Active and passive damping patches on a thin rectangular plate: A refined analytical model with experimental validation

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ABSTRACT

Concurrent placement of active and passive damping is an emerging area of research, though tractable analytical solutions to address such problems for plate-like structures are not readily available. Accordingly, a new analytical model is proposed using a Rayleigh–Ritz scheme for a thin elastic plate with side-by-side active and passive damping patches. Frequency-dependent properties of passive patch viscoelastic material are included in the formulation. The comprehensive model is found to be valid for the cases of either passive or active patches, as well as the combined case. The proposed formulation is verified with a commercial finite element code, and experiments are performed for limiting cases to validate proposed theory and results. The effect of passive patch location on modal loss factor and on the response due to a harmonic disturbance is analytically determined. The phase interaction between the active patches and the disturbance input, including the effect of damping on the required control parameters, is quantified. Finally, the side-by-side combination of active and passive damping is found to yield improved vibration reduction and to enhance the efficacy of active control.

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1. Introduction

A combination of active and passive vibration control methods has recently been proposed, for instance in the form of damping patches, which may include piezoelectric active patches and selective passive constraining layer viscoelastic treatments [1–3]. This problem is gaining importance due to increased lightweighting of structures and low structural damping of modern materials, driving the need for compact and lightweight damping solutions. The state of the art for the “hybrid” patch method was investigated by Benjeddou [1] and Stanway et al. [4], and they concluded that the literature is sparse on analytical and experimental work applied to plate structures. The most relevant technique is “Active Constrained Layer Damping” (ACLD), by Baz and Ro [5,6] and Ray et al. [7,8] for example. The ACLD is found to be an effective method since the active element enhances the shear in the viscoelastic layer, but if it fails, the passive element can still induce damping. This method also allows for simultaneous sensing and actuation. Independent active and passive damping (i.e. side-by-side), on the other hand, is largely ignored. An exception is the work of Lam et al. [9] who analytically consider the side-by-side configuration for a beam structure. The motivation here is to reduce patch thickness and to maximize the active patch control authority with direct bonding to the structure, rather than having the viscoelastic material between. Lam et al.
Appendix A for identification of symbols). As with passive patches, active patches have thickness $h_c$ and can be piezoelectric patches, as illustrated in Fig. 1. The plate (layer 3) has thickness $h_3$. The passive patches (consisting of layers 1 and 2) have thickness $h_1$ and $h_2$, as well as dimensions $L_{xc} \times L_{yc}$ and location $(x_p^q, y_p^q)$ for the $q$th patch ($0 \leq q \leq N_p$). For the sake of simplicity, all passive patches are assumed to consist of the same materials (i.e. $h_1$, $h_2$, as well as $E_1$, $E_2$, $\rho_1$, $\rho_2$, $\eta_1$, and $\eta_2$ are constant for each of the $N_p$ passive patches, where $E$ is Young’s modulus, $\rho$ is density, and $\eta$ is material loss factor; also see Appendix A for identification of symbols). As with passive patches, active patches have thickness $h_c$, as well as dimensions $L_{xc} \times L_{yc}$ and location $(x_c^r, y_c^r)$ for the $r$th patch ($0 \leq r \leq N_c$). Active patches will be used in the “bimorph” patch concept. The sinusoidal response for a discretized system can be written as $\tilde{q}(\omega) = \left[K(\omega) - \omega^2M\right]^{-1}\{\tilde{Q}_d + \tilde{Q}_c\}$. Here, $M$ and $K$ are mass and stiffness matrices, $Q$ are non-conservative generalized forcing vectors, $\omega$ is excitation and/or response frequency (rad/s), $\tilde{q}$ is a generalized response vector, subscripts $c$ and $d$ refer to control and disturbance inputs, respectively, and the tilde ($\sim$) superscript refers to a complex-valued quantity. The frequency-dependent $K(\omega)$ is based on the loss factor concept: $\tilde{E}(\omega) = E(\omega)[1 + j\eta(\omega)]$ [12], where $E$ and $\eta$ are material Young’s modulus and loss factor, respectively. This article intends to investigate the effect of passive patches (contained within $M$ and $K$) on the response, $\tilde{q}$, due to a control input $\tilde{Q}_d$. The side-by-side hybrid damping method (with frequency-dependent material properties) will be applied to thin plate structures, which are both of theoretical interest due to a lack of publications, as well as of practical interest due to the fact that many industrial devices and vehicle components (e.g. panels, covers, brake pads, etc.) may be described as plate-like structures [13,14].

### Table 1

Material properties for various layers of the physical system of Fig. 1.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Material</th>
<th>Young's modulus</th>
<th>Density [kg m$^{-1}$]</th>
<th>Poisson’s ratio</th>
<th>Thickness [mm]</th>
<th>Loss factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Steel$^a$</td>
<td>203 GPa</td>
<td>7600</td>
<td>0.30</td>
<td>0.38</td>
<td>0.005</td>
</tr>
<tr>
<td>2</td>
<td>Adhesive$^b$</td>
<td>(6.2 + 2.2 × 10$^{-2}$) [Hz] MPa</td>
<td>730</td>
<td>0.40</td>
<td>0.94</td>
<td>1.25–3.7 × 10$^{-4}$ [Hz]</td>
</tr>
<tr>
<td>3</td>
<td>Aluminum</td>
<td>63 GPa</td>
<td>2606</td>
<td>0.33</td>
<td>3.2</td>
<td>0.0013</td>
</tr>
<tr>
<td>C</td>
<td>Composite</td>
<td>31 GPa</td>
<td>–</td>
<td>0.35</td>
<td>0.3</td>
<td>–</td>
</tr>
</tbody>
</table>

$^a$ Measured using available stock.  
$^b$ Material provided by Scotch 3M (http://www.scotchbrand.com).  
$^c$ Piezoelectric macro-fiber composite patches provided by Smart Material Corp. (www.smart-material.com).  

[9] present the time domain response using a Ritz–Galerkin approximation by investigating the effect of active and passive patch configuration for a one-dimensional structure. Two dimensional spatial effects of patch configuration in the case of a thin plate, however, are yet to be studied. More recent efforts in this area have dealt mostly with finite element models [10] and control algorithms [11]. Overall, the literature has focused on maximizing attenuation and not on interaction between active and passive patches. Furthermore, much of the literature (in particular the analytical models) ignores the frequency-dependent properties of viscoelastic materials. The goal of this paper is to overcome this void by proposing and experimentally validating a new and comprehensive analytical model of the side-by-side hybrid damping approach for a thin rectangular plate and determining the interactions between active and passive patches.

2. Problem formulation

Consider a thin rectangular plate of dimensions $L_x \times L_y$ with $N_p$ passive constrained layer damping patches and $N_c$ active piezoelectric patches, as illustrated in Fig. 1. The plate (layer 3) has thickness $h_3$. The passive patches (consisting of layers 1 and 2) have thickness $h_1$ and $h_2$, as well as dimensions $L_{xc} \times L_{yc}$ and location $(x_p^q, y_p^q)$ for the $q$th patch ($0 \leq q \leq N_p$). For the sake of simplicity, all passive patches are assumed to consist of the same materials (i.e. $h_1$, $h_2$, as well as $E_1$, $E_2$, $\rho_1$, $\rho_2$, $\eta_1$, and $\eta_2$ are constant for each of the $N_p$ passive patches, where $E$ is Young’s modulus, $\rho$ is density, and $\eta$ is material loss factor; also see Appendix A for identification of symbols). As with passive patches, active patches have thickness $h_c$, as well as dimensions $L_{xc} \times L_{yc}$ and location $(x_c^r, y_c^r)$ for the $r$th patch ($0 \leq r \leq N_c$). Active patches will be used in the “bimorph” patch concept:

$$
\tilde{q}(\omega) = \left[K(\omega) - \omega^2M\right]^{-1}\{\tilde{Q}_d + \tilde{Q}_c\}.
$$

Fig. 1. Active and passive damping patches on a thin rectangular plate with free boundaries.
pair configuration [15], defined as having a patch on the top and bottom surface of layer 3. Active patches are also assumed to be identical in terms of thickness and material properties. Also labeled in Fig. 1 are disturbance force \((x_d, y_d)\) and measurement \((x_0, y_0)\) locations. Note the labeling scheme uses subscripts to denote layers and/or directions and superscripts for patch counting indices. The active and passive patches will be treated separately, as the passive patches are included as part of \(\mathbf{M}\) and \(\mathbf{K}\), whereas the active patches are modeled as forces in the \(\mathbf{Q}\) vector.

The scope of this paper is limited to continuous solutions for classical thin plates. Simplifying assumptions are made to achieve separable solutions to differential equations of motion. Free boundaries are modeled as they are frequently ignored in the literature for analytical models (where simple supports are most common) and because they are most accurately replicated in experimental studies. Abaqus [16] is used as the commercial finite element method (FEM) code to compare vibratory response with analytical and experimental results. The experimental portion of the research focuses on uniform aluminum rectangular plates with free boundary conditions. Piezoelectric actuators used for experimental studies are macro-fiber composite patches and passive damping patches are constrained-layer with a steel constraining layer and a viscoelastic adhesive core. Material properties obtained from experiments and used in the following simulations are listed in Table 1.

The chief objectives of this paper are to develop a refined and experimentally-validated analytical model of a plate with side-by-side active and passive damping patches, and to employ this model to determine the interaction between the active and passive patches. To accomplish this, two prior analytical models [17,18] will be brought together under a common mathematical framework and extended in a comprehensive manner. The major extensions include incorporation of the frequency-dependent eigensolution and an analysis of the combination of simultaneous active and passive patches. This article also expands the work of Lam et al. [9] from a thin beam to a thin plate. The proposed formulation considers both spectrally-varying and spectrally-constant properties to quantify the effect of frequency-dependence. To account for the unknown effects of multiple patches in arbitrary locations, efforts are made in the formulation to maintain generality in terms of patch number, location, and size (e.g. \(N_p\) patches located at \((x_p, y_p)\), etc. as illustrated in Fig. 1). Finally, a tractable and computationally efficient model is desired, and thus, equations should be expressed succinctly (i.e. in matrix form) and should lend themselves to direct investigation of the interaction between the patches.

The Rayleigh–Ritz method is selected as it would allow for evaluation of small, discrete passive damping patches, and to employ this model to determine the interaction between the active and passive patches. To accomplish this, two prior analytical models [17,18] will be brought together under a common mathematical framework and extended in a comprehensive manner. The major extensions include incorporation of the frequency-dependent eigensolution and an analysis of the combination of simultaneous active and passive patches. This article also expands the work of Lam et al. [9] from a thin beam to a thin plate. The proposed formulation considers both spectrally-varying and spectrally-constant properties to quantify the effect of frequency-dependence. To account for the unknown effects of multiple patches in arbitrary locations, efforts are made in the formulation to maintain generality in terms of patch number, location, and size (e.g. \(N_p\) patches located at \((x_p, y_p)\), etc. as illustrated in Fig. 1). Finally, a tractable and computationally efficient model is desired, and thus, equations should be expressed succinctly (i.e. in matrix form) and should lend themselves to direct investigation of the interaction between the patches.

The Rayleigh–Ritz method is selected as it would allow for evaluation of small, discrete passive damping patches as shown by Kung and Singh for beams [3] and plates [18]. The method would also allow for generic, non-conservative forcing [12], which is currently absent from those prior studies. The Kung and Singh method [18] is extended to include a disturbance force and active patches following the piezoelectric actuator patch model from Crawley and de Luis [2] and Clark et al. [19] (which has been extended to plates by Dimitriadis et al. [17] and Fuller et al. [20]). Experimental validation with a laboratory experiment is carried out by comparing spectral responses and modal loss factors, and the active/passive patch interaction for both broadband and single-mode responses is determined in terms of insertion loss and effect on control parameters.

Primary assumptions for analytical portions of the proposed research are as follows. The system is assumed to be linear time-invariant, implying superposition and therefore active reduction by destructive interference is valid [21]. Thin plate theory is used and rotary inertia of all layers (as well as shear deformation in the elastic layers) is ignored. Perfect adhesion is assumed between patches and material substructure (i.e. continuity of strain), and bonding layer thickness is neglected. Furthermore, it is assumed that the presence of the active patches (which are located on the top and bottom of the plate) does not significantly alter the material properties of the base structure. All materials are also presumed to have spatially uniform properties which are spectrally-
invariant (except the viscoelastic layer). Finally, the structural response is at steady state and analyses are performed only in the frequency domain (though equations of motion are derived in the time domain).

3. Development of a refined analytical model

3.1. Equations of motion from energy formulation

The motion of the damped plate (including base layer and passive damping patches) can be described by three $5 \times 1$ generalized displacement vectors ($r_i$, $i=1, 2, 3$) of the following functions, as suggested by Kung and Singh [18]:

$$
\begin{align*}
  r_1^q &= \begin{cases} 
  w \
  u_i \
  v_i \
  \psi_{xz,1} \
  \psi_{yz,1} 
  \end{cases}, \\
  r_2^q &= \begin{cases} 
  w \
  u_i \
  v_i \
  \psi_{xz,2} \
  \psi_{yz,2} 
  \end{cases}, \\
  r_3^q &= \begin{cases} 
  w \
  u_3 \
  v_3 \
  \psi_{xz,3} \
  \psi_{yz,3} 
  \end{cases}, \\
  q &= 1, ..., N_p, 
\end{align*}
$$

(1a-c)

where $w$ is transverse ($z$-direction) motion, $u$ and $v$ are in-plane motion (in $x$- and $y$-directions, respectively), and $\psi_{xz}$ and $\psi_{yz}$ are rotations in the $x$-$z$ and $y$-$z$ planes, respectively. These displacement vectors account for all motions of the base layer (3) as well as layers 1 and 2 for each of the $N_p$ passive patches. The deformations of the three layers in the $x$-$z$ plane are depicted in Fig. 2 (the same relationships hold in the $y$-$z$ plane, except that all $x$ subscripts/variables are replaced with $y$ and in-plane translation $u$ is replaced with $v$). Note that the flexural displacement, $w$, is assumed to be equal for all layers. Also, direct shear deformations in layers 1 and 3 will be assumed negligible (since $G_2=0.3G_1$, $G_3$ where $G$ is the shear modulus). As such, in the first and third layers, the total rotation $\psi$ is equal to the partial derivative of flexure ($\psi_{xz,1}=\psi_{xz,3}=\partial w/\partial x$), whereas in the second layer, the total rotation is defined as $\psi_{xz,2}=\partial w/\partial x-\gamma_{xz,2}$, where $\gamma_{xz}$ is shear strain in the $x$-$z$ plane. The same holds in the $y$-$z$ plane.

The kinetic energy in all layers and for all variables is written compactly in matrix form as

$$
T = \sum_{q=1}^{N_p} \int_0^{L_p} \int_0^{L_p} \left[ \frac{1}{2} (Dr_i^q)^T H_i (Dr_i^q) + \frac{1}{2} (Dr_j^q)^T H_j (Dr_j^q) \right] dx dy + \int_0^{L_p} \int_0^{L_p} \frac{1}{2} (Dr_3^q)^T H_3 r_3 dx dy, 
$$

(2)

where $H$ is a generalized inertia matrix given by

$$
H_i = \rho_i \begin{bmatrix} h_i & 0 \\ h_i & (h_i)^3/12 \\ 0 & (h_i)^3/12 \end{bmatrix}, \\
i = 1, 2, 3.
$$

(3)

Here, integrals for layer 3 are over the entire plate, integrals for layers 1 and 2 are over the $q$th patch area ($A^q_i=L_p^q \times L_p^q$) and summed over all $N_p$ patches, the $T$ operator denotes matrix transpose, and the $\cdot$ operator denotes differentiation with respect to the temporal coordinate.

The strain energy (potential energy) is written similarly as

$$
U = \sum_{q=1}^{N_p} \int_0^{L_p} \int_0^{L_p} \left[ \frac{1}{2} (Dr_i^q)^T E_i (Dr_i^q) + \frac{1}{2} (Dr_j^q)^T E_j (Dr_j^q) \right] dx dy + \int_0^{L_p} \int_0^{L_p} \frac{1}{2} (Dr_3^q)^T E_3 (Dr_3^q) dx dy, 
$$

(4)

where $E$ is a generalized elasticity matrix given by

$$
E_i = \begin{bmatrix} E_{xx,i} & 0 & 0 \\ 0 & E_{yy,i} & 0 \\ 0 & 0 & E_{zz,i} \end{bmatrix}, \\
E_{xx,i} = \frac{E_i (h_i)^3}{12 (1-\nu_i^2)}, \\
E_{yy,i} = \frac{E_i (h_i)^3}{12 (1-\nu_i^2)}, \\
E_{zz,i} = \frac{E_i (h_i)^3}{12 (1-\nu_i^2)}, \\
i = 1, 2, 3.
$$

(5a-d)
where \( \nu \) is the Poisson’s ratio and \( D \) is a differential operator defined by

\[
P^T = \begin{bmatrix}
\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 & 0 & \partial/\partial x & \partial/\partial y \\
0 & 0 & 0 & \partial/\partial x & 0 & \partial/\partial y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial/\partial y & 0 & \partial/\partial x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]  

(6)

The \( E \) matrix may be complex-valued and frequency-dependent (though neglected by Kung and Singh [18]) if appropriate moduli \( (E_2(\omega) \) or \( G_2(\omega) \)) for the viscoelastic layer are used. Note that for the 1st layer, using the relation \( E=2G(1+\nu) \), the modulus matrix can be written as \( E_i(\omega) = E(\omega)[1+j\eta(\omega)]E_i \) where \( E_i \) is real-valued and spectrally-invariant (provided that \( \nu \) is spectrally-invariant).

3.2. Rayleigh–Ritz scheme

In order to determine the unknown motions contained in the displacement vectors \( \mathbf{r}_i \), a weighted summation of \( N_s \) shape functions is assumed:

\[
\begin{align*}
\mathbf{r}_1^i(x, y, t) &= S_1^i(x, y)q(t) = e^{i\omega t} S_1^i(x, y)q, \\
\mathbf{r}_2^i(x, y, t) &= S_2^i(x, y)q(t) = e^{i\omega t} S_2^i(x, y)q, \\
\mathbf{r}_3^i(x, y, t) &= S_3^i(x, y)q(t) = e^{i\omega t} S_3^i(x, y)q,
\end{align*}
\]

(7a–c)

where \( S_i^j \) is a full \( 5 \times N_s \) matrix of shape functions for the \( i \)th layer (at the \( q \)th passive patch):

\[
S_i^j = \begin{bmatrix} S_{i,1}^j & \ldots & S_{i,k}^j & \ldots & S_{i,N_s}^j \end{bmatrix} = \begin{bmatrix} \Phi_{r_1} & \Phi_{r_2} & \Phi_{r_3} & \Phi_{r_4} & \Phi_{r_5} \end{bmatrix}^T,
\]

\[
S_{i,k}^j(x, y) = \{ \phi_{w,k} \phi_{w,k} \phi_{w,k} \phi_{w,k} \phi_{w,k} \}^T, \\
i = 1, 2, 3; \quad q = 1, \ldots, N_p \quad \text{for} \quad i = 1, 2; \quad k = 1, \ldots, N_s.
\]

(8a, b)

Here, \( \Phi \) are \( N_s \times 1 \) vectors of shape functions, \( \phi_{w,k}(x, y) \) is the \( k \)th shape function corresponding to a certain type of motion (i.e. flexure: \( \alpha = w \), etc.), and \( q(t) = q e^{i\omega t} \) is a \( N_s \times 1 \) vector of weighting functions (where \( q \) is a constant-amplitude vector if pure harmonic response is assumed). Note that for all three layers, the same weighting coefficients \( q \) are used, meaning the dimension of the problem is only of order \( N_s \), the number of assumed flexural shape functions, \( \phi_w \). It was shown by Kung and Singh [18] that through use of kinematic relationships and a secondary minimization scheme, the in-plane and rotational shape functions \( (\phi_{un}, \phi_{un}, \phi_{un}, \text{and} \phi_{un}) \) for all three layers can be written in terms of just the flexural shape functions \( \phi_w \), effectively reducing the order of the problem from \( (8N_p + 5)N_s \) to \( N_s \). The kinematic relationships and the minimization scheme [18] are not presented here. Suffice it to say, after the proper mathematics are implemented, all the shape functions can be generated from \( N_s \) flexural shape functions and \( 2N_s \) trial functions for the in-plane motion which must only satisfy the free boundary conditions (namely cosine functions). As the system in question is a fully free plate, flexural shape functions are assumed to be separable (i.e. \( \phi_w(x, y) = X_m(x)Y_n(y) \)) and to consist of the free–free beam functions. Such admissible shape functions are written in a reduced \( N_s \times N_s \) matrix \( \Phi \) (where \( N_s \) is the number of shape functions in the \( x \)-direction, \( N_s \) is the number of shape functions in the \( y \)-direction, and \( N_s \times N_s \) is the total number of shape functions):

\[
\Phi_{uv}(x, y) = \begin{bmatrix} \phi_{w1}(x, y) & \phi_{w2}(x, y) & \ldots & \phi_{w1}(x, y) & \phi_{w2}(x, y) & \ldots \end{bmatrix} = \begin{bmatrix} X_1(x)Y_1(y) & X_1(x)Y_2(y) & \ldots & X_2(x)Y_1(y) & X_2(x)Y_2(y) & \ldots \end{bmatrix}.
\]

(9)

Here, \( X_m(x) \) are the free–free beam modes given by the following, where the \( m \)th shape function has \((m-1)\) nodal lines:

\[
X_m(x) = \begin{cases} \frac{1}{r_1}, & m = 1 \\ \frac{2\pi}{r_2} \left[ \frac{1}{x} - \frac{1}{x^2} \right] - \sin \left( \frac{2\pi x}{L_x} \right) \bigg|_{x = \frac{L_x}{2}}, & m = 2 \\ \frac{r_1}{r_3} \left[ \cosh \beta_m L_x - \cos \beta_m L_x \left( \sinh \beta_m L_x + \sin \beta_m L_x \right) \right] - \left( \sinh \beta_m L_x - \sin \beta_m L_x \right) \left( \cosh \beta_m L_x + \cos \beta_m L_x \right), & 2 < m \leq N_s. \end{cases}
\]

(10a–c)

Note \( X_1 \) and \( X_2 \) are unique to plate motion where deformation can occur independently in each direction. Normalization constant \( C_m \) and wavenumber \( \beta_m \) are given by

\[
C_m = \sqrt{\int_0^{L_x} (X_m(x))^2 \, dx}, \quad \cos \beta_m L_x \cosh \beta_m L_x = 1.
\]

(11a, b)
Shape functions $Y_n(y)$ are defined similarly. The matrix $\Phi$ can then be mapped to a vector such that the $(m,n)$th element of the matrix becomes the $k = (m-1)N_x + n$ element of the vector $(m=1, ..., N_x; n=1, ..., N_y; k=1, ..., N_s)$:

$$\Phi_w(x,y) = \left( \Phi_{w,1}(x,y) \cdots \Phi_{w,k}(x,y) \cdots \Phi_{w,N_t}(x,y) \right)^T.$$  (12)

Based on [18], the shape function vectors for in-plane motion and rotation ($\Phi_u$, $\Phi_v$, $\Phi_{\psi_x}$, $\Phi_{\psi_y}$) are computed from $\Phi_w$ and written in the same way. Finally, substituting Eq. (12) (and the other resulting shape functions) into Eqs. (8a, b), and the result into Eqs. (7a–c), Eqs. (2) and (4) are written as

$$T = \frac{1}{2} q^T M q, \quad U = \frac{1}{2} q^T K q,$$  (13a, b)

where the corresponding matrices $M$ and $K$ (both $N_x \times N_y$) are defined as

$$M = \sum_{q=1}^{N_x} \int_{0}^{l_y} \left( \int_{0}^{l_x} \left[ S_1^T H_1 S_1 + S_2^T H_2 S_2 \right] dx \right. dy + \left. \int_{0}^{l_x} \int_{0}^{l_y} S_1^T H_1 S_1 dx dy \right),$$

$$K = \sum_{q=1}^{N_x} \int_{0}^{l_y} \left( \int_{0}^{l_x} \left[ (D S_1)^T E_1 (D S_1) + (D S_2)^T E_2 (D S_2) \right] dx \right. dy + \left. \int_{0}^{l_x} \int_{0}^{l_y} (D S_1)^T E_1 (D S_1) dx dy \right).$$  (14a, b)

### 3.3. Complex frequency-dependent eigensolution

Applying Lagrange’s equation to Eqs. (13a, b), the resulting matrix equations of motion are $M q + K q = 0$, which lead to the following for harmonic response:

$$[ -\omega^2 M + \tilde{K} ] q = 0.$$  (15)

The $N_x \times N_y$ complex eigenvalue problem (for free vibration response) is formulated from Eq. (15). The eigenvectors of $[M^{-1} \tilde{K}]$ correspond to the normal modes of the system and the eigenvalues $\tilde{\lambda}_i$ are related to natural frequencies $\omega_i$ and modal loss factors $\eta_i$ as

$$\omega_i = \sqrt{\text{Re} [\tilde{\lambda}_i]}, \quad \eta_i = \frac{\text{Im} [\tilde{\lambda}_i]}{\text{Re} [\tilde{\lambda}_i]}.$$  (16a, b)

If the viscoelastic material is allowed to have spectrally-varying properties, leading to $E_2 = \tilde{E}_2(\omega)$ and $K = \tilde{K}(\omega)$, the eigenvalue problem then becomes not only complex-valued but nonlinear. To address this, an iterative method is defined as shown in Fig. 3 (similar to that described by Lin and Lim [22]). Essentially, the $K$ matrix is evaluated at a single frequency; the $i$th natural frequency is estimated from the eigensolution; the $K$ matrix is re-evaluated at this frequency; and the process
is repeated until convergence is achieved. At this point, the loss factor is evaluated and the whole procedure is repeated for each desired mode. As noted in Section 3.1, the frequency-dependent properties can be factored out of the $E_2$ matrix, meaning that the integral in Eq. (14b) need be computed only once in a numerical realization of the algorithm.

4. Forced vibration response under disturbance in the presence of active patch(es)

4.1. Active patch input

Piezoelectric active patches are bonded to the surface of the structure and tend to actuate in the in-plane directions. Crawley and de Luis [2] showed that for a thin beam, if active patches are bonded to the top and bottom surface (i.e. the “bimorph” configuration) and excited out-of-phase, they induce approximately a pure moment at the patch boundaries. Dimitriadis [17] extended this formulation to the case of a thin plate, and similarly found that a rectangular piezoelectric patch with isotropic expansion induces approximately a pure line moment along all four patch boundaries. These results hold if the bonding layer between the substructure and the patch is sufficiently thin and there is perfect adhesion (i.e. continuity of strain) between the substructure and the patch. In order to describe this non-conservative input to the system, a generalized forcing vector, $Q$, must be added to the right hand side of Eq. (15) as

$$Q = \int_0^{L_x} \int_0^{L_y} F(x,y)\Phi(x,y)dx\,dy,$$

where $F(x,y)$ is a distributed non-conservative force. Here, $F$ is decomposed into orthogonal vectors, and transverse forces are multiplied by $\Phi_w$, in-plane forces are multiplied by $\Phi_x$ or $\Phi_y$, etc. For the case of the $r$th rectangular active patch, the equivalent transverse loading due to line moments can be reformulated as transverse dipole forces (as from [17]) and is given by the following (assuming excitation is harmonic with frequency $\omega$):

$$F_c(x,y,t) = \sum_{r=1}^{N_r} \left\{ M_{xc}^r \left[ \delta(x-x_{c1}^r) - \delta'(x-x_{c2}^r) \right] [H(y-y_{c1}^r) - H(y-y_{c2}^r)] ight. \\
+ M_{yc}^r \left[ H(x-x_{c1}^r) - H(x-x_{c2}^r) \right] [\delta(y-y_{c1}^r) - \delta'(y-y_{c2}^r)] \right\} e^{i(\omega t + \theta_c^r)},$$

where $M_{xc}$ and $M_{yc}$ are the moment magnitudes induced by the $r$th active patch about the $y$- and $x$-axes, $\delta'(\cdot)$ is the derivative of the Dirac delta function, $H(\cdot)$ is the Heaviside function, $x_{c1}^r$ and $x_{c2}^r$ are the endpoints of the $r$th active patch (i.e. $x_{c}^r \pm L_{xc}^r$), and $\theta_c^r$ is the phase of the voltage input to the $r$th active patch relative to some reference. From [17], the line

![Fig. 4. Accelerance for a plate with alternate passive damping patch configurations: no patches, reduced model (•••); “minimal” coverage, full model (---); “full” coverage, full model (···).](image-url)

Table 2
Active and passive patch locations.

<table>
<thead>
<tr>
<th>Location index</th>
<th>Patch location $(x, y)$</th>
<th>Patch size $(L_x, L_y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive 1</td>
<td>(0.5, 0.1)</td>
<td>(0.28, 0.15)</td>
</tr>
<tr>
<td>Passive 2</td>
<td>(0.08, 0.5)</td>
<td>(0.09, 0.44)</td>
</tr>
<tr>
<td>Passive 3</td>
<td>(0.5, 0.53)</td>
<td>(0.28, 0.15)</td>
</tr>
<tr>
<td>Passive “minimal”</td>
<td>(0.5, 0.5)</td>
<td>(0.02, 0.02)</td>
</tr>
<tr>
<td>Passive “full”</td>
<td>(0.5, 0.5)</td>
<td>(0.98, 0.98)</td>
</tr>
<tr>
<td>Active 1</td>
<td>(0.5, 0.9)</td>
<td>(0.101, 0.092)</td>
</tr>
</tbody>
</table>
moment magnitudes at the patch boundaries are given to be
\[
M_{dxc}^r = \left(\frac{-Eh^2}{6} \right) \left(\frac{1 - \nu_6}{1 - \nu_3} \right) \left(1 + \nu_3 - (1 + \nu_3)P\right) \left(\frac{d_{31}V_c}{h_c}\right) \left(\frac{\left|U_{xc}\right|^2}{L_x}\right),
\]
\[
P = \frac{6Eh^2}{E^3} \left(\frac{1 - \nu_2^2}{1 - \nu_2^2} \right) h_d h_3 (h_d + h_3) \left(\frac{h_3^2 + 8h_3^2}{h_3^2 + 8h_3^2} + 6h_1 h_1^2\right),
\]
where \(d_{31}\) is the piezoelectric constant (strain per voltage), and \(V_c\) is the input voltage to the active patch. Moment magnitudes \(M_{dxc}^r\) are defined similarly. The active patches used for this research actuate uni-axially, so for patches aligned with the x-axis, \(M_{dxc} = 0\) and \(M_{dyc} \neq 0\). Thus the control effort (i.e. moment strength) from the active patches is a nonlinear function of material properties and geometry but may be assumed to be directly proportional to the control voltage, \(V_c\).

Inserting Eq. (18) into Eq. (17) (with \(\Phi = \Phi_w\)) and making use of the identities \(\int_0^1 \phi(x)\delta(x - x_0) dx = -\frac{d\phi}{dx}\bigg|_{x = x_0}\) and \(\int_0^1 \phi(x)H(x - x_0) dx = \int_{x_c}^{x_c} \phi(x) dx\) [23], the resulting generalized force vector due to the active patches is
\[
\mathbf{Q}_c = \left\{ Q_{c,1} \quad \ldots \quad Q_{c,k} \quad \ldots \quad Q_{c,N_c} \right\}^T,
\]
\[
Q_{c,k} = \sum_{r = 1}^{N_c} \left(M_{dxc}^r \left[ X_m(x_{c2}) - X_m(x_{c1}) \right] \int_{x_{c1}}^{x_{c2}} Y_n(y) \ dy \right) + M_{dyc}^r \left[ Y_n'(x_{c2}) - Y_n'(x_{c1}) \right] \int_{x_{c1}}^{x_{c2}} X_m(x) \ dx \right) e^{\theta_k},
\]
where the mapping \(k = (m-1)N_y + n\) is again used and the ' operator denotes differentiation with respect to the spatial coordinate.

4.2. Disturbance input

The disturbance input is assumed to be a harmonic point load oriented normal to the plate as
\[
F_d(x, y, t) = F_d \delta(x - x_d)\delta(y - y_d) e^{i(\omega t + \theta_d)},
\]
where \(F_d\) is the disturbance force magnitude, \((x_d, y_d)\) is the location of the disturbance input, and \(\theta_d\) is the phase of the disturbance signal relative to some reference. Inserting Eq. (21) into Eq. (17) (with \(\Phi = \Phi_w\)), the resulting generalized force vector due to the disturbance is
\[
\mathbf{Q}_d = \left\{ Q_{d,1} \quad \ldots \quad Q_{d,k} \quad \ldots \quad Q_{d,N_d} \right\}^T,
\]
\[
Q_{d,k} = F_d \phi_k(x_d, y_d) e^{i\theta_k}.
\]

The full matrix equation of motion for the system in question can then be given by
\[
\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}_d + \mathbf{Q}_c.
\]

Assuming harmonic input and response (and allowing for complex modulus, frequency-dependent properties, and phase-linked inputs), the steady-state response vector \(\mathbf{q}\) as a function of \(\omega\) is
\[
\dot{\mathbf{q}}(\omega) = \left(\mathbf{K}(\omega) - \omega^2 \mathbf{M}\right)^{-1} \left[\ddot{\mathbf{Q}}_d + \dot{\mathbf{Q}}_c\right],
\]
where \(\mathbf{M}, \mathbf{K}, \dot{\mathbf{Q}}_c\), and \(\ddot{\mathbf{Q}}_d\) are given by Eqs. (14a,b), (20), and (22), respectively. Actual motions of the structure, given by the vector of displacements, \(\mathbf{r}\), are then calculated from the known shape functions and the calculated weighting coefficients, \(\mathbf{q}\), using Eqs. (7a–c). Here, since \(\mathbf{q}\) must be evaluated at a single frequency, frequency-dependent stiffness \(\mathbf{K}(\omega)\) can be used directly with no iteration needed.

5. Limiting cases of the analytical model

If only passive damping is of interest, the forced response of the structure can be obtained from Eq. (24) by simply setting \(F_c\) (and effectively \(\dot{\mathbf{Q}}_c\)) equal to zero. Because the active patches are assumed to affect the mass and stiffness distribution of the base layer (\(\rho_3(x, y)\) and \(E_3(x, y)\)) only negligibly, the left hand side of Eq. (23) is unchanged. The transverse acceleration, \(\ddot{A}(\omega) = \frac{\ddot{\mathbf{A}}(\omega)}{F_d}(\omega) = -\omega^2 \frac{\mathbf{K}(\omega)}{F_d}(\omega)\), of a plate is shown in Fig. 4 with two limiting cases of passive damping patches: “minimal” coverage with \(L_{xp} = L_{yp} = 0.02\) and “full” coverage, with \(L_{xp} = L_{yp} = 0.98\) (detailed in Table 2). The disturbance input force is located at \(x_d = y_d = 0.45\) and measurement location is \(x_0 = y_0 = 0.025\). In both cases, \(N_p = 1\), \(x_p = y_p = 0.5\), and over-bars denote normalization by plate dimension (i.e. \(x_p = x_p/L_x\)). The disturbance force location is chosen near the center of the plate to excite even-numbered modes (such as the \((2, 2)\) mode) with large displacement at the center, but is slightly offset so that odd-numbered modes (for instance the \((1, 3)\) mode) with nodal lines through the plate center can still be excited. The measurement location is selected near the corner of the plate since free boundaries imply that there will be significant motion for all modes. Problems with edge effects are overcome by employing the hyperbolic terms in the shape functions in
Eq. (10). Observe from Fig. 4 that the model behaves as expected, with increased damping acting to reduce peaks in the accelerance magnitude spectrum.

If only active damping is of interest, the limiting case of the formulation is significantly reduced. Because there is now only one layer, and since flexural deformation is the dominant motion (shear in layer 3 was ignored previously and extension was only needed to compute the shear in layer 2), \( w \) is the only deformation variable of interest. Eqs. (7) and (8) then reduce to

\[
\mathbf{w}(x, y, t) = \mathbf{S}(x, y)\mathbf{q}(t) = e^{j\omega t} \mathbf{S}(x, y)\mathbf{q}, \quad \mathbf{S} = \begin{bmatrix} \phi_{w,1} & \cdots & \phi_{w,k} & \cdots & \phi_{w,N} \end{bmatrix}.
\]

The matrices \( \mathbf{H}_3, \mathbf{E}_3, \) and \( \mathbf{D} \) are also effectively reduced to \((1 \times 1), (3 \times 3), \) and \((3 \times 1), \) respectively. The Rayleigh–Ritz scheme is otherwise identical as the same flexural shape functions are used, other than the omission of the secondary minimization scheme (as this was for in-plane motions). Also, the right hand side of Eq. (23) is unchanged other than the dimension of the generalized force vectors.

Acceleration due to an active patch is normalized by input strain and defined as:

\[
\tilde{H}_c(\omega) = \tilde{\mathbf{H}}(\omega) = -\omega^2 \tilde{\mathbf{Y}}(\omega)
\]

where piezoelectric strain \( \varepsilon_{c} = d_{31} V_c \) for uniaxial extension. The normalized \( A(\omega) \) and \( H_c(\omega) \) transfer function magnitudes for a plate with only active patches are plotted in Fig. 5. The reduced model described by Eq. (25) in which only a single layer (no passive patches) is considered should be approximately equal to the “negligibly” small patch limiting case of the full model. The accelerance from the reduced model is also shown in Fig. 4. The reduced model well approximates the full model with minimally sized patches, and is therefore sufficient for this limiting case.

6. Modal characterization and computational verification of the analytical model

6.1. Physical system properties and modal characterization

The previously described model is applied to a physical system (aluminum base plate, \( L_x = 277 \text{ mm}, L_y = 174 \text{ mm} \)) with material properties as listed in Table 1. Material properties are obtained from direct measurement (\( \rho \) and \( h \)), iterative comparison with finite element models (\( E \)), standard tabulated values (\( \nu, d_{31}, \) and \( E_c \)), and other indirect measurements or bench-type experiments (\( \eta \) and \( G_2 \)) as detailed in Sections 7 and 8.1. The active and passive patch locations for several cases of size and location are also detailed in Table 2, where patch lengths and locations are normalized by plate dimensions.
The first 10 natural frequencies of the base structure (with neither active nor passive patches) are computed using Eqs. (15) and (16) and are tabulated, along with corresponding modal indices, in Table 3. Note that the analytical model gives typically higher values for natural frequency than experiment or the finite element method; this is as expected since the Rayleigh–Ritz method overestimates natural frequencies, which approach asymptotic limits with increasing number of shape functions, $N_s$ [12]. For this study, $N_x = N_y = 8$ separable shape functions will be used, resulting in $N_s = 64$. Here the modal indices, $(m, n)$, correspond to the number of nodal lines parallel to the $y$-axis and to the $x$-axis, respectively. Modal indices can be determined based on a shape-function participation factor. For the $i$th mode (with eigenvalue $\lambda_i$), there is an associated eigenvector, $\Lambda_i$ of size $N_s \times 1$. Each element of $\Lambda_i$ is a weighting coefficient for one of the $N_s$ shape functions, and if an element (say the $k$th) of $\Lambda_i$ is dominant, then the $i$th mode shape tends to resemble the $k$th shape function (and the chosen shape functions were appropriate for the system and boundary conditions). This is formalized by defining a shape-function participation factor (similar to a modal participation factor) $\Gamma_{i,(m,n)}$ of the $(m, n)$th shape function from the matrix in Eq. (9) at the $i$th mode:

$$
\Gamma_{i,(m,n)} = \frac{\left| \tilde{A}_i(k) \right|}{\sum_{r=1}^{N_s} \left| \tilde{A}_i(r) \right|} = \frac{\left| \tilde{A}_i((m-1)N_y + n) \right|}{\sum_{r=1}^{N_s} \left| \tilde{A}_i(r) \right|},
$$

where the mapping $k=(m-1)N_y + n$ is again used to map the $(m, n)$th element of an $N_x \times N_y$ matrix to an $N_s \times 1$ vector. Here, a shape function can be considered dominant if its $\Gamma$ value is approximately 0.5 or more. In such a case, the $i$th mode can be given a modal index $(m, n)$ where $\max(\Gamma_i) = \Gamma_{i,(m+1, n+1)}$ to maintain consistency with $m$ and $n$ referring to number of

![Fig. 6. Accelerance of an undamped plate (modes of interest circled): (a) analytical model (---) vs. computational model (···); (b) analytical model (---) vs. experiment (····), configuration 1.](image-url)
The modes of interest for this article are the (2, 2) mode and the (4, 0) mode (the 8th and 9th modes from Table 3). From Eq. (26), these modes have \( \Gamma_{8, (2, 2)} \) and \( \Gamma_{9, (4, 0)} \) equal to 0.56 and 1.0, respectively, so the designated modal indices are representative of the mode shapes.

Fig. 7. Schematics of three experiments: (a) bench test for material property identification; (b) freely suspended plate with impact hammer; and (c) plate mounted with shaker.

Fig. 8. Frequency-dependent viscoelastic material properties, measured (---) vs. linear regression (---): (a) shear modulus and (b) loss factor.
6.2. Finite element model

A finite element model of the system is developed using the Abaqus [16] commercial finite element software to verify a simple case of the proposed formulation. The eigenvalue problem is solved for a plate consisting of 2D shell elements with no patches (aluminum loss factor of 0.0013 is used, as obtained in Section 8.1) and the natural frequencies are listed in Table 3. The forced response model includes a harmonic point force loading at \( x_d = y_d = 0 \) and transverse accelerance measured at \( x_0 = y_0 = 0.25 \) (reasoning for these locations has been discussed in Section 5). The forced response from the analytical model and the finite element model are plotted in Fig. 6a. The modes of interest (8th mode, (2, 2), and 9th mode, (4, 0)) are circled in Fig. 6a as well. These modes are selected since they are adjacent, well isolated with relatively low modal damping, and have well-defined nodal lines. This allows for judicious placement of patches relative to the mode shapes. Furthermore, the active patches are known to provide more energy to the structure at higher frequencies (see Fig. 5). There is generally good agreement between the finite element model and theory at most of the modes over the frequency range of interest. The few discrepancies occur mostly in terms of peak value (controlled by damping) and natural frequency (as mentioned previously, the analytical method provides an upper bound). Furthermore, the reduced version of the proposed formulation here neglects any in-plane or rotational motion and assumes only transverse displacement, whereas the shell elements in Abaqus account for in-plane motion. The model is also computationally efficient compared to FEM as it requires on the order of minutes vs. hours to compute a 5000 frequency-point forced response while giving similar results.

7. Viscoelastic material property identification

A bench experiment is designed to estimate unknown material properties of the viscoelastic adhesive layer and to quantify the frequency-dependence of these properties. This experiment consists of a traditional direct double-shear test where the viscoelastic adhesive layer is used to bond a free mass between two grounded masses (as shown in Fig. 7a). Harmonic excitation is applied to the free mass from a shaker and acceleration of the free mass is measured. Accelerance is measured and related to dynamic stiffness by \( \tilde{K}(\omega) = -\omega^2 / \tilde{A}(\omega) \), assuming pure harmonic input and response. Assuming
where \( K(\omega) \) is the effective stiffness of the adhesive, \( \eta \) is the viscoelastic loss factor, and \( m \) is the free mass. The frequency-dependent stiffness and loss factor can be extracted as:

\[
K(\omega) = \frac{\text{Re} \left( \tilde{K}(\omega) \right)}{\text{Im} \left( \tilde{K}(\omega) \right)} = \frac{k(\omega) + \eta \omega}{m \omega^2}
\]

and

\[
\eta(\omega) = \frac{\text{Im} \left( \tilde{K}(\omega) \right)}{\text{Re} \left( \tilde{K}(\omega) \right) + m \omega^2}
\]

The shear modulus of the viscoelastic material is then related to the stiffness as

\[
G(\omega) = \frac{k(\omega)}{h}
\]

where \( h \) is the thickness of the strip and \( A \) is the area. The measured \( G(\omega) \) and \( \eta(\omega) \), along with linear regressions, are depicted as a function of frequency (up to 1000 Hz) in Fig. 8a and b, respectively, and their static values and frequency variation are provided in Table 1. Static values of 2.2 MPa and 1.25 for \( G \) and \( \eta \), respectively, are reasonable in comparison to literature [3].

8. Experimental validation of proposed formulation

8.1. Experiment design and undamped plate

An experiment is designed and constructed based on the theoretical model as follows. The accelerance of the plate under free boundaries (freely suspended, termed as “configuration 1” and shown in Fig. 7b) with no patches is measured at \( x_0 = y_0 = 0.025 \) with excitation from an impact hammer at \( x_d = y_d = 0.45 \). The natural frequencies are extracted and tabulated in Table 3. A structural loss factor for aluminum of 0.0013 is assumed based on the mean of the first 9 modal loss factors (omitting the 1st and 7th modes due to poor agreement between successive trials). Modal damping ratios, \( \zeta_m \), are extracted from the spectral data using a least-squares parameter estimation method [25] in the LMS Test.Lab software [26] and related to modal loss factors (\( \eta_m \approx 2\zeta_m^2 \)) with the at-resonance approximation. The standard deviation of loss factor among the 7 modes was 0.00057, or 44 percent of the average \( \eta \) value. The experimental modes correlate well with the model from their respective accelerance spectra, plotted in Fig. 6b. There is some minor frequency shift to the left due to mass loading from the accelerometer however the agreement is in general good, especially in terms of peak value.
To validate the passive damping model, $A(\omega)$ is measured from experimental configuration 1 (with passive patches in location 1) and modal loss factors are extracted. Location 1 is specifically chosen so as to enhance the damping for the modes of interest, (2, 2) and (4, 0). Placement of passive patches at locations of large strain for a particular mode is most effective for increasing the modal loss factor for that mode [27]. Passive patch location 1 corresponds to points of large strain in the $x$-direction ($\epsilon_x = \frac{\partial^2 w}{\partial x^2}$) for both the modes of interest. The modal loss factors, as well as the assumed structural loss factor, are given in Fig. 9, along with the damped and undamped spectra. The passive damping patch results in a significant increase in $\eta$ for the 8th and 9th modes as well as corresponding attenuation of the peaks in the spectrum. Identical experiments are performed for passive damping patches in locations 2 and 3, chosen to address the (2, 2) mode and the (4, 0) mode, respectively. The model is exercised for the 3 passive patch locations using the frequency-dependent material properties. Analytical and experimental modal loss factors are compared for patch locations 2 and 3 in Fig. 10, along with ±20 percent error bars for experimental values. The proposed theory is able to predict the trends in modal loss factor for a given patch location. The agreement seems to be closer at higher modes than at lower, where the analytical model overpredicts the loss factor more significantly. In particular, the expected trends are seen for the modes of interest; a significant increase is seen in $\eta_9$ whereas $\eta_8$ is relatively unchanged, or vice versa.
An error analysis can be performed for $N$ modes by defining the error metric $\xi = \frac{1}{N} \sum_{i=1}^{N} |\eta_{\text{an},i} - \eta_{\text{exp},i}|$, where subscripts “an” and “exp” refer to analytically and experimentally determined, respectively. For passive patch locations 1–3, $\xi = 0.0023$, 0.0023, and 0.0011, respectively, which are approximately the same order as the loss factor of the undamped plate. For every mode compared, the model predicts a slightly higher $\eta$ value than experimentally observed. This is likely due to the fact that the proposed formulation assumes layers 1–3 all have equal deformation in flexure (i.e. $w_1 = w_2 = w_3 = w$). Experimentally, layer 1 may deform less than layer 3 in flexure, resulting in less strain energy being dissipated from the viscoelastic layer and thus a lower effective material $\eta$. If spectrally-constant viscoelastic material properties are used in the model (nominally $G = 6$ MPa, $\eta = 1$), then some $\eta$ values are predicted to be larger than observed while others are smaller than observed. This suggests that spectrally-varying properties result in systematic error while spectrally-constant properties result in random error, lending further credibility to the utility of frequency-dependent properties.

### 8.3. Active patch model validation from measured transfer functions

The vibratory response of the aluminum plate is measured with an accelerometer for experimental configuration 1 with the active patch in location 1 as the only input. The acceleration per input strain transfer function ($H_c$) is measured using a strain gage placed next to the active patch. The vibratory response is plotted in Fig. 11a, showing good agreement between
the theory and measurement, particularly at the modes of interest. The modes in the 500–1000 Hz range do not appear in the model response because they have odd modal x-indices (meaning there is a nodal line at $x = 0.5$). Analytically, the active patch is located exactly at the center of the plate and therefore it cannot excite such modes. In the experiment, however, this patch is likely slightly off-center, meaning these modes are weakly excited. If $\chi_c$ is set to $0.5 + \epsilon$ in the analytical model, where $\epsilon$ is some small nonzero value, a result similar to the experiment is indeed observed with the odd-numbered modes showing weak modal participation. This result is displayed in Fig. 11b.

9. Active and passive patch interaction

9.1. Broadband interaction

Finally, the plate response subject to combined active and passive patches is investigated. Conceptually, the acceleration due to the active patches, $a_c$, should satisfy $\ddot{a}_c(\omega) = -\ddot{a}_d(\omega)$ to cancel vibration at $(x_0, y_0)$. A transfer function, $\Psi$, relating input force to required control strain is defined as follows, given that the $A$ and $H_c$ transfer functions are known from Section 5 to satisfy $\ddot{a}_d(\omega) = \dot{A}(\omega) \ddot{F}_d(\omega)$ and $\ddot{a}_c(\omega) = \dot{H}_c(\omega) \ddot{\epsilon}_c(\omega)$:

$$\Psi(\omega) \equiv \frac{\ddot{\epsilon}_c(\omega)}{\ddot{F}_d(\omega)} = -\frac{\dot{A}(\omega)}{\dot{H}_c(\omega)}.$$  \hspace{1cm} (28)

The $\Psi(\omega)$ transfer function is computed for 5 cases of passive damping patches (0, 1, 2, 3, and 5 patches) at the following locations to target the (4, 0) mode: $(\bar{x}_p, \bar{y}_p) = (0.5, 0.55)$, $(0.1, 0.23)$, $(0.1, 0.77)$, $(0.9, 0.23)$, and $(0.9, 0.77)$. The modal loss factor for the (4, 0) mode, $\eta_{9b}$, is tabulated in Table 4 and is found to monotonically increase with the number of patches, as expected.

The resulting $\Psi(\omega)$ transfer function is plotted in Fig. 12a. Here it is seen that at some modes (namely those with peaks in $\dot{A}(\omega)$ but not in $\dot{H}_c(\omega)$), the required control amplitude is much higher than others. Across the frequency range, passive damping tends to bring the required control amplitude closer to a constant value. This would enhance real-time controller stability as large control amplitudes would not need to be generated. In fact, addition of damping would in some cases make active control possible where voltage limitations would otherwise preclude complete reduction. Furthermore, the added damping reduces the required control effort at resonances which the active patch is only able to weakly excite. Thus additional passive damping should improve active control at multiple modes for which the active patch locations may not have been optimized, as well as potentially reducing energy consumption from the active patch.
Moreover, observe that with little damping, the required control phase is typically close to $0^\circ$ or $\pm 180^\circ$, depending on whether the points $(x_c, y_c)$ and $(x_d, y_d)$ are moving in-phase or out-of-phase for a given mode. As damping is increased, the required phase is typically shifted from $0^\circ$ toward $\pm 180^\circ$ or vice versa. There are, however, some complexities near rapid $0^\circ$ to $\pm 180^\circ$ transitions (generally where participation from mode $n+1$ tends to overtake mode $n$) where the effect of increasing damping patches not only changes $\eta$ but also alters the local stiffness and mass properties, shifting the natural frequencies in a complicated way. Nevertheless, the phase shift occurs monotonically with increasing $\eta$ near the modes of interest.

Similar $\Psi(\omega)$ transfer functions (as from Eq. (28)) are experimentally determined. The broadband excitation is introduced with an electrodynamic shaker as depicted in Fig. 7c, termed as configuration 2. The result for the case of one damping patch in passive patch location 3 is shown along with the analytical prediction of $\Psi(\omega)$ in Fig. 12b. Relatively good agreement is seen between the model and the experiment, especially in terms of the shape of the magnitude and phase curves. Some discrepancy exists close to peaks and phase transitions, largely driven by a disagreement in natural frequencies (due to the upper bound nature of the Rayleigh–Ritz method and mass loading from the shaker, as shown previously). However, the transfer function trends are similar in both plots.

9.2. Single frequency interaction

Active control is then applied to the vibrating plate at a single frequency excitation close to the $(4, 0)$ mode. The disturbance excitation is in the form of a sinusoidal voltage at 1124 Hz, despite the $(4, 0)$ mode occurring in the accelerance spectrum at a higher frequency. The configuration 2 results in significant mass loading for the shaker armature and stinger/load cell assembly of approximately 0.06 kg. In this configuration, the peak acceleration is observed to occur at 1124 Hz, and laser vibrometer measurements have confirmed that most participation is indeed due to the $(4, 0)$ mode (shown in Fig. 13a). A finite element simulation with a 0.06 kg point mass added at $x_d = y_d = 0.45$ shifts the resonance from 1192 Hz to 1130 Hz, verifying the mass loading. Also Fig. 12a shows a rapid phase transition occurs close to the $(4, 0)$ mode. Thus a frequency
slightly to the left is chosen to avoid such complications. The control excitation at 1124 Hz to the active patches is chosen as 0.59 V, such that $|a_c| = |a_d| = 10.7 \text{g_{rms}}$. The relative phase between the two inputs is swept from $-180^\circ$ to $180^\circ$ and the resulting acceleration and sound pressure amplitudes are measured with the combined inputs.

The resulting acceleration (normalized by $a_d$) and sound pressure (normalized by $p_d$) are shown in Fig. 14. The active control results in nearly 50 dB reduction of acceleration and 40 dB reduction of sound pressure at the optimal phase angle ($\theta_c \approx 153^\circ$). At $180^\circ$ opposite of this phase, the response is increased by approximately 6 dB, indicating amplitude doubling, as expected from constructive interference. The model is exercised for the same case (with frequency at 1165 Hz, slightly below the (4, 0) mode, plotted in Fig. 13b), and similar trends are seen: approximately 40 dB reduction in acceleration at an optimal phase and amplitude doubling at $180^\circ$ opposite of this phase. From Fig. 15a, the proposed formulation predicts the optimal phase angle between disturbance force and control strain to be approximately $180^\circ$. The extra phase lag (from $180^\circ$ to $153^\circ$) in the experiment is due to capacitance in the active patch and amplifiers, found to be approximately $\frac{\pi}{30}$ over the entire frequency range. The experiment is repeated with 5 passive patches located on the structure as described previously. The attenuation observed is similar (approximately 42 dB), but the optimal phase angle is now shifted by $-17^\circ$, as shown in Fig. 15b. The theory predicts similar attenuation as well as a shift of $-12^\circ$ in the optimal phase angle, given in Fig. 15a.

The experiment is repeated for 5 cases with damping patches ranging from 0 to 5 patches, as summarized in Table 4. Several experimentally observed trends are also predicted by the model. First, as the number of damping patches increases, the $\eta_9$ increases monotonically both in the experiment and theory. Next, the maximum observed attenuation is typically between 40 and 50 dB for both analytical and experimental methods (notwithstanding two outliers). It should be noted that these values are highly dependent on the control signal value and on the measurement system fidelity. Nevertheless, with no real time control system ("user-in-the-loop feedback" is used), approximately 45 dB reduction is observed independent of the amount of passive damping. From this result, the active and passive methods can be considered to combine approximately on a logarithmic basis (i.e. the curves in Fig. 15a and b have similar shape but increasing damping shifts the curve down). This lends further support to use of the combined active-passive damping method. Finally, a shift in the damping-dependent optimal phase between disturbance and control signals is observed both experimentally and analytically.
10. Conclusion

This article proposes and experimentally validates a new analytical model for the vibratory response of a thin rectangular plate with concurrent active and passive damping patches in the side-by-side configuration. Previously this configuration had only been studied for beams [9]. Unlike recent hybrid damping patches studied with finite element models [10] the proposed formulation compatibly extends two prior analytical models [17,18] under a comprehensive mathematical framework, and is found to be valid for active patches, passive patches, or the combined case. This proposed theory also enhances the previous models by allowing for spectrally-varying viscoelastic material properties, and defines an iterative process to evaluate the nonlinear eigenvalue problem. The use of frequency-dependent stiffness and loss factor is found to result in systematic rather than random errors, which can be explained based on the experiment. The active patch model is also implemented in a more generic sense, as non-conservative forcing in the Rayleigh–Ritz method can be applied in any direction (not just normal to the surface, as from prior work [17,19]). This could be valuable in non-bimorph configurations where in-plane traction is induced in addition to the equivalent moment [15]. Finally, the analytical model is tractable and computationally efficient compared to FEM, and gives spatially continuous, rather than discrete, solutions for motion.

At single frequency, the combination of active and passive damping results in increased vibratory attenuation in both theory and experiment. Unlike prior literature on hybrid damping, the proposed model identifies complicated interaction between the patches in terms of the required control parameters. In particular, with increased passive damping, large control amplitude peaks are reduced, leading to improved stability. This method also potentially improves controllability of modes that would otherwise be uncontrollable due to active patch location, while yielding broadband reduction. Since the proposed formulation is limited to thin plate-like structures and assumes linear, steady-state response, more complicated geometries and nonlinear vibrations would require further work. Nevertheless, the model provides insight into the underlying physics as it suggests trends for the behavior of combined active and passive damping patches.

Acknowledgments

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Appendix A. List of symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>acceleration</td>
</tr>
<tr>
<td>A</td>
<td>area</td>
</tr>
<tr>
<td>A</td>
<td>accelarance transfer function</td>
</tr>
<tr>
<td>C</td>
<td>shape function normalization constant</td>
</tr>
<tr>
<td>dij</td>
<td>piezoelectric strain per charge constant</td>
</tr>
<tr>
<td>e</td>
<td>exponential constant</td>
</tr>
<tr>
<td>E</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>E</td>
<td>generalized elasticity matrix</td>
</tr>
<tr>
<td>f</td>
<td>frequency [Hz]</td>
</tr>
<tr>
<td>F</td>
<td>force</td>
</tr>
<tr>
<td>G</td>
<td>shear modulus</td>
</tr>
<tr>
<td>h</td>
<td>layer thickness</td>
</tr>
<tr>
<td>H</td>
<td>acceleration per unit strain transfer function</td>
</tr>
<tr>
<td>H</td>
<td>generalized inertia matrix</td>
</tr>
<tr>
<td>j</td>
<td>imaginary number, ( \sqrt{-1} )</td>
</tr>
<tr>
<td>k</td>
<td>equivalent scalar stiffness</td>
</tr>
<tr>
<td>K</td>
<td>dynamic stiffness transfer function</td>
</tr>
<tr>
<td>K</td>
<td>stiffness matrix</td>
</tr>
<tr>
<td>L</td>
<td>length</td>
</tr>
<tr>
<td>m</td>
<td>mass</td>
</tr>
<tr>
<td>M</td>
<td>moment</td>
</tr>
<tr>
<td>M</td>
<td>mass matrix</td>
</tr>
<tr>
<td>N</td>
<td>number (patches, shape functions, etc.)</td>
</tr>
<tr>
<td>p</td>
<td>sound pressure</td>
</tr>
<tr>
<td>P</td>
<td>dimensionless active patch parameter</td>
</tr>
<tr>
<td>q</td>
<td>shape function weighting vector</td>
</tr>
<tr>
<td>Q</td>
<td>generalized force vector</td>
</tr>
<tr>
<td>r</td>
<td>generalized displacement vector</td>
</tr>
<tr>
<td>S</td>
<td>full shape function vector or matrix</td>
</tr>
</tbody>
</table>
\( t \)  
kinetic energy

\( u, v \)  
lateral displacements (in \( x-, y\)-directions)

\( U \)  
potential (strain) energy

\( V \)  
voltage

\( w \)  
transverse displacement

\( x, y, z \)  
coordinates, endpoints, or locations

\( X, Y \)  
one-dimensional shape function

\( \beta \)  
wavenumber

\( \gamma \)  
shear strain

\( \Gamma \)  
modal participation factor

\( \varepsilon \)  
normal strain

\( \zeta \)  
damping ratio

\( \eta \)  
loss factor

\( \Theta \)  
phase relative to a reference

\( \lambda \)  
eigenvalue

\( \Lambda \)  
eigenvector

\( \nu \)  
Poisson's ratio

\( \xi \)  
error metric

\( \rho \)  
mass density per unit volume

\( \phi \)  
Rayleigh–Ritz shape function

\( \Phi \)  
reduced shape function vector or matrix

\( \psi \)  
rotation angle

\( \Psi \)  
control strain per unit disturbance force

\( \omega \)  
angular frequency \([\text{rad/s}]\)

**Subscripts**

\( 0 \)  
measurement location

\( 1, 2, 3 \)  
layer indices

\( c \)  
active patch (control) layer or input

\( d \)  
disturbance input

\( i \)  
index for layers and modes

\( k \)  
index for shape functions

\( m, n \)  
modal index (in \( x-, y\)-directions)

\( p \)  
passive patch

\( s \)  
shape function

\( \alpha \)  
generic displacement

\( \_ \)  
vector (under-bar)

\( = \)  
matrix (double under-bar)

**Superscripts**

\( q \)  
index for passive patches

\( r \)  
index for active patches

\( \sim \)  
complex valued

\( \sim \)  
normalized, nondimensional

**Operators**

\( D \)  
differential operator for strain energy

\( H \)  
Heaviside (unit step) function

\( T \)  
matrix transpose

\( \delta \)  
Dirac delta (unit impulse) function

\( \delta' \)  
derivative of \( \delta \) (couple or doublet)
References