Analysis of vibration amplification in a multi-staged clutch damper during engine start-up

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Abstract
Transient vibration amplifications of torsional systems passing through critical speeds have been of interest for a considerable amount of time. However, previous investigations on the piecewise linear system have focused mostly on numerical methods, and thus a reliable analytical method is not available for predicting transient amplification events. This article overcomes this void by developing and utilizing the closed-form solution of a linear single-degree-of-freedom torsional system, given a motion input under a constant acceleration rate, to approximate the transient responses of a piecewise linear system. This system represents a simplified vehicle powertrain system with a multi-staged clutch damper during the engine start-up process under an instantaneous motion input from the flywheel. First, the utility of a single-degree-of-freedom system and the motion input for the start-up process are experimentally and numerically illustrated by vehicle start-up measurements. Second, a closed-form solution of a linear damped torsional oscillator, given instantaneous-frequency excitation, is successfully developed and numerically verified. Finally, the proposed analytical solution of a linear system is utilized to predict the approximate peak-to-peak value of the displacement of a piecewise linear system during transient amplification for a rapid variation in speed.

Keywords
Engine start-up, speed-dependent vibration amplification, driveline transients, analytical methods

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Introduction
The speed-dependent behavior of linear and nonlinear torsional systems passing through critical speeds has been of interest for several decades.1–9 These transient vibrations result from a nonstationary process which has time-varying or instantaneous excitation frequencies. This concept was first introduced by Lewis1 in 1932 where an approximate solution for a single-degree-of-freedom (SDOF) linear system under a uniform acceleration rate was proposed. Several numerical studies have been conducted to investigate various rotor dynamics problems.2–8 In particular, Newland9 stated that an analytical solution is not possible, and thus he numerically examined the amplification for an SDOF linear model for both acceleration cases and deceleration cases with a uniform rate. The response amplitudes under an instantaneous-frequency excitation were compared with the steady-state harmonic response, and the frequencies corresponding to peak response amplitudes were identified in a book by Newland.9 Yet another practical problem where such transient vibration judder responses are observed is in the braking systems of vehicles.10–13 In a recent study, Sen et al.12 have experimentally found resonant amplification in a torsional system, with clearance nonlinearity and with a uniform deceleration rate. Similar vibration amplification problems are seen during the engine start-up process; such powertrain torsional systems usually include a nonlinear multi-staged clutch damper.14 The vibration mode corresponding to transient amplification is usually controlled by the clutch damper whose properties may be approximated by piecewise linear stiffness and damping components.15,16 However, even...
the latest studies regarding the transient torsional amplification in vehicle driveline systems focused
mainly on numerical methods. An analytical method for predicting the transient amplification level
is not available.

The characteristics of a piecewise linear system for a stationary process have been usually examined in the
frequency domain by using a numerical integration technique or semi-analytical methods such as the
harmonic balance method, the describing function method, and the stochastic linearization method. Computational or approximate analytical methods have been utilized to study vibro-impacts in the time domain. For a nonstationary process, numerical or experimental methods have been applied to a piecewise linear system, but their behavior has yet to be fully understood. To overcome such a void in the literature, this article proposes to find analytically a closed-form solution of a linear torsional oscillator (an engine–flywheel–clutch-damper–transmission system) with a motion input (from flywheel) and to approximate the transient amplification level of a piecewise linear system under a high acceleration rate. The practical problem of interest is the transient vibration amplification associated with the engine start-up.14

**Problem formulation**

The first torsional mode of vehicle powertrain systems (say, from 6 Hz to 15 Hz) is often dominant during the engine start-up, it is usually controlled by a dry clutch damper. Note that, during the engine start-up, the downstream driveline subsystem consisting of the transmission main shaft, the propeller shaft, the differential, and the axles is decoupled. Therefore, the focus is on the powertrain subsystem up to a lumped transmission in which the flywheel is the dominant inertia component. Accordingly, an engine–flywheel–clutch–transmission system could be described by a linear four-degree-of-freedom (4DOF) semidefinite system as displayed in Figure 1(a), where \( I_1, I_2, I_3 \), and \( I_4 \) represent the inertias (kg m\(^2\)) of the engine, the flywheel, the clutch, and the lumped transmission respectively. The constant torque \( T_p \) (N m) provides the lubricant-induced drag in the transmission box. The torque \( T_E(t) \) (N m) for a multi-cylinder internal-combustion engine is expressed via a Fourier series as

\[
T_E(t) = T_m + \sum_n T_{an} \sin \left[n \left( \Omega_0 + \frac{\alpha}{2} t \right) + \psi_n \right]
\]

where \( T_m \) (N m) and \( T_{an} \) (N m) are the mean torque and the alternating torque of \( n \)th order respectively. Here, \( \Omega_0 + \alpha t \) is the mean crankshaft speed (rad/s) which is accelerated with a constant rate \( \alpha \) (rad/s\(^2\)), and \( n(\Omega_0 + \alpha t) \) is the instantaneous firing frequency (rad/s). Note that the numerical values of \( \Omega_0 + \alpha t \) and \( \alpha \) are usually given in units of r/min and (r/min)/s respectively. In general, the dominant firing order \( n_{dom} \) is calculated as \( n_{dom} = n_{cyl}/2 \), where \( n_{cyl} \) is the number of cylinders. Since internal-combustion engines are currently downsized to enhance the fuel consumption, such as in hybrid electrical vehicles, a two-cylinder gasoline engine is used as an example and, thus, \( n_{dom} = 1 \). For clarity, only the dominant order is considered first and, thus, equation (1) is rewritten as

\[
T_E(t) = T_m + T_d \sin \left[ (\Omega_0 + \frac{1}{2} \alpha t) t \right]
\]

where \( \psi \) is assumed to be zero without losing any generality. Because of the massive torsional inertia of a flywheel, \( \theta_2(t) \) and \( \theta_3(t) \) are assumed to be unaffected by \( \theta_1(t) \) and \( \theta_3(t) \); this suggests that \( \theta_2(t) \) and \( \theta_3(t) \) may be used as motion excitation terms for a three-degree-of-freedom (3DOF) semidefinite system, as shown in Figure 1(b). Further, since only the clutch mode is of interest, \( I_2 \) and \( I_3 \) are lumped together, and the SDOF definite system is formed as depicted in Figure 1(c).

The chief objectives of this article are as follows:

1. to compare the transient vibration amplification level between the linear and piecewise linear torsional systems describing the engine–flywheel–clutch-damper–transmission system using a numerical method (here the clutch damper is modeled by a dual-staged spring and viscous damping elements);
2. to develop and verify a new closed-form solution for a linear damped torsional oscillator, given an instantaneous frequency (speed) input;
3. to utilize the proposed closed-form solution to approximate the transient amplification of a piecewise linear torsional oscillator (limited experimental validation is included).

This study will also extend the work of Sen et al. who considered the torque input with an instantaneous frequency; this article will consider motion excitation with both instantaneous frequency (speed) and instantaneous amplitude at various orders of the speed.

**Comparison of linear torsional models and experimental validation**

In order to illustrate the utility of an SDOF system that is excited by flywheel motion, the torsional natural frequencies of three linear semidefinite systems are first calculated. The first clutch mode frequency \( f_{clutch} \) is found as follows: 13.9 Hz for the 4DOF semidefinite system, 13.5 Hz for the 3DOF semidefinite system, and 13.5 Hz for the 2DOF semidefinite system. The corresponding (normalized) modes at \( f_{clutch} \) for three models are compared as follows: \([0.01 1 0.99 0.99]^T\) for the 4DOF semidefinite system, \([0.99 1 0.99]^T\) for the 3DOF system, and \([0 1]^T\) for the 2DOF system where the superscript \( T \) indicates the transpose of a column.
The natural frequency calculations suggest that the 2DOF semidefinite system or the SDOF definite system can fully represent the clutch mode of the 4DOF semidefinite system with focus on the relative motion $u_{23}(t) = \frac{u_3(t)}{C_0}u_2(t)$.

The transient responses are numerically examined next with $a = 15$ (r/min)/s, $T_m = 27.4$ N m, $T_a = 300.0$ N m, $T_D = -20$ N m, and $\Omega_0 = 400$ r/min.

First, the absolute velocities $\dot{u}_2(t)$ of the flywheel and the absolute velocities $\dot{u}_3(t)$ of the clutch hub of the 4DOF system are compared. As shown in Figure 2, both $\dot{u}_2(t)$ and $\dot{u}_3(t)$ have a rotational component (the mean crankshaft speed $\Omega_0 + \alpha t$), which is accelerated from 400 r/min to 1000 r/min. However, $\dot{u}_3(t)$ exhibits a transient amplification when the clutch mode is excited by the instantaneous firing frequency $\Omega_0 + \alpha t$; conversely, $\dot{u}_2(t)$ has only a bare minimum amplification which could be neglected. This implies that the flywheel motion may be used as a system excitation.

Second, the $u_{23}(t)$ values from these three systems are compared. The flywheel responses $u_2(t)$ and $u_2(t)$ of the 4DOF semidefinite system are numerically recorded during the calculation process, and then the $u_2(t)$ and $u_2(t)$ time histories are directly applied as the motion input to a 2DOF (or an SDOF) system. Figure 3 shows that $u_{23}(t)$ from an SDOF model is sufficiently close to $u_{23}(t)$ from a 4DOF system. Also, both a 4DOF system model and an SDOF system model yield similar answers for $\dot{u}_{23}$ or $\dot{u}_{23}$; these are not included here, however.

Figure 1. Powertrain system (consisting of an engine–flywheel, a clutch damper and a transmission) described by three linear torsional models: (a) a 4DOF semidefinite system under an engine torque input $T_E(t)$; (b) a 3DOF semidefinite system under flywheel motion inputs $\dot{u}_2(t)$ and $\dot{u}_2(t)$ as obtained from (a); (c) a two-degree-of-freedom (2DOF) semidefinite (or SDOF definite) system under the flywheel motion inputs $\dot{u}_2(t)$ and $\dot{u}_2(t)$ as obtained from (a).
Finally, a vehicle start-up experiment is conducted on a medium-duty pickup truck with a four-cylinder gasoline engine and a six-speed manual transmission. The schematic diagram of the test rig is illustrated in Figure 4; other downstream driveline components including the propeller shaft, the axles, and the differential are not shown as they are decoupled during the engine start-up. The absolute torsional velocities \( \dot{\theta}_2(t) \) and \( \dot{\theta}_3(t) \) (as measured by two magnetic sensors) and key parameters (estimated from the physical dimensions or vehicle measurements) are then applied to the models in Figure 1(b) and (c). Experimental measurement of \( \dot{\theta}_2^{\text{max}} \) (the peak-to-peak value of \( \dot{\theta}_2(t) \)) is compared with the corresponding predictions from linear models. The comparisons in Table 1 further justify the utility of an SDOF powertrain system with an instantaneous motion input (from the flywheel).

**Nonlinear torsional model (with piecewise linear clutch damper)**

A generic piecewise linear torsional oscillator problem is formulated with the angular motion inputs \( \phi(t) \) and \( \phi(t) \), as shown in Figure 5(a). Given the numerical simulation of \( \dot{\theta}_2(t) \) and available measurement of the flywheel speed during the engine start-up process, the analytical forms of \( \phi(t) \) and \( \phi(t) \) are assumed to be

\[
\begin{align*}
\phi(t) &= \dot{\phi}_a(t) + \dot{\phi}_c(t) \\
\dot{\phi}_a(t) &= \Omega_0 + \alpha t \\
\dot{\phi}_c(t) &= \sum_{n=1}^{N} \gamma_n \left( \Omega_0 + \alpha t \right) \cos \left( n\Omega_0 t + \frac{n\pi t^2}{2} + \varphi_n \right) \\
\phi(t) &= \phi_a(t) + \phi_c(t) \\
\dot{\phi}_a(t) &= \Omega_0 t + \frac{n\pi t^2}{2} \\
\phi_c(t) &= \sum_{n=1}^{N} \gamma_n \left[ \sin \left( n\Omega_0 t + \frac{n\pi t^2}{2} + \varphi_n \right) \right]
\end{align*}
\]

It is assumed that \( \Omega_0 + \alpha t \gg \gamma_n \Omega_0 + \alpha t \) or \( \gamma_n \ll 1 \) rad since the speed fluctuations \( \dot{\phi}_a(t) \) are usually much smaller than the mean speed \( \dot{\phi}_a(t) \), which are also found in the \( \dot{\theta}_2(t) \) calculation. In addition, the \( \gamma_n \) values are selected on the basis of the classical paper by Porter which provided dimensionless Fourier coefficients for a multi-cylinder internal-combustion engine.

The real-life clutch dampers usually include multistaged elastic and dissipative properties. For simplicity, this component is approximated in this article by only elements with a two-staged piecewise linear stiffness \( K(\delta(t)) \) and piecewise linear damping \( C(\delta(t)) \), as displayed in Figure 5(b) and (c). Here, assume that \( K = \alpha_1^2 \) and \( C = 2\zeta\alpha_1 \) for the normalized unity torsional inertia, where \( \zeta \) is the viscous damping ratio and \( \theta(t) \) is the angular displacement. Further, \( \alpha_1 \) is selected to be 82.9 rad/s (13.2 Hz) to represent the clutch mode.
The governing equation of the nonlinear SDOF system is given as

\[ \ddot{\delta}(t) + T(\delta(t), \dot{\delta}(t)) = T_D \]  

(5)

where the relative angular displacement \( \delta(t) = \theta(t) - \phi(t) \) is of interest.

The nonlinear torque \( T(\delta(t), \dot{\delta}(t)) \) is given as

\[
T(\delta(t), \dot{\delta}(t)) = \begin{cases} 
\eta K_\chi + K_\delta(\delta(t) - \chi) + C_\delta(t), & \delta(t) > \chi \\
\eta K_\delta + \mu C_\delta(t), & -\chi \leq \delta(t) \leq \chi \\
-\eta K_\chi + K_\delta(\delta(t) + \chi) + C_\delta(t), & \delta(t) < -\chi 
\end{cases}
\]

(6)

where the transition angle is \( \chi \) (rad), and \( \mu \) and \( \eta \) are the dimensionless damping ratio and the dimensionless stiffness ratio respectively between stage I and stage II.

The transition angle \( \chi \) is assumed to be 0.5 rad, and the \( \mu \) or \( \eta \) range is given as \( 0 \leq \mu = \eta \leq 1 \) in this study. Equation (6) is rewritten as

\[
T(\delta(t)) = K_\delta(t) + (1 - \eta)K_\chi \left| \frac{\delta(t) - \chi}{2} \right| - \frac{1}{2} \left( \frac{[\delta(t) - \chi]^2}{2} \right) + C_\delta(t) + (1 - \mu)C_\chi \left( \frac{\text{sgn}[\delta(t) - \chi] - \text{sgn}[\delta(t) + \chi]}{2} \right)
\]

(7)

where \( \text{sgn}[\delta(t) \pm \chi] \) is the sign function. The discontinuity in equation (7) is smoothed using the hyperbolic function \( \tanh\{\sigma[\delta(t) \pm \chi]\} \) where \( \sigma \) (say, \( 10^3 - 10^5 \)) is a regularizing factor.28 A numerical method is utilized to solve equation (5) with a single order \( n = 1 \) as an example. Transient responses of a piecewise linear powertrain system (with \( \eta = \mu = 0.3 \)) are examined, given \( \omega_1 = 13.2\, \text{Hz} = 792 \, \text{r/min} \), \( \alpha = 5 \, \text{r/min/s} \), \( \Omega_0 = 400 \, \text{r/min} \), \( \xi = 0.001 \), \( \gamma_1 = 0.01 \), and \( \phi_1 = 0 \). Since the drag torque \( T_D \) affects the operating point of a piecewise linear system, \( T_D \) is adjusted to ensure that the operating point (red full circle in Figure 5(b)) is located in the transition region from stage II to stage I so as to activate both stages. As shown in Figure 6, \( \delta(t), \dot{\delta}(t), \) and \( \ddot{\delta}(t) \) exhibit similar transient amplification trends between 700 r/min and 800 r/min. This implies that only \( \delta_{p-p} \) (rad) may be used to evaluate the resonant amplification level, where the subscript \( p-p \) indicates the peak-to-peak value of \( \delta(t) \). Further, rapid variations in the speed are considered with \( \alpha \) up to 175 r/min/s. As shown in Figure 7, \( \delta_{p-p} \) for two cases

![Schematic diagram of the start-up experiment for a medium-duty truck with a four-cylinder gasoline engine and a six-speed manual transmission.](image)

**Table 1.** Comparison between the \( \dot{\theta}_{23,p-p} \) value (the peak-to-peak value of \( \dot{\theta}_{23}(t) \)) for the vehicle measurement (during the engine start-up process) and the predictions of the linear models of the engine-flywheel, the clutch damper and the transmission torsional system.

<table>
<thead>
<tr>
<th>Parameter (units)</th>
<th>Experiment</th>
<th>Linear 3DOF semidefinite model</th>
<th>Linear SDOF definite model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{\theta}_{23,p-p}(\text{r/min}) )</td>
<td>513</td>
<td>526</td>
<td>525</td>
</tr>
</tbody>
</table>

SDOF: single-degree-of-freedom.
\[ \eta = \mu = 0.3 \quad \text{and} \quad \eta = \mu = 1.0 \]

are compared. It is clear that, as \( \alpha \) increases up to 25 rad/min/s, the \( \delta_{p-p} \) value of a linear system (\( \eta = \mu = 1.0 \)) is very close to the \( \delta_{p-p} \) value of a nonlinear system. Accordingly, it is reasonable to assume that a linear torsional oscillator may be utilized to approximate a nonlinear system when \( 0.3 < \eta = \mu < 1.0 \).

### Analytical solution for a damped linear torsional oscillator

An analytical solution of the linear damped system is first sought by setting \( \eta = \mu = 1 \) to give

\[ \ddot{\theta}(t) + 2\zeta\omega_1\dot{\theta}(t) + \omega_1^2\theta(t) = 2\zeta\omega_1\dot{\phi}(t) + \omega_1^2\phi(t) \]  

(8)

Since the operating point of the linear system does not affect \( \delta_{p-p} \), \( T_D \) is assumed to be zero for convenience. Also, \( \theta(t) \) is divided into two parts: one is the rotational part \( \theta_1(t) \) which is caused by \( \phi_1(t) \), and the second is the alternating part \( \theta_2(t) \) which is induced by \( \phi_2(t) \). Then, \( \theta_1(t) \) is solved as

\begin{align*}
\theta_1(t) + \omega_1^2\theta_1(t) + 2\zeta\omega_1\theta_1(t) \\
= \omega_1^2(n\Omega_0 t + \frac{n}{2\alpha}t^2) + 2\zeta\omega_1(n\Omega_0 + \alpha t) 
\end{align*}

(9)

First, the system is assumed to rotate with a constant speed \( \Omega_0 \) and, thus, the initial conditions are as follows: \( \theta_i(0) = 0 \) rad, \( \dot{\theta}_i(0) = \Omega_0 \), \( \theta_2(0) = 0 \) rad, and \( \dot{\theta}_2(0) = 0 \) rad/s. Then, \( \theta_1(t) \) is found by independently solving equation (9) with the assumed initial conditions. The general response in the Laplace domain \( 31 \) to an arbitrary excitation \( T(t) \) with the initial conditions \( \theta(0) = \theta_0 \) and \( \dot{\theta}(0) = \dot{\theta}_0 \) is found to be

\[ \Theta(s) = \frac{T(s)}{s^2 + 2\zeta\omega_1 + \omega_1^2} + \frac{s + 2\zeta\omega_1}{s^2 + 2\zeta\omega_1 + \omega_1^2}\theta_0 \\
+ \frac{1}{s^2 + 2\zeta\omega_1 + \omega_1^2}\dot{\theta}_0 \\
\]  

(10)

Let

\[ T(s) = \mathcal{L}\left[ \omega_1^2\left( n\Omega_0 t + \frac{n}{2\alpha}t^2 \right) + 2\zeta\omega_1(n\Omega_0 + \alpha t) \right] \]  

(11)
Now employ the inverse Laplace transformation together with the convolution theorem to yield

$$\theta_d(t) = \Omega_0 t + \frac{1}{\alpha} \sqrt{1 - \xi^2} t^2$$  \hspace{1cm} (12)

Then, $\theta_d(t)$ is solved as

$$\theta_d(t) = \bar{\omega}_d \sqrt{\frac{2}{\pi}} \int_0^t \sin[\omega_d(t - u)] e^{-\zeta \omega_d (t-u)} du$$  \hspace{1cm} (13)

Like equation (12), the convolution theorem is again employed to yield

$$\theta_d(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2\bar{\omega}_d s + \bar{\omega}_d^2} \right\}$$

$$\times \left\{ 2\bar{\omega}_1 \left[ \sum_n \gamma_n(n\Omega_0 + n\alpha t) \cos(n\Omega_0 t + \frac{n}{2} \alpha t^2 + \phi_n) \right] + \omega_d^2 \left[ \sum_n \gamma_n \sin(n\Omega_0 t + \frac{n}{2} \alpha t^2 + \phi_n) \right] \right\}$$  \hspace{1cm} (14)

where $*$ represents the convolution product. With the order of integration and summation switched, equation (14) is rewritten as

$$\theta_d(t) = \sum_n (\theta_{a,A_n} + \theta_{a,B_n})$$  \hspace{1cm} (15)

$$\theta_{a,A_n}(t) = \frac{\omega_d^2 \gamma_n}{2 \omega_d} \sin[\omega_d(t - u)] e^{-\zeta \omega_d (t-u)} \left[ \cos[\frac{n}{2} \alpha u^2 + (n\Omega_0 - \omega_d)u + \omega_d t + \phi_n] \right]$$  \hspace{1cm} (16)

Second, the general solution of an indefinite integral of the product of $\cos(ax^2 + bx + c)$ and $e^{igx}$ over domain $x$ is given as

$$\int \cos(ax^2 + bx + c) e^{igx} \ dx$$

$$= \text{Re} \left\{ e^{iax^2 + bx + c} e^{igx} \right\}$$  \hspace{1cm} (21)

where $\int \cos x \ dx = \text{Re}(\int e^{ix} \ dx)$ relationship is utilized\(^3\)\(^2\) and where $\text{Re}$ is the real part of a complex quantity. Based upon the above solution, $\theta_{a,A_n}(t)$ is calculated by separately solving $\theta_{a,A_n,z}(t)$ and $\theta_{a,A_n,l}(t)$ according to

$$\theta_{a,A_n,l}(t) = \frac{\omega_d^2 \gamma_n}{2 \omega_d} \text{Re} \left\{ -\sqrt{-1} \sqrt{\frac{\pi}{2}} e^{-i(\xi b + \frac{i}{2} a^2 + b + \frac{ig}{2} x)} \text{erf}\left[ \sqrt{\frac{1}{2}} \sqrt{-\frac{1}{2} - \sqrt{-1} f / 2} a + (\xi x) + \frac{1}{2} \frac{b}{2} \right] \right\}$$  \hspace{1cm} (22)

where $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} \ dt$ is the error function\(^3\)\(^2\) and where

$$\text{ERF}_{A_n,z}(u) = \text{erf}\left[ \frac{(-1)^{3/4} (n\Omega_0 - \omega_d)}{2 \sqrt{(n/2) \alpha}} - \frac{\sqrt{-1} (-\xi \omega_d)}{2 \sqrt{(n/2) \alpha}} + (\frac{1}{2})^{3/4} \sqrt{\frac{n}{2}} u \right]$$  \hspace{1cm} (23)
\[ \theta_{a \rightarrow 2}(t) = \frac{a_0^2 \gamma}{\omega_d} \Re \left\{ -\sqrt{-1} \sqrt{\pi} \ e^{-i(n\Omega_0 + \omega_d - \xi \omega_1)^2/2\alpha} \frac{1}{\sqrt{n/2} \alpha} \left[ \text{ERF}_{An\rightarrow 2}(t) - \text{ERF}_{An\rightarrow 2}(0) \right] \right\} \]

\[ \text{ERF}_{An\rightarrow 2}(u) = \text{erf} \left\{ \frac{-1}{\sqrt{2\alpha n}} \left[ -\frac{1}{\sqrt{4\pi \alpha n}} \right] \right\} \]

Then, \( \theta_{a \rightarrow 2}(t) \) is found as

\[ \theta_{a \rightarrow 2}(t) = \theta_{a \rightarrow 2}(t) + \theta_{a \rightarrow 2}(t) \]

Further, \( \theta_{a \rightarrow 2}(t) \) is expanded as

\[ \theta_{a \rightarrow 2}(t) = \theta_{a \rightarrow 2}(t) + \theta_{a \rightarrow 2}(t) \]

The general solution of the indefinite integral of the product of \( \sin(\alpha x^2 + bx + c) \), \( e^{i(\sigma x - n)} \) and \( h + kx \) over domain \( x \) is given by

\[ \int \sin(\alpha x^2 + bx + c) e^{i(\sigma x - n)} (h + kx) \ dx = \int \left\{ \sin \left[ \frac{\pi}{2} \alpha x^2 + (n\Omega_0 + \omega_d) x + \omega_d \right] \right\} \]

Based on the above solution, \( \theta_{a \rightarrow 2}(t) \) and \( \theta_{a \rightarrow 2}(t) \) are calculated as

\[ \theta_{a \rightarrow 2}(t) = \frac{\xi \omega_1 \gamma}{\omega_d} \]

\[ \theta_{a \rightarrow 2}(t) = -\frac{1}{4(\pi n/2)^{1/2}} \left[ \Phi_{a \rightarrow 2}(t) - \Phi_{a \rightarrow 2}(0) \right] \]

\[ \Phi_{a \rightarrow 2}(u) = \Phi_{a \rightarrow 2}(u) \]

\[ \Phi_{a \rightarrow 2}(u) = \Phi_{a \rightarrow 2}(u) \]

\[ \Phi_{a \rightarrow 2}(u) = \Phi_{a \rightarrow 2}(u) \]

\[ \Phi_{a \rightarrow 2}(u) = \Phi_{a \rightarrow 2}(u) \]

\[ \Phi_{a \rightarrow 2}(u) = \Phi_{a \rightarrow 2}(u) \]
\[ F(t) = e^{-\alpha t} \left( \text{erfi} \left( \frac{\xi}{\sqrt{2}} \right) \right) \]

Finally, \( \theta_{u, \text{damped}}(t) \) is found by combining \( \theta_{u, \text{undamped}}(t) \) and \( \theta_{u, \text{damping}}(t) \) as

\[
\theta_{u, \text{damped}}(t) = \frac{\xi \omega_0 n_0}{\omega_d} \text{Im} \left\{ \frac{-1}{4 \left( \frac{\alpha \omega_0}{2} \right)^2} \left[ \Phi_{u, \text{damped}}(t) - \Phi_{u, \text{damped}}(0) + \Phi_{u, \text{undamped}}(t) - \Phi_{u, \text{undamped}}(0) \right] \right\}
\]

The analytical solution of \( \delta(t) \) is then derived from equation (44) as

\[
\delta(t) = \Theta(t) - \phi(t) = \left\{ \Omega_0 t + \frac{1}{2} \alpha t^2 + \sum_n \left[ \gamma_n \sin \left( n \Omega_0 t + \frac{n}{2} \alpha t^2 + \phi_n \right) \right] \right\}
\]

Then, since \( \xi \) and \( \gamma_n \) assume small values, equation (45) is further simplified to

\[
\delta(t) = \sum_n \left[ \theta_{u, \text{undamped}}(t) + \theta_{u, \text{damping}}(t) \right]
\]

Much simpler expressions of the closed-form solutions of \( \Theta(t) \) and \( \delta(t) \) are found for an undamped SDOF system; these are summarized in Appendix 1 for completeness.

**Verification of analytical solution**

Both single-order \( (n = 1) \) and multi-order \( (n = 1.5, 3, 4.5, 6, 7.5, 9) \) excitation cases are examined with \( \alpha = 5 \) (r/min)/s and \( \zeta = 0.001 \) so as to verify the new analytical solution. The solutions of \( \delta(t) \) for \( n = 1 \) are displayed in Figure 8 by using the parameters in the section entitled ‘Nonlinear torsional model (with piecewise linear clutch damper)’; a reasonable match between theory and computation verifies equation (44). Resonant amplifications from both solutions begin at \( \Omega_c (760 \text{ r/min}) \) and reach the same peak-to-peak amplification value at \( \Omega_p (805 \text{ r/min}) \). Note that \( \Omega_p (805 \text{ r/min}) \) is slightly higher than the critical speed \( \Omega_c (792 \text{ r/min}) \).

The multi-order excitation case is examined next for a six-cylinder internal-combustion engine case with \( n = 3 \) as the dominant firing order. The typical values of \( \gamma_n \) and \( \phi_n \) for \( n = 1.5, 3, 4.5, 6, 7.5, 9 \) reported in Table 2 (as extracted from the literature \( 25,26 \)) are

![Figure 8. Verification of the analytical solution for the damped linear SDOF system (Figure 5(a)); (a) analytical solution; (b) numerical solution. rpm: r/min.](image)

<table>
<thead>
<tr>
<th>Parameter (units)</th>
<th>Value for the following ( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_n ) (rad)</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>0.0150</td>
</tr>
<tr>
<td>( \phi_n ) (rad)</td>
<td>0.01</td>
</tr>
</tbody>
</table>
utilized to construct the flywheel motion input in equations (3) and (4). Then equation (8), given the constructed motion input, is solved via the proposed analytical and numerical methods. The solutions of \( \dot{d}(t) \) from both methods are shown in Figure 9, where a reasonable agreement is achieved between the analytical solution and the numerical method. Both transient responses exhibit the same four peaks at these speeds: \( \Omega_{p_1} = 792 \text{ r/min}, \ \Omega_{p_2} = 405.1 \text{ r/min}, \ \Omega_{p_3} = 202.6 \text{ r/min}, \ \Omega_{p_4} = 540.8 \text{ r/min}. \) The peak-to-peak values \( \delta_{p-p} \) of amplification are summarized in Table 3, where a comparison for the \( n = 1 \) case is also included. For the multi-order excitation case, both Figure 9 and Table 3 indicate that \( \dot{d}(t) \) has the highest amplification at \( n = 3 \), as expected; other amplifications occur at the orders \( n = 1.5, n = 6, \) and \( n = 9. \) Additionally, the speeds \( \Omega_{p} \) corresponding to the local peaks are slightly higher than the corresponding critical speeds \( \Omega_c. \) Both

<table>
<thead>
<tr>
<th>Excitation order ( n )</th>
<th>Critical speed ( \Omega_c ) (r/min)</th>
<th>Analytical closed-form expression</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-order ( (n = 1) ) excitation</td>
<td>792</td>
<td>805.1</td>
<td>2.8</td>
</tr>
<tr>
<td>Excitation order ( n )</td>
<td>Critical speed ( \Omega_c ) (r/min)</td>
<td>Analytical closed-form expression</td>
<td>Numerical solution</td>
</tr>
<tr>
<td>Multiple-order excitation</td>
<td>88 ((n = 9))</td>
<td>93.2</td>
<td>2.3</td>
</tr>
<tr>
<td>\ 132 ((n = 6))</td>
<td>138.4</td>
<td>3.1</td>
<td>137.7</td>
</tr>
<tr>
<td>\ 264 ((n = 3))</td>
<td>272.9</td>
<td>11.6</td>
<td>272.1</td>
</tr>
<tr>
<td>\ 528 ((n = 1.5))</td>
<td>540.8</td>
<td>3.7</td>
<td>538.9</td>
</tr>
</tbody>
</table>

Figure 9. Verification of the analytical solution for the damped linear SDOF system (Figure 5(a)) with multi-order motion excitation, given \( \zeta = 0.001 \) and \( \alpha = 5 \) (r/min)/s: (a) analytical solution; (b) numerical solution. See Table 2 for the \( n, g_n, \) and \( \phi_n \) values.

rpm: r/min.

Figure 10. Short-time Fourier transform of \( \dot{d}(t) \) for the linear damped SDOF system (Figure 5(a)) with multi-order motion excitation, given \( \zeta = 0.001 \) and \( \alpha = (r/min)/s. \) The colored areas represent the amplitude in decibels with reference to 1.0 rad. rpm: r/min.

Figure 9 and Table 3 suggest that the new analytical solution is valid for the multi-order excitation case.

The time–frequency domain representation of the numerically obtained \( \delta(t) \) is examined for the speed order analysis during the acceleration process. Specifically, the short-time Fourier transform technique is employed, and the result is displayed in Figure 10. Here, the \( n = 3 \) order has the highest amplitude (with the darkest color), and the other three dominant orders are \( n = 1.5, 6, \) and 9. The horizontal line at 13.2 Hz represents the natural frequency \( \omega_1. \) As the speed increases with \( \alpha = 5 \) (r/min)/s, the four order lines intersect with \( \omega_1 \) at four speeds; the four critical speeds \( \Omega_c \) are observed as darker regions.

Finally, in order to examine the application of the linear-system-based theory, the peak-to-peak value \( \delta_{p-p} \) found from the closed-form solution is compared with the results of a nonlinear SDOF powertrain system (with \( \eta = \mu = 0.3 \)) by using the parameters in the section entitled 'Nonlinear torsional model (with
Conclusion

The chief contribution of this article is the development of a new closed-form solution that approximates the transient vibration amplification of a nonlinear multi-staged clutch damper during the engine start-up; a damped linear torsional oscillator with instantaneous-frequency excitation is utilized to find this solution. The results of an SDOF powertrain system with a flywheel motion input for the start-up process is numerically verified and experimentally validated by comparing the peak displacements. The previous analytical work by Sen et al.\textsuperscript{12,13} with an instantaneous torque input is extended by developing a closed-form solution with a motion input that includes instantaneous frequency and amplitude terms. The chief limitation is the applicability of the error function algorithm, since it loses high accuracy in specific regions; this has obviously hindered historical calculations as well. Future work should focus on the transient analyses of multi-degree-of-freedom nonlinear driveline systems for both conventional vehicles and hybrid vehicles.\textsuperscript{14}

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Declaration of conflict of interest

The authors declare that there is no conflict of interest.

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References

Appendix I

Analytical solution for an undamped linear torsional oscillator

For an undamped SDOF system ($\zeta = 0$), equation (8) is rewritten as the two equations

\[ \dot{\theta}_1(t) + \omega_1^2 \theta_1(t) = \omega_1^2 \Omega_0 t + \frac{1}{2} \omega_1^2 \alpha t^2 \]  
\[ \dot{\theta}_2(t) + \omega_1^2 \theta_2(t) = \omega_1^2 \sum_{n} \gamma_n \sin \left( n\Omega_0 t + \frac{n}{2} \alpha t^2 + \varphi_n \right) \]  

First, the system is assumed to rotate with a constant speed $\Omega_0$, and thus the initial conditions are $\theta_1(0) = 0$ rad, $\dot{\theta}_1(0) = \Omega_0$, $\theta_2(0) = 0$ rad and $\dot{\theta}_2(0) = 0$ rad/s. Then, $\theta_1(t)$ is found by independently solving equation (47) with the assumed initial conditions. The general response in the Laplace domain to an arbitrary excitation $T(t)$ with the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$ is found as

\[ \Theta(s) = \frac{T(s)}{s^2 + 2\xi \omega_1 s + \omega_1^2} + \frac{s + 2\xi \omega_1}{s^2 + 2\xi \omega_1 s + \omega_1^2} \theta_0 \]  
\[ + \frac{1}{s^2 + 2\xi \omega_1 s + \omega_1^2} \dot{\theta}_0 \]  

Let

\[ T(s) = \mathcal{L} \left( \omega_1^2 \Omega_0 t + \frac{1}{2} \omega_1^2 \alpha t^2 \right) \]  

Now employ the inverse Laplace transformation and convolution theorem to yield

\[ \theta_1(t) = \Omega_0 t + \frac{1}{2} \alpha t^2 + \frac{\alpha}{\omega_1} \left[ \cos(\omega_1 t) - 1 \right] \]  

Two issues are observed with this analytical solution of $\theta_1(t)$. First, $\theta_1(t)$ follows $\phi(t)$ since the $\dot{\theta}_1 + \omega_1^2 \theta_1$ term is present in equation (51). Second, there is an oscillatory part $\left( \alpha/\omega_1 \right) \left[ \cos(\omega_1 t) - 1 \right]$ even though the amplitude $\alpha/\omega_1^2$ is relatively small because $\omega_1 = 82.9$ rad/s and $\alpha$ normally ranges from 0.1 rad/s$^2$ to 15 rad/s$^2$ (from 1 (r/min)/s to 150 (r/min)/s). As a consequence, $\left( \alpha/\omega_1^2 \right) \left[ \cos(\omega_1 t) - 1 \right]$ is neglected. Next, $\theta_1(t)$ is analytically solved from equation (48) to give

\[ \theta_2(t) = \int_{0}^{t} \frac{\sin(\omega_1 (t-u))}{\omega_1} \, du \]  
\[ \omega_1^2 \sum_{n} \gamma_n \sin \left( n\Omega_0 t + \frac{n}{2} \alpha t^2 + \varphi_n \right) \]  

where $u$ is a dummy variable.

By using trigonometric identities, equation (52) is rewritten as
\[ \theta_a(t) = \frac{\omega_1}{2} \sum_n \gamma_n[\theta_{a,n1}(t) - \theta_{a,n2}(t)] \]  

(53)

\[ \theta_{a,n1}(t) = \int_0^t \cos \left[ \frac{\pi}{2} au^2 + (n\Omega_0 - \omega_1)u + \omega_1 t + \varphi_n \right] \, du \]  

(54)

\[ \theta_{a,n2}(t) = \int_0^t \cos \left[ \frac{\pi}{2} au^2 + (n\Omega_0 + \omega_1)u - \omega_1 t + \varphi_n \right] \, du \]  

(55)

The order of integration and summation is switched in equation (53) so as to evaluate separately the convolution at each order \( n \). Thus, \( \theta_{a,n1}(t) \) and \( \theta_{a,n2}(t) \) are separately calculated at each \( n \) by using the formulas

\[ C(x) + iS(x) = \frac{1 + i}{2} \text{erf} \left[ \frac{\sqrt{\pi}}{2} (1 - i)x \right] \]  

(56)

\[ \int \cos(au^2 + bx + c) \, dx = \frac{\sqrt{\pi}}{2} \cos \left[ \frac{b^2}{4a} - c \right] C[b + 2ax] \sqrt{\pi} \right] + \sin \left( \frac{b^2}{4a} - c \right) S[b + 2ax] \sqrt{\pi} \right] \]  

\[ \theta_{a,n1}(t) = \Psi_{a,n1}(t) - \Psi_{a,n1}(0) \]  

(58)

\[ \Psi_{a,n1}(u) = \sqrt{\frac{\pi}{\alpha \pi}} \left\{ \cos \left[ \frac{(n\Omega_0 - \omega_1)^2}{2\alpha} \right] - \omega_1 t + \varphi_n \right\} C \left( \frac{n\Omega_0 - \omega_1 + n\alpha u}{\sqrt{\alpha \pi}} \right) + \sin \left[ \frac{(n\Omega_0 - \omega_1)^2}{2\alpha} \right] - \omega_1 t + \varphi_n \right\} S \left( \frac{n\Omega_0 - \omega_1 + n\alpha u}{\sqrt{\alpha \pi}} \right) \]  

(59)

\[ \theta_{a,n2}(t) = \Psi_{a,n2}(t) - \Psi_{a,n2}(0) \]  

(60)

\[ \Psi_{a,n2}(u) = \sqrt{\frac{\pi}{\alpha \pi}} \left\{ \cos \left[ \frac{(n\Omega_0 + \omega_1)^2}{2\alpha} \right] + \omega_1 t + \varphi_n \right\} C \left( \frac{n\Omega_0 + \omega_1 + n\alpha u}{\sqrt{\alpha \pi}} \right) + \sin \left[ \frac{(n\Omega_0 + \omega_1)^2}{2\alpha} \right] + \omega_1 t + \varphi_n \right\} S \left( \frac{n\Omega_0 + \omega_1 + n\alpha u}{\sqrt{\alpha \pi}} \right) \]  

(61)

where \( C(x) \) and \( S(x) \) represent the Fresnel integrals, which are related to the error function \( \text{erf}(x) \).

By substituting equations (58) and (60) into equation (53), an analytical solution of \( \theta_a(t) \) is found as

\[ \theta_a(t) = \frac{\omega_1}{2} \sum_n \gamma_n \left[ \frac{\Psi_{a,n1}(t) - \Psi_{a,n1}(0)}{-\Psi_{a,n2}(0) - \Psi_{a,n2}(0)} \right] \]  

(62)

Combining equations (51) and (62) yields \( \theta(t) \) as

\[ \theta(t) = \theta_a(t) + \theta_b(t) \]  

\[ = \Omega_0 t + \frac{1}{2} \alpha t^2 + \frac{\alpha}{\omega_1} \left[ \cos(\alpha_1 t) - 1 \right] \]  

\[ + \frac{\omega_1}{2} \sum_n \gamma_n \left[ \frac{\Psi_{a,n1}(t) - \Psi_{a,n1}(0)}{-\Psi_{a,n2}(0) - \Psi_{a,n2}(0)} \right] \]  

(63)

Then, \( \delta(t) \) is found as

\[ \delta(t) = \theta(t) - \phi(t) \]  

\[ = \frac{\omega_1}{2} \sum_n \gamma_n \left[ \frac{\Psi_{a,n1}(t) - \Psi_{a,n1}(0)}{-\Psi_{a,n2}(0) - \Psi_{a,n2}(0)} \right] \]  

(64)

For a lightly damped system, equation (64) may be efficiently utilized to approximate the transient amplification of a piecewise linear system under a rapid acceleration rate.