Spectrogram of \( \bar{\theta}_{01}(t) \) from the experiment with a reference value of 1.0 rad/s; (d) Spectrogram of \( \bar{\theta}_{01}(t) \) from the 4DOF model with a reference value of 1.0 rad/s.
Here, $T_m(\Omega)$ is assumed to be time-invariant, and a single order instantaneous excitation frequency in $T_a(\Omega)$ is assumed to linearly vary with a constant rate $(\alpha \sigma)$ with a time-invariant phase $\varphi$ though it is assumed to be zero without losing any generality. For the run-up, $\sigma > 0$ and for the run-down, $\sigma < 0$. The assumption of a constant $\varphi$ is based on the constant slope of the instantaneous firing frequency over the focused region as given in the STFT result of $\tilde{\varphi}_0(\Omega)$ in Fig. 5(a). Also, $\varphi_m$ and $\varphi_a$ determine the amplitudes of $T_m(\Omega)$ and $T_a(\Omega)$, respectively. The nonlinear torque transmitted through a symmetric clutch damper $T(\varphi(\Omega), \tilde{\varphi}(\Omega))$ is simplified from the asymmetric expressions of Eq. (1) as follows, where $\varphi = \varphi_c = \varphi_d$, $T_h = T_{H_{l,d}} = T_{H_{l,c}} = T_h$:

$$
T(\varphi(\Omega), \tilde{\varphi}(\Omega)) = T_1(\varphi(\Omega)) + T_2(\varphi(\Omega)) + T_3(\varphi(\Omega)),
$$

$$
T_1(\varphi(\Omega)) = \omega_1^2 \left[ \varphi(\Omega) + \frac{(1-\eta)}{2} \left[ (\varphi(\Omega) - \varphi) \text{sgn}(\varphi(\Omega) - \varphi) - (\varphi(\Omega) + \varphi) \text{sgn}(\varphi(\Omega) + \varphi) \right] \right] ,
$$

$$
T_2(\varphi(\Omega)) = T_p \frac{\text{sgn}(\varphi(\Omega) - \varphi) + \text{sgn}(\varphi(\Omega) + \varphi)}{2} ,
$$

$$
T_3(\varphi(\Omega)) = T_{\varphi} \frac{\text{sgn}(\varphi(\Omega))}{2} .
$$

Here, $T_1(\varphi(\Omega))$ represents the torque from the piecewise linear damper which has two symmetric stages of stiffness $(\eta \varphi_1^2$ and $\varphi_1^2)$; here, $\eta \varphi_1^2$ represents the stiffness of pre-damper with a compliant spring, and $\varphi_1^2$ is the main stiffness (for transmitting the torque to the driveline) [15–16]. Since $\eta$ is normally less than 1 as the pre-damper is relatively more compliant than the main spring, $0 \leq \eta \leq 1$ is of interest. The $T_2(\varphi(\Omega))$ term represents the symmetric preload at $\pm \varphi$, and the $T_3(\varphi(\Omega))$ term describes one stage and symmetric hysteresis. The dimensionless form of Eq. (11)–(15) are found by defining the following:

$$
t = \varphi_1 \Omega, \quad \theta = \frac{\varphi}{\varphi_1}, \quad \sigma = \frac{\alpha}{\varphi_1^2}, \quad \Omega_0 = \frac{\Theta_0}{\varphi_1} , \quad \gamma_a = \frac{\gamma_a}{\varphi_1}, \quad \gamma_m = \frac{\gamma_m}{\varphi_1}, \quad T_p = \frac{T_p}{\varphi_1 \omega_1^2}, \quad T_h = \frac{T_h}{\varphi_1 \omega_1^2}, \quad \omega_1 \frac{\varphi_1}{\varphi_1} = 1, \quad \varphi = \frac{\varphi}{\varphi_1} = 1,
$$

$$
\varphi(T) = \varphi(\Omega), \quad \varphi(T) = \frac{d\varphi}{d\Omega}, \quad \varphi(T) = \frac{d\varphi}{d\Omega} = \varphi(\Omega) = \varphi(\Omega) \frac{d\varphi}{d\Omega},
$$

$$
(16a–k)
$$

$$
\varphi(T) = \varphi(\Omega) = \varphi(\Omega) \frac{d\varphi}{d\Omega} = \varphi(\Omega) \frac{d\varphi}{d\Omega} = \varphi(\Omega) \frac{d\varphi}{d\Omega} = \varphi(\Omega) \frac{d\varphi}{d\Omega}.
$$

(17a, b)
7. Numerical examination of nonlinear characteristics

Eq. (20) is solved by using a numerical integration method (MATLAB [35]), and the transient envelope $E(t)$ is then estimated via Hilbert transform [28]. For the envelope study, two metrics are of particular interest: the maximum amplification $E_{\text{max}}$ and the peak frequency $\Omega_p$ at which $E_{\text{max}}$ is reached [3]. Note that the operating point $\theta_m$ is assumed to be $\gamma_m/\eta = \phi$ (transition point on the driving side) to activate all nonlinearities. The comparison between nonlinear (Case 1 of Table 3A) and linear (Case 2

![Figure 8](image)

Fig. 8. Comparison of transient envelopes $E(t)$ and phase planes between nonlinear and linear SDOF systems; refer to Table 3A for Cases 1 and 2. Figures (a) and (b) are for the run-up process and (c) and (d) are for the run-down process where $\Omega_0 + at$ is the instantaneous excitation frequency. Key: nonlinear SDOF system; linear SDOF system.
Fig. 9. Comparison of spectrograms of $\theta(t)$ between nonlinear and linear SDOF systems with a reference value of 1.0; refer to Table 3A for Cases 1 and 2. Figures (a and b) are for the run-up process and (c and d) are for the run-down process; (a, c) are for a nonlinear system with $|\alpha| = 0.001$, $\zeta = 0.015$; (b, d) are for a linear system. Here, $\omega$ is the instantaneous frequency of response $\theta(t)$ and $\Omega_0 + \alpha t$ is the instantaneous excitation frequency.

Fig. 10. Spectrograms of $\theta(t)$ of nonlinear SDOF system with a reference value of 1.0; refer to Table 3A for Case 1. Figures (a and b) are for the run-up process and (c and d) are for the run-down process; (a, c) are with $|\alpha| = 0.005$, $\zeta = 0.015$; (b, d) are with $|\alpha| = 0.001$, $\zeta = 0.075$. Here, $\omega$ is the instantaneous frequency of response $\theta(t)$ and $\Omega_0 + \alpha t$ is the instantaneous excitation frequency.
of Table 3A) systems in Fig. 8 where $\Omega_0 + \alpha t$ is the instantaneous excitation frequency (speed) indicates that the nonlinear case has direction dependent transient envelops $E(t)$ with respect to $\omega_1 = 1$ (especially for $E_{\text{max}}$ and $\Omega_p$) for both run-up and run-down processes. In addition, the STFT results of $\dot{\theta}(t)$ for Cases 1 and 2 in Fig. 9 suggest that the linear system (Fig. 9b and d) only has the primary order ($\Omega_0 + \alpha t$), and the horizontal line at 1 indicates the natural frequency ($\omega_1$) of the linear system. Conversely, multiple orders $q(\Omega_0 + \alpha t)$ exist in $\dot{\theta}(t)$ for the nonlinear case, where $q$ is a real and positive constant. Therefore, $T(\dot{\theta}(t), \dot{\theta}(t))$ introduces both the super-harmonic ($q > 1$) and sub-harmonic ($q < 1$) responses though the $q=1$ and $q > 1$ terms are dominant.

The effect of $\alpha$ and $\zeta$ on the nonlinear responses are examined for Case 1 as an example. The STFT results of $\dot{\theta}(t)$ in Figs. 9 and 10 clearly show that the nonlinear transient responses are more sensitive to a change of $\alpha$ than $\zeta$, and a higher $\alpha$ suppresses the formation of super- and sub-harmonic components. In addition, the contribution of each nonlinearity on the multiple harmonic response components is also numerically investigated by three cases with $|\alpha| = 0.001$, $\zeta = 0.015$. As shown in Fig. 11, only primary and minor super-harmonic components are found for preload and hysteresis cases; and piecewise linear stiffness curve generates most of the super- and sub-harmonic components.

![Fig. 11. Spectrograms of $\theta(t)$ of nonlinear SDOF system with $\alpha = 0.001$, $\zeta = 0.015$ with a reference value of 1.0. Figures (a–c) are for the run-up process and (d–f) are for the run-down process; (a, d) are for Case 3; (b, e) are for Case 4; (c, f) are for Case 5; refer to Table 3A for Cases 3–5. Here, $\omega$ is the instantaneous frequency of response $\theta(t)$ and $\Omega_0 + \alpha t$ is the instantaneous excitation frequency.](image-url)
8. Semi-analytical solutions for a slowly varying process

The general nonlinear characteristics of the clutch damper and its specific effect on engine start-up are analytically examined for slowly and rapidly varying processes, respectively. Since the nonlinear characteristics are more dominant for smaller values of $\alpha$ (say $|\alpha| = 0.001$), a slowly varying process is considered first to analytically estimate the transient envelope $E(t)$. Given the extremely slow rate $\alpha$, the nonlinear response $\theta(t)$ is assumed to be quasi-periodic which indicates that the harmonic balance method (HBM) can be utilized for such a quasi-stationary or slowly varying process. First, a new variable is introduced as

$$\tau = \alpha t.$$  \hfill (24)

Therefore, the governing equation is written as follows:

$$\tilde{\theta}(t) + 2\zeta \dot{\theta}(t) + T(\theta(t), \theta(t)) = \gamma_m + \gamma_d \sin \left(\Omega_0 + \frac{1}{2} \tau\right)t.$$  \hfill (25)

Since primary and super-harmonic components are more dominant than sub-harmonic components, $\theta(t)$ is assumed as follows where $\theta_m(\tau)$ is the amplitude of the nth harmonic given $\tau$ and $n$ is a positive integer to describe primary $(n=1)$ and super harmonic $(n > 1)$ components:

$$\theta(t) = \theta_m + \sum_{n=1}^{\infty} \theta_m(\tau) \sin \left(n \left(\Omega_0 + \frac{1}{2} \tau\right)t + \psi_n\right).$$  \hfill (26)

The strong form residual is defined as below at a given $\tau$:

$$r(t) = R(t) - \dot{\theta}(t) - 2\zeta \ddot{\theta}(t) = -T(\theta(t), \dot{\theta}(t)) \to 0 \quad \forall \tau.$$  \hfill (27)

Since the system excitation $T(t)$ and the nonlinear response $\theta(t)$ are assumed to be quasi-periodic due to a slowly varying $\tau$, the frequency domain analysis is conducted by applying the Fourier transform on both sides of strong form residual as follows:

$$\mathfrak{A}(r(t)) = \mathfrak{A}(T(t)) - 2\zeta \mathfrak{A}(\dot{\theta}(t)) - \mathfrak{A}(T(\theta(t), \dot{\theta}(t))) \to 0 \quad \forall \tau.$$  \hfill (28)

Then, the residual is minimized in the Newton-Raphson form as below where $\mathbf{\Theta}(\tau) = \mathfrak{A}(\theta(t))$ is the Fourier component vector of $\theta(t)$ at a given $\tau$:

$$\frac{R(\tau)}{\mathbf{\Theta}(\tau)} = -\frac{\partial R(\tau)}{\partial \mathbf{\Theta}(\tau)} \Delta \mathbf{\Theta}(\tau).$$  \hfill (29)

Here, $R(\tau)$ is the residual from the initial guess, and $\Delta \mathbf{\Theta}(\tau) = -\left(\partial R(\tau)/\partial \mathbf{\Theta}(\tau)\right)^{-1} R(\tau)$ is the correction factor. The Newton-Raphson method is employed to find $\mathbf{\Theta}(\tau)$ during an iterative process [25-26]. The Fourier transformation of the quasi-periodic $\theta(t)$ is expressed as follows, where $\mathfrak{A}^\mathcal{T}$ is the discrete Fourier transform matrix:

$$\mathfrak{A}(\theta(t)) \to \begin{bmatrix} \theta_0(t_0) \\ \theta_1(t_1) \\ \vdots \\ \theta_{N-2}(t_{N-2}) \\ \theta_{N-1}(t_{N-1}) \end{bmatrix} = \begin{bmatrix} 1 & \sin((\Omega_0 + \frac{1}{2})t_0) & \cos((\Omega_0 + \frac{1}{2})t_0) & \sin(2(\Omega_0 + \frac{1}{2})t_0) & \cos(2(\Omega_0 + \frac{1}{2})t_0) & \cdots & \(\Omega_0 + \frac{1}{2})t_0) & \cos(2(\Omega_0 + \frac{1}{2})t_0) & \cdots \\ 1 & \sin((\Omega_0 + \frac{1}{2})t_1) & \cos((\Omega_0 + \frac{1}{2})t_1) & \sin(2(\Omega_0 + \frac{1}{2})t_1) & \cos(2(\Omega_0 + \frac{1}{2})t_1) & \cdots & \(\Omega_0 + \frac{1}{2})t_1) & \cos(2(\Omega_0 + \frac{1}{2})t_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sin((\Omega_0 + \frac{1}{2})t_{N-2}) & \cos((\Omega_0 + \frac{1}{2})t_{N-2}) & \sin(2(\Omega_0 + \frac{1}{2})t_{N-2}) & \cos(2(\Omega_0 + \frac{1}{2})t_{N-2}) & \cdots & \(\Omega_0 + \frac{1}{2})t_{N-2}) & \cos(2(\Omega_0 + \frac{1}{2})t_{N-2}) & \cdots \\ 1 & \sin((\Omega_0 + \frac{1}{2})t_{N-1}) & \cos((\Omega_0 + \frac{1}{2})t_{N-1}) & \sin(2(\Omega_0 + \frac{1}{2})t_{N-1}) & \cos(2(\Omega_0 + \frac{1}{2})t_{N-1}) & \cdots & \(\Omega_0 + \frac{1}{2})t_{N-1}) & \cos(2(\Omega_0 + \frac{1}{2})t_{N-1}) & \cdots \end{bmatrix} = \mathfrak{A}(\theta(t)) \mathbf{\Theta}(\tau) \mathfrak{A}(\tau) \Delta \mathbf{\Theta}(\tau).$$  \hfill (30)

Since $\theta(t)$ is assumed to be quasi-periodic at a given $\tau$, the vector $\theta = \{\theta(t_0), \theta(t_1), \ldots, \theta(t_{N-1})\}$ $T$ is quasi-periodic as well, where $T$ indicates the transpose of a vector. The symbol $N$ represents the number of sampling points of $\theta(t)$ within one period. In addition, the column vector $\mathbf{\Theta}(\tau)$ contains the Fourier amplitude of each component which is calculated by the Discrete Fourier Transform (DFT) algorithm. A new dimensionless time variable is introduced as $\tau = ((\Omega_0 + \tau/2)t)$ and $\Theta \in [0, 2\pi]$. Then, $\theta(t)$ is represented as $\theta(t) = (d\theta/d\tau)(d\tau/dt) = (\Omega_0 + \tau\theta')(\theta)$. Similarly, $\dot{\theta}(t)$ is described as $\theta'(t) = (\Omega_0 + \tau^2\theta''(\theta)) \cdot (\theta' \cdot \theta'' \cdot \theta''')$ within one period are given as below:

$$\begin{align*}
\mathbf{\Theta} = \mathfrak{A}(\tau), \\
\theta_0 = \mathfrak{A}(\tau), \\
\theta_1 = \mathfrak{A}(\tau), \\
\dot{\theta}_0 = \mathfrak{A}(\tau), \\
\dot{\theta}_1 = \mathfrak{A}(\tau).
\end{align*}$$  \hfill (31)

$$\dot{\theta}_0 = \begin{bmatrix} 1 & \sin(\theta_0) & \cos(\theta_0) & \sin(2\theta_0) & \cos(2\theta_0) & \cdots \\ 1 & \sin(\theta_1) & \cos(\theta_1) & \sin(2\theta_1) & \cos(2\theta_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sin(\theta_{N-2}) & \cos(\theta_{N-2}) & \sin(2\theta_{N-2}) & \cos(2\theta_{N-2}) & \cdots \\ 1 & \sin(\theta_{N-1}) & \cos(\theta_{N-1}) & \sin(2\theta_{N-1}) & \cos(2\theta_{N-1}) & \cdots \end{bmatrix} \mathbf{\Theta}.$$  \hfill (32)
Therefore, the vector presentations of \( \theta(t) \), \( \dot{\theta}(t) \), and \( \ddot{\theta}(t) \) are given as follows:

\[
\theta(t) = \Xi \Xi^T \Theta(t), \quad \dot{\theta}(t) = (\Omega_0 + \tau) \Xi \Lambda \Theta(t), \quad \ddot{\theta}(t) = (\Omega_0 + \tau)^2 \Xi \Gamma \Theta(t).
\]

Regarding the system excitation, the Fourier transform is described as follows where \( T_e \) is the vector form of \( T_e(t) \) within one period:

\[
\mathcal{X} T_e(t) \rightarrow T_e = \Xi E.
\]

The column vector \( E \) represents the excitation torque amplitude vector. Since only one mean part and one harmonic component are considered, \( E = [\gamma_m, \gamma_a, 0, 0, 0, \ldots] \). In addition, the column vector \( \mathcal{X}(r) \), the Fourier components of the nonlinear term \( T(\theta(t), \dot{\theta}(t)) \), is defined as

\[
T = \Xi \mathcal{X}(r).
\]

Then, the residual in the time domain is described in the discrete form by using the Fourier series expansion as below where \( \Xi^+ \) is the pseudo-inverse of \( \Xi \):

\[
\Xi R(t) = \Xi E - (\Omega_0 + \tau)^2 \Xi \Gamma \Theta(t) - 2 \zeta (\Omega_0 + \tau) \Xi \Lambda \Theta(t) - \Xi \mathcal{X}(r) = 0,
\]

\[
R(t) = E - (\Omega_0 + \tau)^2 \mathcal{X}(r) - 2 \zeta (\Omega_0 + \tau) \mathcal{X}(r) - \Xi^+ T = 0.
\]

In order to utilize the Newton–Raphson algorithm, the Jacobian matrix is calculated as below:

\[
J(t) = \frac{\partial R(t)}{\partial \Theta(t)} = - (\Omega_0 + \tau)^2 \Gamma - 2 \zeta (\Omega_0 + \tau) \Lambda - \frac{\partial (\Xi^+ T)}{\partial \Theta(t)}.
\]

The term \( \frac{\partial (\Xi^+ T)}{\partial \Theta(t)} \) is calculated as follows:

\[
\frac{\partial (\Xi^+ T)}{\partial \Theta(t)} = \Xi^+ \frac{\partial T}{\partial \Theta(t)} = \Xi^+ \frac{\partial T}{\partial \theta(t)} \frac{\partial \theta(t)}{\partial \Theta(t)} = \Xi^+ \frac{\partial T}{\partial \theta(t)} \frac{\partial \Xi}{\partial \Theta(t)}.
\]

\[
\frac{\partial T}{\partial \theta(t)} = \text{diag} \left[ \frac{\partial T}{\partial \theta(t)} (t_0), \frac{\partial T}{\partial \theta(t)} (t_1), \ldots, \frac{\partial T}{\partial \theta(t)} (t_{N-2}), \frac{\partial T}{\partial \theta(t)} (t_{N-1}) \right].
\]

The next prediction (for iteration \( k+1 \)) is adjusted by the following:

\[
\Theta(t)_{k+1} = \Theta(t)_k - \frac{R(t)_k}{J(t)_k} = \Theta(t)_k + \Delta \Theta(t)_k.
\]

This iteration process stops when the error criterion is satisfied, as defined below, where \( \varepsilon \) is an extremely small value:

\[
\left\| \frac{R(t)}{J(t)} \right\| < \varepsilon.
\]
Therefore, at given $\tau = \alpha t$, column vector $\Theta(\tau)$ is found as follows:

$$\Theta(\tau) = \Theta(at) = \Theta(\Omega_0 + at).$$  \hspace{1cm} (44)

This equation implies that for a slowly varying excitation frequency $\Omega_0 + \alpha t$, the amplitude of each harmonic ($\Theta$) from the nonlinear response $\theta(t)$ is successfully found. Further, the transient envelope $E(t)$ in the time domain is calculated as follows where $\Theta_n(at)$ is the $n$th component of the column vector $\Theta(at)$, and $\Theta_0(at)$ is the mean (DC) part of $\theta(t)$:

$$E(t) = \sqrt{\sum_{n=1}^{2n_{\text{max}}+1} \Theta_n^2(at) + \Theta_0(at)}.$$  \hspace{1cm} (45)

The proposed HBM for $|\alpha| = 0.001$ is utilized to examine the transient envelope of each nonlinearity by using the parameters for Cases 6–9 of Table 3B. Note that a higher value of $\zeta$ is needed to ensure the numerical convergence. Fig. 12 compares $E(t)$ estimated by the HBM and the Hilbert transform method, and the results indicate that piecewise linear stiffness curve introduces a direction dependent envelope, and the HBM accurately captures primary and super harmonic amplifications from these nonlinearities.

9. Approximate formulas for a rapidly varying process

A rapidly varying process with a much higher $|\alpha|$ (say up to 0.03) is considered next since this rate is closer to the actual engine start-up or shut-down process [2]. Three nonlinearities are separately examined with a focus on $E_{\text{max}}$ and $\Omega_p$ by using the parameters of Table 3C. Case 10 results in Fig. 13 suggesting that a higher value of hysteresis ($T_h$) decreases $E_{\text{max}}$, and this trend is consistent with $\alpha$ up to 0.03 for both run-up and run-down processes (Fig. 13a and b). However, hysteresis does not have a significant effect on $\Omega_p$ (Fig. 13c and d). A change in preload $T_p$ does not influence $E_{\text{max}}$ or $\Omega_p$ as seen in Fig. 14 with Case 11. The effect of the stiffness ($\eta$) is examined by Cases 12 and 13 for run-up and run-down processes, respectively. Although $\eta$ does not affect $E_{\text{max}}$, it has a dominant effect on $\Omega_p$ (Fig. 15c and d). In order to approximate $E_{\text{max}}$ and incorporate the effects of $T_h$, the following formula is proposed based on a prior expression for a linear system that is

![Fig. 12. Verification of the HBM for a slowly varying process with $|\alpha| = 0.001$. Figures (a–c) are for the run-up process and (d–f) are for the run-down process; (a) is for Case 6; (d) is for Case 7; (b, e) are for Case 8; (c, f) are for Case 9; refer to Table 3B for Cases 6–9. Key: $\ast$ HBM; • numerical prediction using Hilbert transform.](image)
reported in [29], where $T_a = \omega_1^2 \gamma_a$:

$$E_{\text{max}} = \frac{(T_a - (T_h/2))}{2\zeta + \sqrt{(\alpha/2)}} + \theta_m.$$  

Likewise, the formulas for $\Omega_p$ are as follows based on the stiffness sensitivity for the run-up or run-down process:

For run–up process: $\Omega_p = \sqrt{\frac{1 + \eta}{2}} \left(\sqrt{1 - \zeta^2} + \sqrt{2(\pi - 1)|\alpha + \zeta^2|}\right).$  

For run–down process: $\Omega_p = \frac{1 + \eta}{2} \left(\sqrt{1 - \zeta^2} - \sqrt{2(\pi - 1)|\alpha + \zeta^2|}\right).$

---

Fig. 13. The effect of hysteresis ($T_h$) on $E_{\text{max}}$ and $\Omega_p$ for a rapidly varying process. Figures (a, c) are for the run-up process and (b, d) are for the run-down process; refer to Table 3C for Case 10. Key: $\square$, $T_h = 0$; $\bigcirc$, $T_h = 0.05$; $\bigtriangleup$, $T_h = 0.1$.

Fig. 14. The effect of preload ($T_p$) on $E_{\text{max}}$ and $\Omega_p$ for a rapidly varying process. Figures (a, c) are for the run-up process and (b, d) are for the run-down process; refer to Table 3C for Case 11. Key: $\square$, $T_p = 0$; $\bigcirc$, $T_p = 0.025$; $\bigtriangleup$, $T_p = 0.05$. 

The $E_{\text{max}}$ calculation with Eq. (46) is compared with numerical predictions by using the parameters of Case 10. Fig. 16 clearly shows that Eq. (46) is able to approximate $E_{\text{max}}$ reasonably well when $|\alpha|$ is up to 0.03 for both run-up and run-down processes. Also, Eqs. (47) and (48) are verified in Fig. 17 with the parameters of Cases 12 and 13 of Table 3C for run-up and run-down respectively. Again, a relatively close match between calculations using Eqs. (47) and (48) and numerical predictions is achieved.

10. Conclusion

This article has examined the transient vibration amplification in a nonlinear powertrain system during the start-up or shut-down process. Numerical, analytical, and experimental methods are used to investigate the role of nonlinear characteristics of a
multi-staged clutch damper. Specific contributions are as follows. First, a new 4DOF nonlinear model is successfully developed and experimentally validated for a transient event during the engine start-up process. Second, the vibro-impacts within the multi-staged clutch damper are correlated with those in the transmission by using new transient metrics in the context of both 4DOF and SDOF models. Third, the transient amplification envelope for a slowly varying process is successfully estimated by utilizing the harmonic balance method; it is also well approximated for a rapidly varying process by extending formulas based on prior work [29]. The main limitation of this article is that only a minimal order system of the vehicle powertrain is utilized and the effect of the transmission temperature on the transient vibration events is not considered.

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Appendix A

List of symbols

\[ b \] backlash size
\[ \zeta \] damping ratio
\[ C \] viscous damping coefficient
η  stiffness ratio between clutch stages
E  transient envelope
θ  torsional displacement
E  excitation torque amplitude vector
ϕ  transition point of clutch damper
F  gear mesh force
φ  phase of torque excitation
H  clutch damper hysteresis
ω  natural frequency
I  torsional inertia
Ωp  excitation frequency at which E_{\text{max}} is reached
J  Jacobian matrix

Subscripts
K  torsional or gear mesh stiffness
0, 1, 2, 3 index of torsional elements
q  order of harmonics
I, II, III index of clutch damper stages
r  strong form residual
a  alternating part
T  torque
c  coast side of clutch damper
α  constant acceleration rate
d  driving side of clutch damper
β  variation of design parameters
D  drag torque
γ  torque amplitude
e  torsional excitation
δ  relative translation displacement within a gear pair
g  gear mesh stiffness
DFT  discrete Fourier transform
h_i, h_o headset input/output gear
HBM  harmonic balance method
h  hysteresis
m  mean part
min  minimum value
max  maximum value
p  clutch damper preload
pp  peak to peak value
r  stiffness reference value
u_i, u_o unloaded input/output gear

Superscripts
K  clutch torsional spring stiffness
+  pseudo-inverse of a matrix
−  dimensional parameter

Abbreviations

DOF  degree-of-freedom
STFT  short-time Fourier transform

References
