Nonlinear dynamic properties of hydraulic suspension bushing with emphasis on the flow passage characteristics

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Abstract
Hydraulic bushings, which are often employed in vehicle suspension systems, exhibit significant excitation-dependent properties. However, previous analyses were mainly based on the linear system theory. To overcome this void, nonlinear characteristics of common hydraulic bushing configurations are examined in this article, with focus on the component properties as excited by sinusoidal or step displacements of various amplitudes. First, a nonlinear model for a laboratory prototype with a long passage and a short passage (in parallel) is developed using a lumped-parameter approach. Then the system parameters and nonlinearities are identified using experimental and computational methods, with an emphasis on characterization of the flow passage resistances. Steady-state harmonic and transient step experiments are conducted on the prototype, and the dynamic pressures inside two fluid chambers and the force transmitted to the base are measured. Numerical solution of the nonlinear model shows that the proposed model predicts both steady-state sinusoidal responses and transient responses well for single-passage and dual-passage configurations; significant improvement over a corresponding linear model is observed. Finally, approximate analytical and semi-analytical solutions of the nonlinear model are obtained by using the harmonic balance method.

Keywords
Suspension bushings, vibration and ride control, hydraulic device, parameter identification, analytical methods

Date received: 26 December 2013; accepted: 31 October 2014

Introduction
Hydraulic bushings are widely used in vehicle suspension systems for improved ride and handling performance and to isolate vibration and structure-borne noise. Their design features have been described in many patents but without any analytical justifications. Further, very few scholarly articles have addressed their characterization and modeling issues. For instance, Sauer and Guy have numerically simulated a hydraulic bushing with a parallel inertia track and bypass track (a controlled leakage path), but no results were provided. Lu and Ari-Gur have derived a simple expression for the natural frequency of hydraulic bushing based on a linear model, without presenting any experimental validation. Sevensson and Hakansson have proposed empirical nonlinear models, but such models are not accurate, especially at the low excitation amplitudes. Arzanpour and Golnaraghi have presented a linear model of the hydraulic engine subframe bushing, but their approach is virtually identical with that for a fixed decoupler hydraulic engine mount.

Although hydraulic bushings exhibit significant frequency-dependent and amplitude-sensitive properties, previous articles and designs essentially employed a linear system theory to examine their spectral characteristics. Conversely, hydraulic shock absorbers and hydraulic engine and transmission mounts have received much attention regarding their nonlinear properties. Therefore, this article intends to overcome this particular void in the literature and to examine rigorously the nonlinear dynamic...
properties of hydraulic bushings. Accordingly, the specific objectives (with focus on the component only) are as follows.

1. Formulate the nonlinear model of common fluid-filled bushing configurations, and identify fluid passage nonlinearities via experiments on a laboratory prototype.
2. Validate the nonlinear model by comparing harmonic and transient response predictions with measurements.

This article will also extend the linear system formulation for time domain responses as proposed in a recent publication.20

Nonlinear fluid system model

A typical hydraulic bushing usually consists of inner and outer metal sleeves, a rubber element, and two hydraulic chambers filled with a mixture of anti-freeze and water.1,2,20 The two chambers are usually connected by one or more flow passages which communicate fluid between two chambers when a relative deflection of the inner and outer metal sleeves causes the pressures to vary. Note that, although hydraulic bushings are somewhat similar to the hydraulic engine mount, their application, design, construction, and dynamic performance are different.1,2,20,21 Hydraulic mounts, which are usually equipped with fixed or free decouplers, are widely employed to support vehicle engines and to control motion mostly at lower frequencies. The excitation is applied to the mount top. Its lower chamber is very compliant, and thus the pressure in the lower chamber can usually be neglected in comparison with the pressure in the upper chamber. In contrast, fluid-filled bushings are mainly used in automotive suspension and chassis systems. They usually do not have any decoupler mechanisms. The outer metal sleeve can be assumed to be fixed in most applications, and the displacement excitation is applied to the inner metal part. The two fluid chambers are almost identical, and some bushing designs have a leakage (bypass) path in parallel with an inertia track to communicate fluid under high-amplitude deflection.1,20 Overall, hydraulic engine mount models cannot be directly applied to fluid-filled bushings because of the differences discussed above.

In the fluid-filled bushing model, the hydraulic elements are represented by several control volumes, and incompressible flow is assumed. Moreover, only radial damping is considered because the relative displacement between the inner metal part and the outer metal part is assumed to be only in the radial direction. The fluid model for a bushing with a long flow passage in parallel with a short flow passage is developed in Figure 1(a). Note that the internal long fluid passage (the inertia track) is shown as external tubing for clarity. Static displacement ($x_{st}$) and dynamic displacement ($x(t)$ from the static equilibrium) excitations are applied to the inner metal part while the outer metal sleeve is fixed relative to the inner sleeve. The rubber element $r$ is assumed to be represented by a Kelvin–Voigt model with an excitation displacement-dependent rubber stiffness $k_r(x)$ and a viscous damping coefficient $c_r(x)$. Unlike the fluid chambers in the work reported by Chai et al.,20 the two fluid chambers 1 and 2 are described by nonlinear compliance elements $C_1(p_1)$ and $C_2(p_2)$ respectively, with effective pumping areas $A_1$ and $A_2$. Since the fluid passages are usually made of hard plastic or thin rubber-coated metal and can accommodate only small fluid volumes, they are represented by nonlinear fluid inertances $I_1(q_1)$ and $I_2(q_2)$ and nonlinear fluid

![Figure 1. Lumped-parameter nonlinear model of a hydraulic bushing: (a) a fluid system model consisting of two compliance elements (denoted by # 1 and # 2), a long passage (say inertia track, # i) and a short passage with an orifice-like restriction (# s); (b) alternative rubber path formulation that could replace the nonlinear Kelvin–Voigt model of the rubber path in (a).]
Utilized to characterize bushing nonlinearities and to under a mean load volumetric flow rate. Note that \( ds \) end of a metal tube. Also, the short flow passage with its opening is controlled by a ball valve placed at the bends, etc.\(^{27} \)

Under the static condition, the mean force \( f_m \) transmitted to the outer sleeve under a static displacement \( x_m \) is given by

\[
f_m = k_s(x_m) + (A_2 - A_1)p_m
\]

where \( p_m \) is the fluid chamber pressure under static equilibrium. The dynamic displacement excitation \( x(t) \), from the static equilibrium, is applied to the bushing under a mean load \( f_m \) and it induces flow through both passages. By applying the momentum equation to each flow passage, the equations obtained are

\[
\begin{align*}
 p_1(t) - p_2(t) &= f_s(q_1(t)) + R_s(q_1(t)) \quad (2a) \\
 p_1(t) - p_2(t) &= f_s(q_2(t)) + R_s(q_2(t)) \quad (2b)
\end{align*}
\]

where \( q_1(t) \) and \( q_2(t) \) are the dynamic volumetric flow rates through the long track and the short passage respectively and \( p_1(t) \) and \( p_2(t) \) are the dynamic pressures inside the two fluid chambers 1 and 2 respectively.

The application of the continuity equation to two compliance elements yields

\[
\begin{align*}
 -A_1\ddot{x}(t) - q_1(t) - q_2(t) &= C_1(p_1(t)) \quad (3a) \\
 A_2\ddot{x}(t) + q_1(t) + q_2(t) &= C_2(p_2(t)) \quad (3b)
\end{align*}
\]

When only one passage is used (say, the long inertia track), the governing equations are obtained by setting \( q_1(t) = 0 \) and \( R_s = \infty \). With respect to the total dynamic force transmitted to the outer sleeve, \( f_T(t) \) is divided into the rubber path (subscript \( r \)) and the hydraulic path (subscript \( h \)) according to

\[
\begin{align*}
 f_T(t) &= f_m + f_{Tr}(t) + f_{Th}(t) \quad (4a) \\
 f_{Tr}(t) &= k_s(x)(x(t) + c_s(x)) \quad (4b) \\
 f_{Th}(t) &= A_2 p_2(t) - A_1 p_1(t) \quad (4c)
\end{align*}
\]

**Experimental studies with sinusoidal displacement excitation**

A laboratory device, as described by Chai et al.,\(^{20} \) is utilized to characterize bushing nonlinearities and to develop a nonlinear mathematical model. As shown in Figure 2(a),\(^{20} \) this device consists of two similar chambers filled with water and a customized mid-plate which accommodates an external long flow passage and a short flow passage. The long flow passage, which simulates the inertia track, is facilitated via an external metal tube of inside diameter \( d_i \) and length \( l_i = 9d_i \), and its opening is controlled by a ball valve placed at the end of a metal tube. Also, the short flow passage with diameter \( d_s = 3d_i/2 \) and length \( l_s = 9d_i \) is formed by utilizing a needle valve that is embedded in the mid-plate. By adjusting the orifice diameter \( d_o \) (inside the needle valve), a short passage with different flow restrictions can be constructed (see the paper by Chai et al.\(^{20} \) for more details). To simulate typical real-life fluid-filled bushing designs, five configurations of the prototype, as illustrated in Figure 2(b) to (d), are evaluated. First, configuration B1 is investigated when the two hydraulic chambers are connected by only one long passage. Second, configuration B2 is examined when the long passage is closed and the effective diameter \( d_o \) of a restriction in the short passage is set to \( d_o = 2d_i/3 \) with a fully opened needle valve. Like configuration B2, configuration B3 is formed when the restriction diameter is reduced to \( d_o = 2d_i/9 \). Note that configuration B3 is designed to simulate an abrupt reduction in the flow area via a needle valve. Finally, the combination of a long passage and a short passage in parallel is examined by using configuration B4 with \( d_o = d_o = 2d_i/9 \), and configuration B5 with \( d_o = d_o = 2d_i/9 \).

The steady-state sinusoidal experiments are first conducted on the above-mentioned configurations using a non-resonant elastomer MTS 831.50 test machine.\(^{28} \) A sinusoidal displacement excitation \( x(t) = A_s \sin (2\pi f t) \) is applied to the prototype device under a mean load, where \( A_s \) is the zero-to-peak amplitude and \( f \) is the excitation frequency (Hz). The dynamic transmitted force \( f_{dr}(t) \) and the dynamic pressures \( p_1(t) \) and \( p_2(t) \) inside the two chambers are measured from 1 Hz to 60 Hz in 1 Hz increments for two amplitudes \( X = 0.1 \) mm and \( 1.0 \) mm, where \( X = 2A_s \) is the peak-to-peak value. The magnitude \( |K_d| \) and the loss angle \( \phi_K \) of the dynamic stiffness \( K_d \) are defined as the amplitude ratio and the phase difference between \( f_{dr}(t) \) and \( x(t) \) respectively. Accordingly, define

\[
K_d(\omega) = |K_d(\omega)| \cos[\phi_K(\omega)] + i|K_d(\omega)| \sin[\phi_K(\omega)]
\]

where \( i = \sqrt{-1} \) and \( \omega = 2\pi f \) is the circular frequency (rad/s).

The measured dynamic stiffness spectra for configurations B1, B2, B3, and B4 with \( X = 0.1 \) mm and \( 1.0 \) mm are shown in Figure 3. Here \( |K_d| \) is normalized by its static stiffness \( k_{static} \). The frequency ratio \( \Omega \) is defined as the excitation frequency (Hz) divided by a reference frequency, at which the loss angle of configuration B1 achieves its maximum value (about 10 Hz). Significant amplitude and frequency dependences are observed. It can be observed that the loss angle of configuration B1 reaches its peak at \( \Omega^*_a \) as the frequency is increased and then gradually decreases with increasing \( \Omega \), while \( |K_d| \) continues to increase until the frequency reaches \( \Omega^*_K \), which corresponds to maximum \( |K_d| \), and then settles down. For configurations B2 and B4, their peak loss angles are at higher frequencies but in a broader range than for configuration B1, while the magnitude and the loss angle of configuration B3 are relatively low compared with those of other configurations. Furthermore, both \( |K_d(\omega)| \) and \( \phi_K(\omega) \) decrease for a higher value of \( X \), and their peaks become less abrupt; thus both \( \Omega^*_K \) and \( \Omega^*_a \) both decrease. Accordingly, the nonlinear system parameters must be judiciously identified to model the frequency- and amplitude-dependent...
properties. The methods and procedures for estimating the model parameters, including $k_r$ and $c_r$, the hydraulic chamber compliances $C_1$ and $C_2$, the effective pumping areas $A_1$ and $A_2$, the inertances $I_i$ and $I_s$ of each flow passage, and the resistances $R_i$, $R_o$, and $R_0$ of each fluid passage, will be discussed in detail in the following section although the focus will be on determination of the flow resistance terms.

Identification of the nonlinear system parameters

The rubber path is approximated by the Kelvin–Voigt model, as shown in Figure 1(a), with $k_r$ and $c_r$ as the rubber stiffness and the viscous damping elements respectively. To examine the rubber path properties, the dynamic stiffness $K_{dr}$ of a bushing with fluid drained is measured for $X = 0.1$ mm, 0.5 mm, and 1.0 mm (peak-to-peak value) from 1 Hz to 60 Hz, as shown in Figure 4. Observe that $|K_{dr}|$ decreases as $X$ is increased, but $\phi_r$ slightly increases. Nevertheless, both $|K_{dr}|$ and $\phi_r$ gradually increase at higher frequencies. Based on the assumption that $K_{dr} = k_r + i\omega c_r$ from the Laplace transform of $f_T(t) = k_r x(t) + c_r \dot{x}(t)$ (equation (4b)), both $k_r$ and $c_r$ are first estimated from very-low-frequency responses. Since $\phi_r$ varies by only about 1–2° from 1 Hz to 60 Hz, $c_r(X)$ is assumed to be only amplitude dependent. Accordingly, empirical values of $k_r(\omega, X)$ are estimated by interpolating the measured $K_{dr}$ data.

In order to estimate $C_1$ or $C_2$, a laboratory experiment, as shown in Figure 5(a), is carried out to measure a change $\Delta V$ in the chamber’s internal volume due to the increase $\Delta p$ in the pressure. By applying an external force to the cylinder piston with cross-sectional area $A_{piston}$, $\Delta V$ is calculated as $A_{piston} \Delta x_{piston}$, where $x_{piston}$ is the piston displacement. $\Delta p$ is measured by a pressure transducer. Then the fluid compliance around an operating point $\omega$ is estimated as $C_{\omega} \approx \Delta V/\Delta p |_{\omega}$. From the
Figure 3. Measured dynamic stiffness spectra of the four bushing configurations (a) B1, (b) B2, (c) B3, and (d) B4 for two excitation amplitudes. Key: ———, $X = 0.1$ mm; – – –, $X = 1.0$ mm.

Figure 4. Dynamic stiffnesses of the rubber path (bushing with the fluid drained) with three excitation amplitudes. Key: ———, $X = 0.1$ mm; ———, $X = 0.5$ mm; – – –, $X = 1.0$ mm.

Figure 5. Estimation of the effective chamber compliance. (a) The bench experiment for chamber compliance measurement and (b) the measured $\Delta p$ versus $\Delta V$ plot for chamber compliance under a mean load. Key: ——— measured; – – –, linear curve-fit.
measured $\Delta V$ versus $\Delta p$ data in Figure 5(b), it is seen that, as the pressure is increased from 0 kPa to around 180 kPa, the variation in $\Delta V/\Delta p$ is less than 15%. Therefore, $C_1$ and $C_2$ are assumed to be linear, and their values are estimated on the basis of a linear polynomial fit of measurements. The effective pumping areas $A_1$ and $A_2$ are examined next by employing a structural finite element code, as shown in Figure 6(a). A static load $f_m$ is applied to the top metal part of the elastomeric chamber, and the corresponding deflection $x$ and the chamber’s internal volume change $\Delta V$ are calculated. Then $A_1$ and $A_2$ are estimated using $\Delta V/x$. The results in Figure 6(b) show that the variations in $A_1$ and $A_2$ are less than 3% when the load is varied from 100 N to 1000 N. Thus, $A_1$ and $A_2$ are also assumed to be constant values.

The inertances of a long passage and a short passage are calculated as $I_l = pl_l/A_l$ and $I_s = pl_s/A_s$, respectively by assuming unsteady turbulent flow. However, the transition length of 115$d_l$ necessary to achieve a fully developed laminar flow is longer than $l_l$. In addition, turbulence may be generated around the pipe fittings and the sharp entrances or exits. Thus, a constant value $I_l = pl_l/A_l$ based on geometric $l_l$ and $A_l$ values is employed and no coefficient is applied. Similarly, the flow restriction in a short passage requires a much lower Reynolds number to develop laminar flow fully, and thus a constant $I_s = pl_s/A_s$ is assumed for the short passage as well.

Assume that the flow resistance obtained from steady flow calculations or measurements is applicable for an oscillatory flow condition at lower frequencies. A bench experiment is conducted to measure the steady flow rate $q$ through each fluid passage under a pressure differential $\Delta p$. As shown in Figure 7(a), the inlet (embedded in the mid-plate) is connected to a flow source (city water) by a smooth plastic tube, while the outlet is open to the ambient atmosphere; the properties of the anti-freeze mixture are close to those of water at room conditions. The flow source is controlled by a regulator, and the pressure at the inlet is measured with a pressure gauge. By adjusting the flow source and regulator, steady flow in a pressure range 6.9–206.8 kPa can be generated. The steady flow rate $q$ is estimated by measuring the accumulated fluid volume (using a

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**Figure 6.** Estimation of the effective pumping area: (a) finite element analysis of the fluid chamber; (b) computed $\Delta V$ versus $x$ plot from the finite element model.

**Figure 7.** Steady flow experiments (a) over a higher pressure differential range and (b) over a lower pressure differential range.
graduated cylinder) over a known time duration. In order to reduce the test error, multiple measurements of $q$ with a specific $D_p$ are taken and an averaged value is obtained. Experimental results for a long passage and a short passage with effective restriction diameters $d_{o1}$ and $d_{o2}$ are shown in Figure 8. It is evident that the relationship of $D_p$ to $q$ is nonlinear for each passage.

Accordingly, the nonlinear resistance $R_i(q_i)$ of a long passage is defined on the basis of a fully developed turbulent flow in a smooth circular pipe$^{27}$ as

$$R_i(q_i) = \frac{D_p q_i}{l_i} = \lambda_i \frac{0.242 \sqrt[0.75]{\mu^{0.25}} \rho^{0.75}}{d_i^{0.75}} q_i^{0.75}$$ (5)

where an empirical scaling factor $\lambda_i \approx 1.4$ is employed to account for momentum losses at the tube bends and fittings, and $\rho$ and $\mu$ are the water density and viscosity respectively. The nonlinear resistance $R_s(q_s)$ for a short passage is defined by using the well-known orifice formula$^{27}$ with an empirical factor $\lambda_o$

$$R_s(q_s) = \lambda_o \frac{\rho}{2C_d^2 A_{o1}} |q_s|$$ (6)

where $C_d$ is the discharge coefficient and $A_{o1} = \pi d_{o1}^2 / 4$ is the geometric cross-sectional area. When the effective diameter is reduced to $d_{o2}$, the short passage should act more like a sharp-edge orifice element, and the nonlinear resistance is

$$R_o(q_o) = \frac{\rho}{2C_d^2 A_{o2}^2} |q_o|$$ (7)

where $\lambda_o \approx 1.0$ and $C_d$ should be about 0.61 for an ideal orifice.

The $\Delta p$ versus $q$ relationship predicted by the above equations is compared with the measurements in Figure 8. For a long passage, a curve estimated by equation (7) with $C_d = 0.61$ is also compared with that from equation (5) in Figure 8(a). It is observed that equation (5) gives a much better representation of the measured data. Figure 8(b) compares the measured $\Delta p$ versus $q$ relation for a short passage with equation (6) predictions for three $\lambda_o$ values. It is evident that the ideal orifice formula ($\lambda_o = 1.0$) underestimates the
short-passage resistance. The curve with $\lambda_3=1.3$, which is a quadratic relation provided by the needle valve supplier,\textsuperscript{32} yields a better representation of the measurements. To account for minor momentum losses around the valve fittings, $\lambda_4=1.7$ is utilized for dynamic analysis. Finally, Figure 8(c) shows that equation (7) with $C_d=0.61$ yields a good prediction for the steady-state flow characteristics of the orifice-like element.

### Estimation of linearized system parameters

The nonlinear resistance formulation of the previous section can be linearized by assuming that the variation around an operating point is small. For instance, by linearizing the $\Delta p$ versus $q$ relations in equations (5) to (7) at an operating point $q_{op}$, the linear resistance parameters for configurations B1, B2, and B3 are\textsuperscript{29}

$$R_i = \frac{d(\Delta p)}{dq_i} \bigg|_{q_{op}}$$

$$= \lambda_1 \frac{0.4235 \mu_{0.25} \rho_{0.75}^2}{d_{15}^{0.75}} q_{op}^{0.75}$$

(8a)

$$R_s = \lambda_2 \frac{\rho}{C_d A_{o1}^{0.5}} q_{op}$$

(8b)

$$R_o = \frac{\rho}{C_d A_{o2}^{0.5}} q_{op}$$

(8c)

Accordingly, a simplified bench experiment is conducted over the low pressure range, as shown in Figure 7(b). By adjusting the flow rate of the source,
a steady water level $h$ can be maintained, where $\Delta p = \rho gh$ and $g$ is the acceleration due to gravity. Then the linear resistance is estimated as $d_Dp/d\dot{q}$ around $q_{op}$ (where $q_{op} = 10 \text{ ml/s}, 8 \text{ ml/s}, \text{ and } 3 \text{ ml/s}$ for $R_i$, $R_s$, and $R_o$ respectively). Note that this experiment is limited to a lower $\Delta p$ range owing to the difficulty of maintaining a high water level.

By using the linear resistance terms and assuming that $k_r$ and $c_r$ are constant values estimated from the dynamic stiffness $K_{dr}$ measurements at lower frequencies, equations (2) to (4) are linearized as

\begin{align}
-A_1 \ddot{x}(t) - q_i(t) - q_s(t) &= C_1 \dot{y}_i(t) \quad (9a) \\
A_2 \ddot{x}(t) + q_i(t) + q_s(t) &= C_2 \dot{y}_s(t) \quad (9b) \\
p_i(t) + p_s(t) &= I_1 \dot{q}_i(t) + R_i q_i(t) \quad (9c)
\end{align}

\begin{align}
p_i(t) - p_s(t) &= I_s \dot{q}_i(t) + R_s q_i(t) \\
f_{Td}(t) &= f_{Tr}(t) + f_{Th}(t) \\
&= [k_r x(t) + c_r \dot{x}(t)] + [A_2 p_2(t) - A_1 p_1(t)]
\end{align}

where $f_{Td}(t)$ is the dynamic transmitted force. By transforming equations (9) and (10) to the Laplace ($s$) domain and assuming zero initial conditions, the cross point dynamic stiffness $K_d(s)$ is defined as

\begin{align}
K_d(s) &= \frac{F_{Td}}{X}(s) \\
&= \frac{\alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}{\beta_3 s^4 + \beta_2 s^3 + \beta_1 s + \beta_0} \quad (11a) \\
\alpha_4 &= c_r C_1 C_2 I_s \\
\alpha_3 &= k_r C_1 C_2 I_s + c_r C_1 C_2 (I_s R_s + I_s R_f) \quad (11c)
\end{align}

Figure 10. Comparison of the nonlinear model and the measurements for the harmonic responses of configuration B4 (a) with $X = 0.1 \text{ mm}$ at 15 Hz and (b) with $X = 1.0 \text{ mm}$ at 15 Hz. Key: ———, nonlinear model prediction; – – –, measurements.

Table 1. Fourier amplitudes of $f_T(t)$ and $\Delta p(t)$ for configuration B2 given sinusoidal excitation with two excitation amplitudes. Here, the predicted results are from the nonlinear model.

<table>
<thead>
<tr>
<th>$\Delta p$ (kPa)</th>
<th>Measured</th>
<th>Predicted</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0.99\times10^{-2}$</td>
<td>$1.11\times10^{-3}$</td>
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<table>
<thead>
<tr>
<th>$F_T$ (N)</th>
<th>Measured</th>
<th>Predicted</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2.59\times10^{-2}$</td>
<td>$2.81\times10^{-2}$</td>
</tr>
</tbody>
</table>

Fourier amplitudes of B2, with $X = 1.0 \text{ mm}$ (peak-to-peak value) and $\Omega = 10 \text{ Hz}$

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>$\Delta p$ (kPa)</th>
<th>$F_T$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\Omega$</td>
<td>$36.95\times10^{-2}$</td>
<td>$25.98\times10^{-2}$</td>
</tr>
<tr>
<td>$3\Omega$</td>
<td>$1.92\times10^{-2}$</td>
<td>$1.57\times10^{-2}$</td>
</tr>
<tr>
<td>$4\Omega$</td>
<td>$1.45\times10^{-2}$</td>
<td>$1.03\times10^{-2}$</td>
</tr>
<tr>
<td>$5\Omega$</td>
<td>$1.34\times10^{-2}$</td>
<td>$1.01\times10^{-2}$</td>
</tr>
<tr>
<td>$6\Omega$</td>
<td>$2.94\times10^{-2}$</td>
<td>$2.15\times10^{-2}$</td>
</tr>
<tr>
<td>$7\Omega$</td>
<td>$7.16\times10^{-2}$</td>
<td>$4.69\times10^{-2}$</td>
</tr>
</tbody>
</table>
\( \alpha_2 = k_r C_1 C_2 (I_R s + I_R i) + c_r C_1 C_2 R_s \) (11d) 
\[ \alpha_1 = k_r C_1 C_2 R_i (I_R s + (C_1 + C_2) (I_i + I_s)) + c_r (C_1 + C_2) (R_i + R_s) \] (11e) 
\[ \alpha_0 = k_r (C_1 + C_2) (R_i + R_s) \] (11f) 
\( \beta_3 = C_1 C_2 I_s \) (11g) 
\( \beta_2 = C_1 C_2 (I_R s + I_R i) \) (11h) 
\[ \beta_1 = C_1 C_2 R_i (C_1 + C_2) (I_i + I_s) \] (11i) 
\( \beta_0 = (C_1 + C_2) (R_i + R_s) \) (11j)

For a bushing with only one long passage (inertia track), \( K_d(s) \) is simplified by setting \( R_s \rightarrow \infty \) in equation (11) to give

\[ K_d(s) = \frac{\alpha_0 s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_0}{b_2 s^2 + b_1 s + b_0} \] (12a) 
\[ a_3 = c_c C_1 C_2 I_i \] (12b) 
\[ a_2 = C_1 C_2 (k_r I_i + c_r R_i) + (A_1^2 C_1 + A_2^2 C_2) I_i \] (12c) 
\[ a_1 = c_r (C_1 + C_2) + (A_2^2 C_1 + A_1^2 C_2) + k_r C_1 C_2 R_i \] (12d) 
\[ a_0 = k_r (C_1 + C_2) + (A_1 - A_2)^2 \] (12e) 
\[ b_2 = C_1 C_2 I_i \] (12f) 
\[ b_1 = C_1 C_2 R_i \] (12g) 
\[ b_0 = C_1 + C_2 \] (12h)

The sinusoidal stiffness \( K_d(\omega) \) is obtained by substituting \( s = i \omega \). Also, harmonic responses in the time domain can be obtained by using the convolution method or numerical simulations as described by Chai et al.\textsuperscript{20} The linear model predictions will be compared with the nonlinear model responses and the experimental measurements in the following section.

**Validation of the nonlinear model using sinusoidal measurements**

The governing nonlinear equations (equations (2) to (4)) are represented by

\[ \dot{p}_1(t) = -\frac{1}{C_1} [\dot{q}_1(t) + \dot{q}_2(t) + A_1 \ddot{x}(t)] \] (13a) 
\[ \dot{p}_2(t) = \frac{1}{C_2} [\dot{q}_1(t) + \dot{q}_2(t) + A_2 \ddot{x}(t)] \] (13b) 
\[ \dot{q}_1(t) = \frac{1}{I_s} [p_1(t) - p_2(t) - R_s (\Delta p, q_1, q_2)] \] (13c) 
\[ \dot{q}_2(t) = \frac{1}{I_s} [p_1(t) - p_2(t) - R_s (\Delta p, q_1, q_2)] \] (13d) 
\[ \dot{f}_{T_d}(t) = \left( \frac{A_1}{C_1} + \frac{A_2}{C_2} \right) [q_1(t) + q_2(t)] + \dot{k}_r \ddot{x}(t) + c_c(x) \ddot{x}(t) \] (13e)
where \( \gamma = \frac{A_1^2}{C_1} + \frac{A_2^2}{C_2} \). Based on the identified system parameters, this nonlinear model is numerically solved using an explicit fourth- and fifth-order Runge–Kutta method. The steady-state harmonic responses of configurations B1 and B2, as predicted by the nonlinear model, are compared with the measured pressure and transmitted force at 10 Hz in Figure 9. Excellent agreement between the nonlinear model and the measurements of configuration B1 is observed in Figure 9(a) with \( X = 0.1 \) mm. Since the maximum \( \Delta p \) between the two fluid chambers is below 22 kPa with \( X = 0.1 \) mm, the model with a linear \( R_s \) value also agrees well with measurements. When \( X \) is increased to 1.0 mm, the nonlinear model still yields a very good match with measurements, as shown in Figure 9(b). Conversely, the linear model overestimates the measured \( \Delta p(t) \) and \( f_2(t) \) responses. Since the maximum \( \Delta p \) between the two fluid chambers is now around 90 kPa with \( X = 1.0 \) mm, the linear resistance value needs to be re-estimated over the higher \( \Delta p \) range; the effective \( R_s \) value with \( X = 1.0 \) mm is found to be about three times the \( R_s \) value with \( X = 0.1 \) mm. For the steady harmonic responses of configuration B2, as seen in Figure 9(c) and (d) for 10 Hz excitation, the nonlinear model provides excellent agreement with the measurements at both low excitation amplitudes and high excitation amplitudes, especially when compared with \( \Delta p(t) \) predicted by the linear model with a constant \( R_s \) value.

The fast Fourier transform results of the predicted time histories and the measured time histories are compared in Table 1. Measured \( \Delta p(t) \) signals of configuration B2 contain significant odd superharmonics. For instance, the third harmonic magnitude is around 24% of the fundamental frequency (10 Hz) with \( X = 0.1 \) mm. Overall, both Figure 9 and Table 1 show that the proposed nonlinear model is capable of estimating the odd superharmonic terms and harmonic distortions in \( \Delta p(t) \). The sinusoidal responses for configurations B4 and B5 with a long passage and a short passage (in parallel) are finally examined using the nonlinear model. For instance, the harmonic time histories of configuration B4 at 15 Hz with \( X = 0.1 \) and 1.0 mm are shown in Figure 10. Good agreement between the nonlinear model and the measurements validates the nonlinear formulation for both single-passage configurations and dual-passage configurations.

**Construction of the analytical sinusoidal solution to the nonlinear model**

Although numerical methods may yield harmonic responses for a specific design, they are computationally time intensive, especially when responses at many frequencies are desired. Moreover, the starting transients in the simulations may lead to inaccurate spectral predictions as it is difficult to identify when the steady
state begins at a particular frequency. Thus, approximate analytical solutions are needed to calculate the amplitude-dependent frequency responses efficiently. First, a general form of \( R_i(q) \) for all passages is assumed as

\[
R_i(q) = l_i r_i^2 C_2 dA_i^2 q_j \approx R_i/c_j
\]

where \( R_i/c_j = l_i r_i^2 C_2/dA_i^2 \) is a resistance coefficient and \( l_i r_i^4 = 4.75 \) is a scaling coefficient, although the r.m.s. error of equation (14) for the \( R_i/c_j \) calculation is higher than that of equation (5). Then, on the basis of equations (13a) to (13d), the equations for a single-flow-passage configuration are obtained as

\[
A_1 \ddot{x}(t) - q(t) = C_1 \dot{p}_1(t) \tag{15a}
\]
\[
A_2 \ddot{x}(t) + q(t) = C_2 p_2(t) \tag{15b}
\]
\[
p_1(t) - p_2(t) = Iq(t) + R^* q(t)|q(t)| \tag{15c}
\]

where \( I \) and \( R^* \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R_i/c_j = l_i r_i^2 C_2/dA_i^2 \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
\]

where \( \alpha \) and \( \alpha_0 \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R^* = \rho/2C_2 A_i^2 q_j \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
 k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
\]

where \( \alpha \) and \( \alpha_0 \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R^* = \rho/2C_2 A_i^2 q_j \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
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\[
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\[
 k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
\]

where \( \alpha \) and \( \alpha_0 \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R^* = \rho/2C_2 A_i^2 q_j \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
 k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
\]

where \( \alpha \) and \( \alpha_0 \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R^* = \rho/2C_2 A_i^2 q_j \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
 k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
\]

where \( \alpha \) and \( \alpha_0 \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R^* = \rho/2C_2 A_i^2 q_j \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
 k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
\]

where \( \alpha \) and \( \alpha_0 \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R^* = \rho/2C_2 A_i^2 q_j \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
 k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
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where \( \alpha \) and \( \alpha_0 \) are the inertance and the resistance respectively of each passage; for the short-flow-passage configurations, \( R^* = \rho/2C_2 A_i^2 q_j \) and \( R^* = \rho/2C_2 A_i^2 q_j \). By eliminating \( p_1(t) \) and \( p_2(t) \), a single-degree-of-freedom nonlinear model in terms of \( x(t) \) and \( q(t) \) is derived as

\[
 k q(t) + 2 R^* q(t) |q(t)| \text{sgn} q(t) + I q(t) = \alpha x(t) \dot{x}(t) + \alpha_0 x(t) \ddot{x}(t) \tag{16}
\]
\[
\begin{align*}
q(t) &= kQ_1 \sin(\omega t) + R^2 Q_1^2 \sin(2\omega t) \frac{\text{sgn}(\sin(\omega t))}{2} \\
&= \frac{kQ_1}{C_0} I_v^2 Q_1 \sin(\omega t) + \frac{R^2 Q_1^2}{C_3} v Q_2^1 \sin(2\omega t) \frac{\text{sgn}(\sin(\omega t))}{2} \\
&= \alpha_1 A_x \omega \cos(\omega t) \cos \phi + \sin(\omega t) \sin \phi
\end{align*}
\]

where \(\text{sgn}\) is the sign function.

The nonlinear term in equation (17) is expanded using the Fourier series as

\[
\begin{align*}
\sin(2\omega t) \frac{\text{sgn}(\sin(\omega t))}{2} &= \sum_{n=-1}^{\infty} a_n \cos(n\omega t) \\
a_n &= \frac{1}{\pi} \left( \frac{1}{2 + n} + \frac{1}{2 - n} \right) \left[ 1 - (-1)^n \right]
\end{align*}
\]

The coefficient \(a_1 = 8/3\pi\) when \(n = 1\). Substituting equation (18) into equation (17) and then equating the coefficients of \(\sin(\omega t)\) and \(\cos(\omega t)\) yield the two relations

\[
\begin{align*}
kQ_1 - i\omega Q_1 &= \alpha_1 A_x \omega \sin \phi \\
8R^2 Q_1 &= \alpha_1 A_x \omega \cos \phi
\end{align*}
\]

By solving the above, the amplitude \(Q_1\) and the phase \(\phi\) at the frequency \(\omega\) are obtained as

\[
Q_1^2 = \frac{-(k - i\omega^2)^2 + \sqrt{(k - i\omega^2)^4 + 4(8\alpha_1 A_x R^2 \omega^2/3\pi)^2}}{2(8R^2 \omega^2/3\pi)^2}
\]

Figure 14. The predicted and the measured forces transmitted through the rubber path in a step-like experiment. Key: ———, measured; – – –, predicted by the original Kelvin–Voigt model of Figure 1(a); – – –, predicted by the modified model of Figure 1(b).

Figure 15. Comparison of the predicted forces and the measured forces for configuration B1 in the transient experiments for (a) a step-up response with \(A_e = A_{xt}\) and (b) a step-down response with \(A_e = 2A_{xt}\). Key: ———, nonlinear model; – – –, linear model; – – –, measurements.
\[ \phi = \arctan \left[ \frac{\sqrt{2(k - I\omega^2)}}{-\sqrt{(k - I\omega^2)^2} + \sqrt{(k - I\omega^2)^4 + 4(8\alpha, A_r R^* \omega^3 / 3\pi)^2}} \right] \]  

Figure 16. Comparison of the predicted forces and the measured forces for configuration B2 in the transient experiments for (a) a step-up response with \( A_e = A_{xt} \) and (b) a step-down response with \( A_e = 2A_{xt} \). Key: ———, nonlinear model; – – –, linear model; ———, measurements.

Figure 17. Comparison of the predicted forces and the measured forces for configuration B4 for (a) a step-up response with \( A_e = A_{xt} \) and (b) a step-down response with \( A_e = 2A_{xt} \). Key: ———, nonlinear model; ———, measurements.

Note that the \( k - I\omega^2 \) term mainly controls the resonance frequency. An increase in \( k - I\omega^2 \) causes the peaks to be more abrupt and to shift to the left. Also, the \( 8R^*\omega^3 / 3\pi \) term dictates the damping; it is observed that the peak amplitude decreases when \( R^* \) is increased.

Next, the sinusoidal volume flow rate is defined as

\[ Q(\omega) = Q_1 \cos \phi + iQ_1 \sin \phi \]  

Then relations between \( q(t) \), \( p_1(t) \), \( p_2(t) \) and \( f_2(t) \) from equations (13e), (15a), and (15b) are employed, the Laplace transform is applied and \( s = io \) is substituted.
to yield, at \( \omega \) (for a given excitation amplitude \( X \)), the dynamic stiffness and pressure amplitude expressions

\[
\begin{align*}
K_d(\omega) & = k_x(\omega, X) + i \omega c_x(\omega, X) + \gamma + \left( \frac{A_1}{C_1} + \frac{A_2}{C_2} \right) \frac{Q(\omega)}{i \omega} \\
P_1(\omega) & = -\frac{A_x A_1}{C_1} - \frac{Q(\omega)}{i C_1 \omega}
\end{align*}
\]

(22a)

(22b)

\[
P_2(\omega) = \frac{A_x A_2}{C_2} + \frac{Q(\omega)}{i C_1 \omega}
\]

(22c)

Although \( K_d(\omega) \) can be approximated by \( Q(\omega) \), the effect of \( k_x \) will still induce a difference between the numerical method and the analytical method. Thus, the \( Q(\omega) \) values (peak to peak) are calculated by using equation (20) from 1 Hz to 60 Hz, with \( X = 0.1 \text{ mm} \) and 1.0 mm, and then compared with the numerical integration results in Figure 11. Both methods yield almost the same results for the long passage (configuration B1), although the peak magnitude predicted by equation (20) with \( X = 1.0 \text{ mm} \) is slightly higher. However, some discrepancies are observed for configuration B2, as shown in Figure 12, especially with the \( X = 1.0 \text{ mm} \) case. This can be explained by referring to Figure 9 and Table 1. Since configuration B2 exhibits more significant superharmonics than configuration B1 does, the usage of only one harmonic term is not sufficient. Nevertheless, the analytical solution still provides a very good approximation.

**Semi-analytical solution using a multi-term harmonic balance method**

A multi-term harmonic balance method (MHBH)\(^{33,34} \) is utilized next to calculate the steady-state response of the nonlinear equation (16), given a sinusoidal excitation of \( x(t) = A_s \sin(\omega t) \). First, using a truncated Fourier series with \( N \) odd harmonics, the solution for \( q(t) \) is written

\[
q(t) = \sum_{n=1}^{N} \left\{ a_{2n+1} \sin[(2n+1)\omega t] + b_{2n+1} \cos[(2n+1)\omega t] \right\}
\]

(23)

where \( n \) is a positive integer. The even harmonics are neglected because the Fourier transforms of the measured \( \Delta p(t) \) and \( f_x(t) \) signals show that the odd harmonics are dominant, as also suggested by equation (18).

By defining \( \theta = \omega t \), \( q'(\theta) = dq/d\theta \), \( q''(\theta) = d^2q/d\theta^2 \), and \( x'(\theta) = dx/d\theta \), the derivatives of \( q(t) \) and \( x(t) \) are obtained as \( \dot{q}(t) = \omega q' \), \( \ddot{q}(t) = \omega^2 q'' \), and \( \dot{x}(t) = \omega x' \). Then equation (16) is converted into

\[
kq(\theta) + 2R^2 q(\theta)\omega q'(\theta)\text{sgn}(q) + I_0 \omega^2 q''(\theta) = \alpha_x \omega x'(\theta)
\]

(24)

Suppose that the discrete \( \theta \) signal consists of \( L \) data points, say from \( \theta_1 \) to \( \theta_L \). Equation (23) is discretized as

\[
q = \mathbf{\Psi} \mathbf{a}
\]

where \( a \) is the vector of Fourier coefficients, and \( \mathbf{\Psi} \) is the discrete Fourier transform matrix defined as

\[
\mathbf{\Psi} = \begin{bmatrix}
\sin(\theta_1) & \cos(\theta_1) & \sin(3\theta_1) & \ldots \\
\sin(\theta_2) & \cos(\theta_2) & \sin(3\theta_2) & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\sin(\theta_L) & \cos(\theta_L) & \sin(3\theta_L) & \ldots
\end{bmatrix}
\]

(25)

By using a hyperbolic tangent function to approximate the discontinuous sign function, the nonlinear term is expressed using the truncated Fourier series as \( q' / \tanh(\sigma q) = \mathbf{\Psi} \mathbf{b} \) where \( \mathbf{b} \) is the Fourier coefficient vector of the nonlinear function and \( \sigma \) is a regularizing (smoothening) factor. The derivatives of \( q \) with respect to \( \theta \) are obtained as \( q' = \mathbf{\Psi} \mathbf{D} \mathbf{a} \) and \( q'' = \mathbf{\Psi} \mathbf{D}^2 \mathbf{a} \), where \( \mathbf{D} \) is the differential operator matrix defined as

\[
\mathbf{D} = \begin{bmatrix}
0 & -1 & 0 & \cdots \\
1 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & -3 & 0 & \cdots \\
2N + 1 & 0 & \cdots & \cdots
\end{bmatrix}
\]

(26)

By substituting the above Fourier series expansions and derivatives of \( q \) into equation (24), the discretized equations in the time domain are given in the matrix form

\[
k \mathbf{\Psi} \mathbf{a} + 2R^2 \omega \mathbf{\Psi} \mathbf{b} + I_0 \omega^2 \mathbf{D}^2 \mathbf{a} = \alpha_x \omega \mathbf{\Psi} \mathbf{D} \mathbf{X}
\]

(27)

where \( \mathbf{X} = [A_x, 0, \ldots, 0]^T \) is the input vector corresponding to the displacement excitation \( x(t) = A_x \sin(\omega t) \). The vector \( \mathbf{a} \) can be found by balancing the right-hand side and the left-hand side of the equation. First, by pre-multiplying equation (27) with the pseudo-inverse of \( \mathbf{\Psi} \) as \( \mathbf{\Psi}^+ = (\mathbf{\Psi}^T \mathbf{\Psi})^{-1} \mathbf{\Psi}^T \), the residue function \( \mathbf{R} \) is defined as

\[
\mathbf{R} = k \mathbf{a} + 2R^2 \omega \mathbf{b} + I_0 \omega^2 \mathbf{D}^2 \mathbf{a} - \alpha_x \omega \mathbf{D} \mathbf{X}
\]

(28)

Then the correct solution of \( \mathbf{a} \) is calculated when \( \mathbf{R} \) is minimized. The Broyden method,\(^{35} \) which requires only the calculation of the initial Jacobian matrix, is applied to minimize \( \mathbf{R} \). The Jacobian matrix \( \mathbf{J} \) is calculated as

\[
\mathbf{J} = \frac{\partial \mathbf{R}}{\partial \mathbf{a}} = k + 2R^2 \omega \frac{\partial \mathbf{b}}{\partial \mathbf{a}} + I_0 \omega^2 \mathbf{D}^2
\]

(29a)
\[ \frac{\partial h}{\partial a} = \Psi + qg' \tanh(\sigma q) \frac{\partial q}{\partial a} \]

\[ = \Psi + \left\{ q \tanh(\sigma q) \frac{\partial q}{\partial a} + q \tanh(\sigma q) \frac{\partial q}{\partial a} \right\} (29b) \]

Note that \( \frac{\partial q}{\partial a} = \Psi \) and \( \frac{\partial q}{\partial a} = \Psi D \). The initial Jacobian matrix \( J_0 \) is obtained by substituting \( a_0 \) into equation (29), and a first approximation of \( a \) is calculated using the Newton method\(^{15} \) as \( a_1 = a_0 - J_0^{-1}R_0 \). The Jacobian matrix for the \( n \)th iteration is updated as

\[ J_m = J_{m-1} + UW^T \]  
\[ W = a_m - a_{m-1} \]  
\[ W = \frac{R_m - R_{m-1}}{W} - J_{m-1}W \]  

The inverse of the Jacobian matrix is calculated by using the Sherman–Morrison formula\(^{15} \) \( J_m^{-1} = [I - J_m^{-1}UW^T/(1 + W^TJ_m^{-1}U)]J_m^{-1} \). Then, \( a \) is updated at each step with \( a_{m+1} = a_m - J_m^{-1}R_m \).

The iterations continue until \( a \) converges to a satisfactory solution when \( \| R \| < \varepsilon \), i.e. the \( L^2 \) norm of vector \( R \) is less than a very small number \( \varepsilon \).

\[ Q(\omega) \] and \( \phi_q(\omega) \), which are calculated by the multiterm harmonic method for \( N = 3 \) with \( X = 0.1 \text{ mm and 1.0 mm} \), are shown in Figures 11 and 12 from 1 Hz to 60 Hz. The difference between the MHBM solutions and the numerical integrations is negligible for both configuration B1 and configuration B2, but there are minor discrepancies for the response of configuration B2 with \( X = 1.0 \text{ mm} \). Unlike the analytical solution of the previous section, the \( Q(\omega) \) spectra given by the MHBM are much closer to the numerical results. The phase angle spectra of the three methods are very close since they are calculated on the basis of the phase difference between the fundamental harmonic terms of \( x(t) \) and \( q(t) \). Overall, both single-term solutions and multiterm solutions approximate the harmonic responses well. Moreover, \( \Delta p(t) \) and \( f_r(t) \) can be easily calculated, given \( q(t) \) and equation (13).

### Study of the transient responses using experiments and the nonlinear model

A step-up and step-down displacement excitation, as shown in Figure 13(a), shows the ideal input signal which can be expressed in terms of the unit step function \( u(t) \) as \( x_0(t) = A_{x(t)} + A_{x(t)}u(t - t_1) - 2 A_{x(t)}u(t - t_2) + A_{x(t)}u(t - t_3) \)

Since it is extremely difficult to generate an ideal step signal given the practical limitations of an actuator, only smoothed step events are realized in the experiment, such as the step-up-like excitation shown in Figure 13(b). Since the step-down excitation has twice the step amplitude \( A_s \) of the step-up excitation, the measured \( f_r(t) \) responses are normalized by \( A_s \) and then superposed in the same plot for easier comparison. Figure 13(c) shows the normalized force response of configuration B1. Comparison shows that the two step-up responses (with \( A_s = A_{x(t)} \)) exhibit similar transients, although their asymptotic values differ according to the initial loading levels; this also suggests that \( k_s \) and \( c_s \) are highly dependent on \( f_m \). However, the normalized step-down response (with \( A_s = 2A_{x(t)} \)) has a longer oscillation period and lower oscillation peak value than the step-up event does (with \( A_s = A_{x(t)} \)). This again implies that the dynamic responses depend highly on the step amplitude, and thus the proposed nonlinear model is applied next to analyze the transient force measurements.

Since the Kelvin–Voigt model cannot represent the exponential decay term in \( f_r(t) \), a Maxwell element (with stiffness \( k_e \) and damping \( c_e \)) is added to the rubber path model, as shown in Figure 1(b). Load-dependent \( k_e(\omega) \) is estimated from the asymptotic value of \( f_r(t) \), while \( k_s \) and \( c_s \) are identified by a curve-fit procedure of the measurements. The force response predicted by the modified rubber path model, given a step-like excitation, is compared with the measured \( f_r(t) \) in Figure 14. It can be observed that the modified model (with Maxwell element) yields much better agreement with the measurement than does the original Kelvin–Voigt model.

The nonlinear model predictions of the step-up response and the step-down response of configuration B1 are compared in Figure 15 with the transmitted force measurements; a linear model with an effective resistance \( R_{ei} \) is also compared. Here \( R_{ei} \) is estimated on the basis of the settling time, the oscillation period, and the peak values of the step response at each amplitude.\(^{20} \) Similarly, the effective resistances \( R_{es} \) and \( R_{ez} \) are estimated from the step responses of configurations B2 and B3 respectively.\(^{20} \) As shown in Figure 15, the predicted \( f_r(t) \) matches the measured forces well, and it yields much better agreement of the oscillatory peaks and periods than the linear model does. Similarly, the predicted forces and the measured forces of configuration B2 are compared in Figure 16, and excellent agreement is observed for both step-up responses and step-down responses. In particular, the nonlinear model represents the higher-frequency small-amplitude oscillations well in the \( f_r(t) \) responses of configuration B2, which are neglected in the linear model predictions. Furthermore, configurations with a long passage and a short passage (in parallel) are investigated. Figure 17 compares the nonlinear model and the measurements of configuration B4, and excellent predictions for both step-up responses and step-down responses are observed. These comparisons confirm that the nonlinear model represents the nonlinearities well in the transient responses for both single-passage configurations and dual-passage bushing configurations. Some discrepancies between the predictions and the measurements exist, especially for higher \( \Delta p \) conditions, although the error is less than 20%. These may be caused by a variation in the assembly and disassembly
process of the prototype device and by the $\Delta p$ measurement errors when $q$ is high. Nevertheless, the effect of nonlinearities is more evident in the transient responses when compared with the sinusoidal excitations because a step response induces a broader operating range. For instance, $\Delta p$ is around 220 kPa in the step-down responses of configuration B1 but only up to 22 kPa and 90 kPa for $X = 0.1$ mm and 1.0 mm respectively in steady-state harmonic responses. Thus the linear model with constant effective parameters is incapable of accurately predicting the step response although it is appropriate for steady-state harmonic responses over a narrow operating range.

**Conclusion**

Unlike previous literature that has focused on only the linear system analysis of fluid-filled bushings, this article has developed a refined nonlinear model for a bushing with a long passage and a short passage and identified key nonlinear parameters using theory and experiments. The system nonlinearities are found to be dominated by the fluid resistance terms, and thus these are defined on the basis of the turbulent flow formulation. Five configurations of the laboratory prototype device are examined by using the nonlinear models. The nonlinear model is first solved by using the numerical integration method, and its predictions match both the steady-state harmonic measurements and the transient measurements well for either a single-passage configuration or a multi-passage configuration. Finally, this article obtained an analytical approximation and a semi-analytical solution of the nonlinear flow rate model by using the harmonic balance method, given a sinusoidal excitation. These solutions may be utilized to estimate the amplitude-dependent responses efficiently in the frequency domain. Such nonlinear models should lead to better industrial design and tuning processes. Future work is needed to characterize the amplitude dependence of the rubber path and to examine the bushings in the context of specific vehicle suspension systems.

**Acknowledgement**

The authors thank C Gagliano and PC Detty for their help with experimental studies.

**Declaration of conflict of interest**

The authors declare that there is no conflict of interest.

**Funding**

This work was supported by the Transportation Research Center Inc., Honda R&D Americas, Inc., and YUSA Corporation of the Smart Vehicle Concepts Center and the National Science Foundation, Industry & University Cooperative Research Program.

**References**


Appendix 1

Notation

\( a, b \) equation coefficients
\( \alpha, \beta \) equation coefficients
\( \gamma \) static stiffness of the hydraulic chambers
\( \theta \) angular displacement
\( \lambda \) scaling factor for the nonlinear resistance model
\( \mu \) fluid viscosity
\( \rho \) density
\( \phi \) phase angle
\( \psi \) discrete Fourier transform matrix
\( \omega \) circular frequency (rad/s)
\( \Omega \) frequency (Hz)

\( J \) Jacobian matrix
\( k \) stiffness
\( K \) dynamic stiffness
\( l \) length of fluid passage
\( L \) number of data points
\( n \) index
\( N \) number of harmonics
\( p \) pressure in the time domain
\( P \) dynamic pressure in the \( s \) or \( \omega \) domain
\( q \) flow rate in the time domain
\( Q \) dynamic flow rate in the \( s \) or \( \omega \) domain
\( R \) fluid resistance
\( R \) residue vector
\( s \) Laplace domain variable
\( t \) time
\( V \) volume
\( x \) displacement excitation in the time domain
\( X \) peak-to-peak value of displacement excitation amplitude
\( X \) excitation vector

\( * \) peak magnitude or peak phase angle
\( + \) pseudo-inverse
\( / \) normalized value

\( d \) dynamic
\( e \) effective or equivalent
\( h \) hydraulic path
\( i \) long flow passage (inertia track)
\( K \) magnitude of the dynamic stiffness
\( [K] \) mean or static value
\( o \) orifice-like element
\( r \) rubber or rubber path
\( s \) short flow passage with restriction
\( T \) transmitted
\( x \) displacement excitation
\( \phi \) phase angle
\( 1 \) chamber 1
\( 2 \) chamber 2